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Spectra of k coalescence of complete graphs

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Abstract: Coalescence or overlap of graphs is a significant operation involving two graphs, due to a nice expression, for its chromatic polynomial. The spectra of vertex (1 coalescence) and edge (2 coalescence) for a pair of complete graphs are obtained by Jog, Kotambari in 2016. Here we generalise the concept and obtain the adjacency, Laplacian and signless Laplacian spectra for a pair of complete graphs.

Key words: Laplacian, signless Laplacians, coalescence, Subdivision graph and line graph

1. Introduction

The basic symmetric matrices in Spectral graph theory we come across are, the adjacency, Laplacian and signless Laplacian matrix. Their eigenvalues along with multiplicity giving spectra correspondingly. Due to several applications, to various fields,it still remains as thrust area of research although in a different direction. The chromatic polynomial, came into existence in a bid to solve, the famous four color conjecture. Recently chromatic number of integer distance graphs is considered for analysis. One can refer [14] for nice connection between chromatic number and set theory analysis, topology and number theory. Based on greedy coloring there is also another coloring of graph known as first fit coloring [15].

Chromatic polynomial of coalescence, has a simple expression in terms of chromatic polynomials of individual graphs [1]. In view of this, Jog and Kotambari deduced the adjacency, Laplacian and signless Laplacian spectra of vertex and edge coalescence of complete graphs [13]. In this paper,we define k coalescence and generalise all the results obtained in [13]. For detailed work on Laplacian spectra one can refer [2–7]. For some work on the signless Laplacian spectra refer[8–10].

Another interesting connection of the coefficients of characteristic polynomials of an arbitrary matrix in terms of bell polynomials are given in [16]

Example: G_1 and G_2 are non isomorphic coalescence's of H_1 and H_2 in K_4.

1.1. On the k coalescence of complete graphs

Definition: The line graph $L(G)$ of a graph G is a graph with vertex set as edge set of given graph and two elements adjacent in it if they are incident (have a point in common) in G. If G has order n and size m(number of edges), then $L(G)$ has order m and size depending on structure of G.

Definition:The subdivision graph $S(G)$ of a graph G is a graph obtained from G by inserting a vertex into

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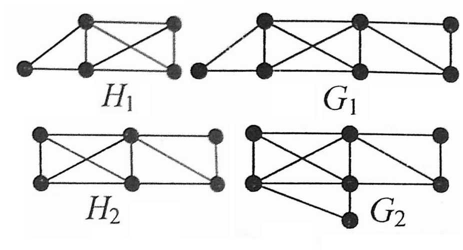

every edge of G. If G has order n and size m (number of edges), then $S(G)$ has order $m+n$ and size $2n$.
Definition: k **coalescence of graphs;** For a pair of connected graphs G_1 and G_2 with n_1, n_2 vertices and m_1, m_2 edges respectively having an induced complete graph of order k with $n_1, n_2 \geq k$, the graph obtained by merging k vertices on kC_2 edges of induced K_k is called as k coalescence denoted by $G_1 O_k G_2$. The graph $G_1 O_k G_2$ is of order $'n_1+n_2-k'$ with $'m_1+m_2-kC_2$'edges.

Now we consider k coalescence of complete graphs in particular. Let K_{n_1} and K_{n_2} be the complete graphs of order n_1 and n_2. The graph $K_{n_1} O_k K_{n_2}$ is of order $'n_1+n_2-k'$ with $'n_1C_2+n_2C_2-kC_2$' edges. It will have $'n_1-k'$ vertices of degree $'n_1-1'$, $'n_2-k'$ vertices of degree $'n_2-1'$ and remaining $'k'$' vertices of degree $'n_1+n_2-k-1'$.

Example:

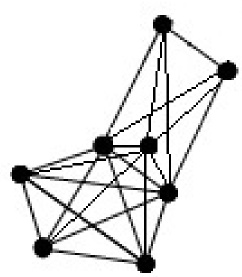

Figure : $K_6 O_3 K_5$

We denote the characteristic polynomial of G as, $P(G:\lambda)$ and spectra of a graph having eigenvalue λ_i with multiplicity m_i as, $\langle \lambda_1^{m_1}, \lambda_2^{m_2}, ..., \rangle$. We require following Lemma.

Lemma 1.1. *Let A, B,I(identity matrix) and J(matrix of all $1's$) are square matrices of same order q, then the block determinant of order pq,*
$|AI_p + B(J_p - I_p)| = |A-B|^{p-1}|A+(p-1)B|$.

Theorem 1.1. $P(K_{n_1} O_k K_{n_2} : \lambda_c)$ *of the k coalescence $K_{n_1} O_k K_{n_2}$ is given by,*

$$|\lambda_c I - A(K_{n_1} O_k K_n)| = (\lambda_c+1)^{n_1+n_2-k-3}[\lambda_c^3 - (n_1+n_2-k-3)\lambda_c^2 + [(n_1-k-1)(n_2-k-1)$$
$$+(k-1)(n_1+n_2-2k-2)-k(n_1+n_2-2k)]\lambda_c + k(n_1-k)(n_2-k-1)+k(n_2-k)(n_1-k-1)-(k-1)(n_1-k-1)(n_2-k-1)]$$

Proof. The adjacency matrix of $K_{n_1}O_kK_{n_2}$ has the form

$$A(K_{n_1}O_kK_{n_2}) = \begin{bmatrix} A(K_k) & J_{k\times n_1-k} & J_{k\times n_2-k} \\ J_{n_1-k\times k} & A(K_{n_1-k}) & O_{n_1-k\times k} \\ J_{n_1-k\times k} & O_{n_2-k\times k} & A(K_{n_2-k}) \end{bmatrix} \tag{1}$$

where A is adjacency matrix, J is a matrix with all elements $1's$ and O is a matrix with all zeros. $P(K_{n_1}O_kK_{n_2} : \lambda_c)$ is given by,

$$|\lambda_c I - A(K_{n_1}O_kK_{n_2})| = \begin{vmatrix} \lambda_c I - A(K_k) & -J_{k\times n_1-k} & -J_{k\times n_2-k} \\ -J_{n_1-k\times k} & \lambda_c I - A(K_{n_1-k}) & O_{n_1-k\times k} \\ -J_{n_1-k\times k} & O_{k\times n_1-k} & \lambda_c I - A(K_{n_2-k}) \end{vmatrix}$$

Adding to first k columns addition of next n_1 columns multiplied by $\frac{1}{\lambda_c-(n_1-k-1)}$, followed by addition of next n_2 columns multiplied by $\frac{1}{\lambda_c-(n_2-k-1)}$ we get,

$$|\lambda_c I - A(K_{n_1}O_kK_{n_2})| = \begin{vmatrix} B & -J_{k\times n_1-k} & -J_{k\times n_2-k} \\ O & \lambda_c I - A(K_{n_1-k}) & O_{n_1-k\times k} \\ O & O_{k\times n_1-k} & \lambda_c I - A(K_{n_2-k}) \end{vmatrix}$$

where B stands for
$B = \lambda_c I - \frac{n_1-k}{\lambda_c-(n_1-k-1)} + \frac{n_2-k}{\lambda_c-(n_2-k-1)} A(K_k)$;
On direct diagonal expansion we have,
$|\lambda_c I - A(K_{n_1}O_kK_{n_2})|$
$= |\lambda_c I - A(K_{n_1-k})||\lambda_c I - A(K_{n_2-k})|[|\lambda_c - \frac{n_1-k}{\lambda_c-(n_1-k-1)} \frac{n_2-k}{\lambda_c-(n_2-k-1)} I_k| - |1 + \frac{n_1-k}{\lambda_c-(n_1-k-1)} + \frac{n_2-k}{\lambda_c-(n_2-k-1)}(J_k - I_k)|]$
On applying Lemma 1.1 and simplifying we finally arrive at the result. □

Theorem 1.2. *The Laplacian spectrum of the k coalescence $K_{n_1}O_kK_{n_2}$ is given by*
$\mathcal{L}_{spec}(K_{n_1}O_kK_{n_2}) = \langle 0^1, k^1, (n_1+n_2-k)^k, n_1^{n_1-k-1}, n_2^{n_2-k-1} \rangle$.

Proof. The Laplacian matrix of $K_{n_1}O_kK_{n_2}$ has the form

$$\mathcal{L}(K_{n_1}O_kK_{n_2}) = \begin{bmatrix} A & B & C \\ B^T & E & F \\ C^T & F^T & I \end{bmatrix}$$

where A,B,C,E,F, and I stands for
$A = (n_1+n_2-k)I_k - A(K_k)$; $B = -J_{k\times n_1-k}$; $C = -J_{k\times n_2-k}$; $E = (n_1-1)I_{n_1-k} - A(K_{n_1-k})$; $F = O_{n_1-k\times k}$;
$I = (n_2-1)I_{n_1-k} - A(K_{n_2-k})$
The characteristic polynomial of $L(K_{n_1}O_kK_{n_2})$ is the Laplacian polynomial is given by

$$|\mu_c I - \mathcal{L}(K_{n_1}O_kK_{n_2})| = \begin{vmatrix} A' & -B & -C \\ -B^T & E' & -F \\ -C^T & -F^T & I' \end{vmatrix}$$

where $A' = [\mu_c - (n_1+n_2-k)]I_k - A(K_k)$; $E' = [\mu_c - (n_1-1)]I_{n_1-k} + A(K_{n_1-k})$; $I' = [\mu_c - (n_2-1)]I_{n_1-k} +$

$A(K_{n_2-k})$
Subtracting from each of first k columns addition of next n_1-k columns multiplied by $\frac{1}{\mu_c-k}$, followed by addition of next (n_2-k) columns multiplied by $\frac{1}{\mu_c-k}$ we get,

$$|\mu_c I - \mathcal{L}(K_{n_1}O_kK_{n_2})| = \begin{vmatrix} A'' & B'' & C'' \\ -B'' & E'' & F'' \\ -C'' & -F'' & I'' \end{vmatrix}$$

where
$A'' = [\mu_c - (n_1+n_2-k)I_k - \frac{n_1+n_2-2k}{\mu_c-k}]I_k - \frac{n_1+n_2-2k}{\mu_c-k} + A(K_k)$; $B'' = J_{k\times n_1-k}$; $C'' = J_{k\times n_2-k}$; $E'' = [\mu_c-(n_1-1)]I_{n_1-k} + A(K_{n_1-k})]$; $F'' = O_{n_1-k\times k}$; $I'' = [\mu_c-(n_2-1)]I_{n_2-k} + A(K_{n_2-k})]$
On diagonal expansion and simplification in a similar manner we arrive at,
$|\mu_c I - \mathcal{L}(K_{n_1}O_kK_{n_2})| = [\mu_c-(n_2+n_1-k)]^{k-1}[\mu_c-(n_2+n_1-k-1) - \frac{n_1+n_2-2k}{\mu_c-k}k + (k-1)][\mu_c-k]^2[\mu_c - n_1]^{n_1-k-1}[\mu_c-n_2]^{n_2-k-1}$
$= \mu_c[\mu_c-k][\mu_c-(n_2+n_1-k)]^k][\mu_c-n_1]^{n_1-k-1}[\mu_c-n_2]^{n_2-k-1}$
Hence the theorem. □

Corollary 1.1. *[7] The spanning tree count of $K_{n_1}O_kK_{n_2}$ is given by,*

$\tau(K_{n_1}O_kK_{n_2}) = \frac{k(n_1+n_2-k)^k n_1^{n_1-k-1} n_2^{n_2-k-1}}{n_1+n_2-k} = k(n_1+n_2-k)^{k-1}n_1^{n_1-k-1}n_2^{n_2-k-1}$, *where* $n_1, n_2 \geq k+1$
Further if $n_1 = n_2 = k+1$ *we get the number of spanning trees in* $K_{k+1}O_kK_{k+1}$, *as* $k(k+2)^{k-1}$

Theorem 1.3. *The signless Laplacian spectrum of $K_{n_1}O_kK_{n_2}$ is,*

$Q_{spec}(K_{n_1}O_kK_{n_2}) = \langle (n_1+n_2-k-2)^{k-1}, (n_1-2)^{n_1-k-1}, (n_2-2)^{n_2-k-1}, \alpha_i^1,\ i=1,2,3\rangle$
where α_i *satisfy*
$[\gamma_c^3 - (3n_1+3n_2-2k-7)\gamma_c^2 + [(n_1+n_2-2)(2n_1-k-2) + (3n_21+_2-k-4)(2n_2-k-2) - k(n_1-k) - k(n_2-k)]\gamma_c + [k(n_1-k)(2n_1-k-2) + k(n_2-k)(2n_2-k-2) - (n_1+n_2-2)(2n_1-k-2)(2n_2-k-2)] = 0$

Proof. The signless Laplacian matrix of $K_{n_1}O_kK_{n_2}$ has the form

$$Q(K_{n_1}O_kK_{n_2}) = \begin{bmatrix} A & B & C \\ B^T & E & F \\ C^T & F^T & I \end{bmatrix}$$

where A,B,C,E,F, and I stands for
$A = (n_1+n_2-k-1)I_k + A(K_k)$; $B = J_{k\times n_1-k}$; $C = J_{k\times n_2-k}$; $E = (n_1-1)I_{n_1-k} + A(K_{n_1-k})$; $F = O_{n_1-k\times k}$; $I = (n_2-1)I_{n_1-k} + A(K_{n_2-k})$
The characteristic polynomial of $Q(K_{n_1}O_kK_{n_2})$ is the signless Laplacian polynomial, given by

$$|\gamma_c I - Q(K_{n_1}O_kK_{n_2})| = \begin{vmatrix} A' & -B & -C \\ -B^T & E' & -F \\ -C^T & -F^T & I' \end{vmatrix} \quad (2)$$

where A', E' and I' stand for
$A' = [\gamma_c - (n_1+n_2-k-1)]I_k - A(K_k)$; $B' = -J_{k\times n_1-k}$; $C' = J_{k\times n_2-k}$; $E' = [\gamma_c-(n_1-1)]I_{n_1-k} - A(K_{n_1-k})$;

$F' = O_{n_1-k\times k}\,;\; I' = [\gamma_c - (n_2-1)]I_{n_1-k} - A(K_{n_2-k}))$

Adding to first k columns addition of next n_1-k columns multiplied by $\frac{1}{\gamma_c-(2n_1-k-2)}$, followed by addition of next $'n_2-k'$ columns multiplied by $\frac{1}{\gamma_c-(2n_2-k-2)}$ we get,

$$|\gamma_c I - Q(K_{n_1}O_kK_{n_2})| = \begin{vmatrix} A'' & B'' & C'' \\ B''^T & E'' & F'' \\ C''^T & F''^T & I'' \end{vmatrix}$$

where A'', B'', E'', F'', and I'' stand for

$A'' = [\gamma_c-(n_1+n_2-k)-\frac{n_1-k}{\gamma_c-(2n_1-k-2)}-\frac{n_2-k}{\gamma_c-(2n_2-k-2)}]I_k-\frac{n_1-k}{\gamma_c-(2n_1-k-2)}-\frac{n_2-k}{\gamma_c-(2n_2-k-2)}-A(K_k)\,;\; C'' = -J_{k\times n_2-k}\,;$

$E'' = [\gamma_c-(n_1-1)]I_{n_1-k} - A(K_{n_1-k})\,;\; F'' = O_{n_1-k\times k}\,;\; I'' = [\gamma_c-(n_2-1)]I_{n_2-k} - A(K_{n_2-k})$

On diagonal expansion and simplification in a similar manner we arrive at,

$|\gamma_c I - Q(K_{n_1}O_kK_{n_2})| = [\gamma_c-(n_2+n_1-k-2)]^{k-1}[\gamma_c-(n_1-2)]^{n_1-k-1}[\gamma_c-(n_2-2)]^{n_2-k-1}[\gamma_c-(n_1+n_2-2)-\frac{k(n_1-k)}{\gamma_c-(2n_1-k-2)}-\frac{k(n_2-k)}{\gamma_c-(2n_2-k-2)}]$

Hence the theorem. □

Corollary 1.2. *[9] If $S(K_{n_1}O_kK_{n_2})$ denotes subdivision graph of $K_{n_1}O_kK_{n_2}$ then $P(S(K_{n_1}O_kK_{n_2})$ is,*

$|\lambda_c I - S(K_{n_1}O_kK_{n_2})| = \lambda_c^{(n_1C_2+n_2C_2-kC_2)-(n_1+n_2-k)}[\lambda_c^2-(n_1+n_2-k-2)]^{k-1}[\lambda_c^2-(n_1-2)]^{n_1-k-1}[\lambda_c^2-(n_2-k-1)]^{n_2-k-1}[\lambda_c^6-(3n_1+3n_2-2k-7)]\lambda_c^4+[(n_1+n_2-2)(2n_1-k-2)+(3n_1+n_2-k-4)(2n_2-k-2)-k(n_1-k)-k(n_2-k)]\lambda_c^2+[k(n_1-k)(2n_1-k-2)+k(n_2-k)(2n_2-k-2)-(n_1+n_2-2)(2n_1-k-2)(2n_2-k-2)]$.

So that the adjacency spectra of $S(K_{n_1}O_kK_{n_2})$ is,

$A_{spec}[S(K_{n_1}O_kK_{n_2})] = \langle a^f, b^g, c^h, d^i, e^j\rangle$

where, $a=0\,;b=\pm\sqrt{n_1+n_2-k-2}\,;c=\pm\sqrt{n_1-2}\,;d=\pm\sqrt{n_2-2}\,;e=\alpha_i(i=1,2,...6)\,;\; f=(n_1C_2+n_2C_2-kC_2)-(n_1+n_2-k)\,;\; g=k-1\,;\; h=n_1-k-1\,;\; i=n_2-k-1\;\; j=1$

Where $\alpha_i\;\;(i=1,2,....6)$ *satisfy*

$\lambda_c^6-(3n_1+3n_2-2k-6)\lambda_c^4+[(n_1+n_2-2)(2n_1-k-2)+(3n_1+n_2-k-4)(2n_2-k-2)-k(n_1-k)-k(n_2-k)]\lambda_c^2+[k(n_1-k)(2n_1-k-2)+k(n_2-k)(2n_2-k-2)-(n_1+n_2-2)(2n_1-k-2)(2n_2-k-2)]=0$

Corollary 1.3. *[8] If $L(K_{n_1}O_kK_{n_2})$ denotes line graph of $K_{n_1}O_kK_{n_2}$ then $P(L(K_{n_1}O_kK_{n_2}):\lambda_c)$,*

$|\lambda_c I - L(K_{n_1}O_kK_{n_2})| = (\lambda_c+2)^{(n_1C_2+n_2C_2-kC_2)-(n_1+n_2-k)}[\lambda_c-(n_1+n_2-k-6)]^{k-1}[\lambda_c-(n_1-6)]^{n_1-k-1}[\lambda_c-(n_2-6)]^{n_2-k-1}[(\lambda_c+2)^3-(3n_1+3n_2-2k-6)(\lambda_c+2)^2]+[(n_1+n_2-2)(2n_1-k-2)+(3n_1+n_2-k-4)(2n_2-k-2)-k(n_1-k)-k(n_2-k)](\lambda_c+2)+[k(n_1-k)(2n_1-k-2)+k(n_2-k)(2n_2-k-2)-(n_1+n_2-2)(2n_1-k-2)(2n_2-k-2)]$

Hence the adjacency spectra of the line graph, $L(K_{n_1}O_kK_{n_2})$ is,

$A_{spec}[L(K_{n_1}O_kK_{n_2})] = \langle a^f, b^g, c^h, d^i, e^j\rangle$

where, $a=-2\,;b=n_1+n_2-k-2\,;c=n_1-6\,;d=n_2-6\,;e=\alpha_i(i=1,2,3)\,;\; f=(n_1C_2+n_2C_2-kC_2)-(n_1+n_2-k)\,;g=k-1\,;h=n_1-k-1\,;i=n_2-k-1\,;j=1$, *where* $\alpha_i\;\;(i=1,2,3)$ *satisfy*

$[(\lambda_c+2)^3-(3n_1+3n_2-2k-7)(\lambda_c+2)^2+[(n_1+n_2-2)(2n_1-k-2)+(3n_1+n_2-k-4)(2n_2-k-2)-k(n_1-k)-k(n_2-k)](\lambda_c+2)+[k(n_1-k)(2n_1-k-2)+k(n_2-k)(2n_2-k-2)-(n_1+n_2-2)(2n_1-k-2)(2n_2-k-2)]=0$

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