# MODULE 5

# UNSTEADY STATE HEAT CONDUCTION

## 5.1 Introduction

To this point, we have considered conductive heat transfer problems in which the temperatures are independent of time. In many applications, however, the temperatures are varying with time, and we require the understanding of the complete time history of the temperature variation. For example, in metallurgy, the heat treating process can be controlled to directly affect the characteristics of the processed materials. Annealing (slow cool) can soften metals and improve ductility. On the other hand, quenching (rapid cool) can harden the strain boundary and increase strength. In order to characterize this transient behavior, the full unsteady equation is needed:

$$\frac{1}{a} \cdot \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\overset{\wedge}{q}}{k} \tag{5.1}$$

where $\alpha = \dfrac{k}{\rho c}$ is the thermal diffusivity. Without any heat generation and considering spatial variation of temperature only in x-direction, the above equation reduces to:

$$\frac{1}{a} \cdot \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2} \tag{5.2}$$

For the solution of equation (5.2), we need two boundary conditions in x-direction and one initial condition. Boundary conditions, as the name implies, are frequently specified along the physical boundary of an object; they can, however, also be internal – e.g. a known temperature gradient at an internal line of symmetry.

## 5.2 Biot and Fourier numbers

In some transient problems, the internal temperature gradients in the body may be quite small and insignificant. Yet the temperature at a given location, or the average temperature of the object, may be changing quite rapidly with time. From eq. (5.1) we can note that such could be the case for large thermal diffusivity $\alpha$.

A more meaningful approach is to consider the general problem of transient cooling of an object, such as the hollow cylinder shown in figure 5.1.
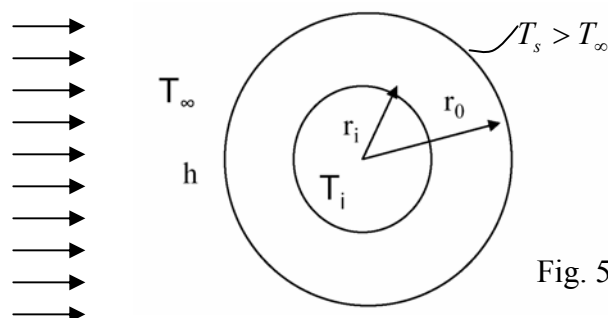


Fig. 5.1

For very large $r_i$, the heat transfer rate by conduction through the cylinder wall is approximately

$$q \approx -k(2\pi r_o l)\left(\frac{T_s - T_i}{r_o - r_i}\right) = k(2\pi r_o l)\left(\frac{T_i - T_s}{L}\right) \tag{5.3}$$

where $l$ is the length of the cylinder and $L$ is the material thickness. The rate of heat transfer away from the outer surface by convection is

$$q = \overline{h}(2\pi r_o l)(T_s - T_\infty) \tag{5.4}$$

where $\overline{h}$ is the average heat transfer coefficient for convection from the entire surface. Equating (5.3) and (5.4) gives

$$\frac{T_i - T_s}{T_s - T_\infty} = \frac{\overline{h}L}{k} = \text{Biot number} \tag{5.5}$$

The Biot number is dimensionless, and it can be thought of as the ratio

$$\mathbf{Bi} = \frac{\textbf{resistance to internal heat flow}}{\textbf{resistance to external heat flow}}$$

Whenever the Biot number is small, the internal temperature gradients are also small and a transient problem can be treated by the "lumped thermal capacity" approach. The lumped capacity assumption implies that the object for analysis is considered to have a single mass-averaged temperature.

In the derivation shown above, the significant object dimension was the conduction path length, $L = r_o - r_i$. In general, a characteristic length scale may be obtained by dividing the volume of the solid by its surface area:

$$L = \frac{V}{A_s} \tag{5.6}$$

Using this method to determine the characteristic length scale, the corresponding Biot number may be evaluated for objects of any shape, for example a plate, a cylinder, or a sphere. As a thumb rule, if the Biot number turns out to be less than 0.1, lumped capacity assumption is applied.

In this context, *a dimensionless time*, known as the **Fourier number**, can be obtained by multiplying the dimensional time by the thermal diffusivity and dividing by the square of the characteristic length:

$$\textbf{dimensionless time} = \frac{\alpha t}{L^2} = \textbf{Fo} \tag{5.7}$$

## 5.3 Lumped thermal capacity analysis

The simplest situation in an unsteady heat transfer process is to use the lumped capacity assumption, wherein we neglect the temperature distribution inside the solid and only deal with the heat transfer between the solid and the ambient fluids. In other words, we are assuming that the temperature inside the solid is constant and is equal to the surface temperature.
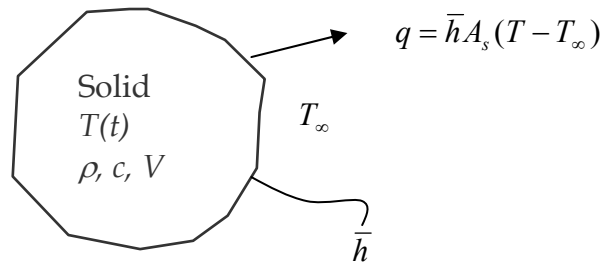


Fig. 5.2

The solid object shown in figure 5.2 is a metal piece which is being cooled in air after hot forming. Thermal energy is leaving the object from all elements of the surface, and this is shown for simplicity by a single arrow. The first law of thermodynamics applied to this problem is

$$\left(\begin{array}{l}\text{heat out of object} \\ \text{during time } dt\end{array}\right) = \left(\begin{array}{l}\text{decrease of internal thermal} \\ \text{energy of object during time } dt\end{array}\right)$$

Now, if Biot number is small and temperature of the object can be considered to be uniform, this equation can be written as

$$\overline{h}A_s\left[T(t) - T_\infty\right]dt = -\rho c V dT \tag{5.8}$$

or,
$$\frac{dT}{\left(T - T_\infty\right)} = -\frac{\overline{h}A_s}{\rho c V}dt \tag{5.9}$$

Integrating and applying the initial condition $T(0) = T_i$,

$$\ln\frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{\overline{h}A_s}{\rho c V}t \tag{5.10}$$

Taking the exponents of both sides and rearranging,
$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \tag{5.11}$$

where
$$b = \frac{\overline{h}A_s}{\rho c V} \qquad (1/s) \tag{5.12}$$

**Rate of convection heat transfer** at any given time t:

$$\dot{Q}(t) = hA_s \left[ T(t) - T_\infty \right]$$

**Total amount of heat transfer** between the body and the surrounding from t=0 to t:

$$Q = mc \left[ T(t) - T_i \right]$$

**Maximum heat transfer** (limit reached when body temperature equals that of the surrounding):

$$Q = mc \left[ T_\infty - T_i \right]$$

## 5.4 Numerical methods in transient heat transfer: The Finite Volume Method

Consider, now, unsteady state diffusion in the context of heat transfer, in which the temperature, $T$, is the scalar. The corresponding partial differential equation is:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + S \tag{5.13}$$

The term on the left hand side of eq. (5.13) is the storage term, arising out of accumulation/depletion of heat in the domain under consideration. Note that eq. (5.13) is a partial differential equation as a result of an extra independent variable, time (t). The corresponding grid system is shown in fig. 5.3.
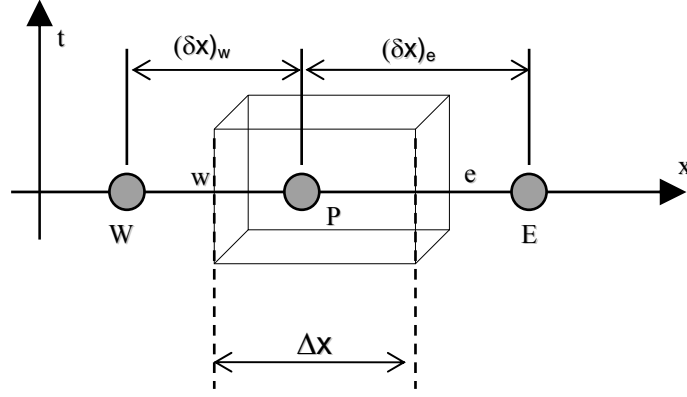
Fig. 5.3: Grid system of an unsteady one-dimensional computational domain

In order to obtain a discretized equation at the nodal point P of the control volume, integration of the governing eq. (5.13) is required to be performed with respect to time as well as space. Integration over the control volume and over a time interval gives

$$\int_t^{t+\Delta t}\left(\int_{CV}\left(\rho c\frac{\partial T}{\partial t}\right)dV\right)dt = \int_t^{t+\Delta t}\left(\int_{cv}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dV\right)dt + \int_t^{t+\Delta t}\left(\int_{CV}SdV\right)dt \qquad (5.14)$$

Rewritten,

$$\int_w^e\left(\int_t^{t+\Delta t}\rho c\frac{\partial T}{\partial t}dt\right)dV = \int_t^{t+\Delta t}\left(\left(kA\frac{\partial T}{\partial x}\right)_e - \left(kA\frac{\partial T}{\partial x}\right)_w\right)dt + \int_t^{t+\Delta t}\left(\overline{S}\Delta V\right)dt \qquad (5.15)$$

If the temperature at a node is assumed to prevail over the whole control volume, applying the central differencing scheme, one obtains:

$$\rho c\left(T^{new}_P - T^{old}_P\right)\Delta V = \int_t^{t+\Delta t}\left(\left(k_e A\frac{T_E - T_P}{\delta x_e}\right) - \left(k_w A\frac{T_P - T_W}{\delta x_w}\right)\right)dt + \int_t^{t+\Delta t}\left(\overline{S}\Delta V\right)dt \qquad (5.16)$$

Now, an assumption is made about the variation of $T_P$, $T_E$ and $T_w$ with time. By generalizing the approach by means of a weighting parameter $f$ between 0 and 1:

$$\int_t^{t+\Delta t}\phi_P dt = \phi_P\Delta t = \left[f\phi_P^{new} - (1-f)\phi_P^{old}\right]\Delta t \qquad (5.17)$$

Repeating the same operation for points E and W,

$$\rho c\left(\frac{T_P^{new} - T_P^{old}}{\Delta t}\right)\Delta x = f\left[\left(k_e\frac{T_E^{new} - T_P^{new}}{\delta x_e}\right) - \left(k_w\frac{T_P^{new} - T_W^{new}}{\delta x_w}\right)\right]$$
$$+ (1-f)\left[\left(k_e\frac{T_E^{old} - T_P^{old}}{\delta x_e}\right) - \left(k_w\frac{T_P^{old} - T_W^{old}}{\delta x_w}\right)\right] + \overline{S}\Delta x \qquad (5.18)$$

Upon re-arranging, dropping the superscript "new", and casting the equation into the standard form:

$$a_P T_P = a_W \left[ f T_W + (1-f) T_W^{old} \right] + a_E \left[ f T_E + (1-f) T_E^{old} \right] + \left[ a_P^0 - (1-f) a_W - (1-f) a_E \right] T_P^{old} + b$$

(5.19)

where

$$a_P = \theta(a_W + a_E) + a_P^0 ; \quad a_P^0 = \rho c \frac{\Delta x}{\Delta t} ; \quad a_W = \frac{k_w}{\delta x_w} ; \quad a_E = \frac{k_e}{\delta x_e} ; \quad b = \overline{S} \Delta x$$

(5.20)

The time integration scheme would depend on the choice of the parameter $f$. When $f = 0$, the resulting scheme is "explicit"; when $0 < f \le 1$, the resulting scheme is "implicit"; when $f = 1$, the resulting scheme is "fully implicit", when $f = 1/2$, the resulting scheme is "Crank-Nicolson" (Crank and Nicolson, 1947). The variation of $T$ within the time interval $\Delta t$ for the different schemes is shown in fig. 5.4.
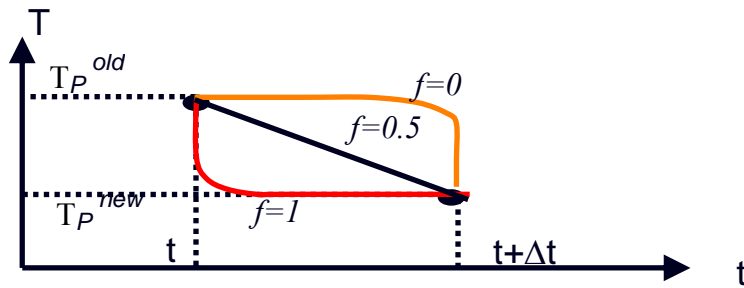


Fig. 5.4: Variation of T within the time interval Δt for different schemes

Explicit scheme

Linearizing the source term as $b = S_u + S_p T_P^{old}$ and setting $f = 0$ in eq. (5.19), the explicit discretisation becomes:

$$a_P T_P = a_W T_W^{old} + a_E T_E^{old} + \left[ a_P^0 - (a_W + a_E) \right] T_P^{old} + S_u$$

(5.21)

where

$$a_P = a_P^0 ; \quad a_P^0 = \rho c \frac{\Delta x}{\Delta t} ; \quad a_W = \frac{k_w}{\delta x_w} ; \quad a_E = \frac{k_e}{\delta x_e}$$

(5.22)

The above scheme is based on backward differencing and its Taylor series truncation error accuracy is first-order with respect to time. For stability, all coefficients must be positive in the discretized equation. Hence,

$$a_P^0 - (a_W + a_E - S_P) > 0$$

or, $\rho c \dfrac{\Delta x}{\Delta t} - (\dfrac{k_w}{\delta x_w} + \dfrac{k_e}{\delta x_e}) > 0$

or. $\rho c \dfrac{\Delta x}{\Delta t} > \dfrac{2k}{\Delta x}$

or, $\Delta t < \rho c \dfrac{(\Delta x)^2}{2k}$ (5.23)

The above limitation on time step suggests that the explicit scheme becomes very expensive to improve spatial accuracy. Hence, this method is generally not recommended for general transient problems. Nevertheless, provided that the time step size is chosen with care, the explicit scheme described above is efficient for simple conduction calculations.

Crank-Nicolson scheme

Setting $f = 0.5$ in eq. (5.19), the Crank-Nicolson discretisation becomes:

$$a_P T_P = a_E \left( \dfrac{T_E + T_E^{old}}{2} \right) + a_W \left( \dfrac{T_W + T_W^{old}}{2} \right) + \left[ a_P^0 - \dfrac{a_E}{2} - \dfrac{a_W}{2} \right] T_P^0 + b \qquad (5.24)$$

where

$$a_P = \dfrac{1}{2}(a_E + a_W) + a_P^0 - \dfrac{1}{2} S_P ; \quad a_P^0 = \rho c \dfrac{\Delta x}{\Delta t} ; \quad a_W = \dfrac{k_w}{\delta x_w} ; a_E = \dfrac{k_e}{\delta x_e} ; \quad b = S_u + \dfrac{1}{2} S_p T_p^{old} \qquad (5.25)$$

The above method is implicit and simultaneous equations for all node points need to be solved at each time step. For stability, all coefficient must be positive in the discretized equation, requiring

$$a_P^0 > \dfrac{a_E + a_W}{2}$$

or, $\Delta t < \rho c \dfrac{(\Delta x)^2}{k}$ (5.26)

The Crank-Nicolson scheme only slightly less restrictive than the explicit method. It is based on central differencing and hence it is second-order accurate in time.

The fully implicit scheme

Setting $f = 1$ in eq. (5.19), the fully implicit discretisation becomes:

$$a_P T_P = a_E T_E + a_W T_W + a_P^0 T_P^{old} \qquad (5.27)$$

where $a_P = a_P^0 + a_E + a_W - S_P ; \quad a_P^0 = \rho c \dfrac{\Delta x}{\Delta t} ; \quad a_W = \dfrac{k_w}{\delta x_w} ; \quad a_E = \dfrac{k_e}{\delta x_e}$ (5.28)

A system of algebraic equations must be solved at each time level. The accuracy of the scheme is first-order in time. The time marching procedure starts with a given initial field of the scalar $\phi^0$. The system is solved after selecting time step $\Delta t$. For the implicit scheme, all coefficients are positive, which makes it unconditionally stable for any size of time step. Hence, the implicit method is recommended for general purpose transient calculations because of its robustness and unconditional stability.