MODULE 4

MULTI-DIMENSIONAL STEADY STATE HEAT CONDUCTION

4.1 Introduction

We have, to this point, considered only One Dimensional, Steady State problems. The reason for this is that such problems lead to ordinary differential equations and can be solved with relatively ordinary mathematical techniques.

In general the properties of any physical system may depend on both location (x, y, z) and time (τ). The inclusion of two or more independent variables results in a partial differential equation. The multidimensional heat diffusion equation in a Cartesian coordinate system can be written as:

$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\hat{q}}{k} \qquad (1)$$

The above equation governs the Cartesian, temperature distribution for a three-dimensional unsteady, heat transfer problem involving heat generation. To solve for the full equation, it requires a total of six boundary conditions: two for each direction. Only one initial condition is needed to account for the transient behavior. For 2D, steady state ($\partial/\partial t = 0$) and without heat generation, the above equation reduces to:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \qquad (2)$$

Equation (2) needs 2 boundary conditions in each direction. There are three approaches to solve this equation:

- **Analytical Method**: The mathematical equation can be solved using techniques like the method of separation of variables.
- **Graphical Method**: Limited use. However, the conduction shape factor concept derived under this concept can be useful for specific configurations. (see Table 4.1 for selected configurations)
- **Numerical Method**: Finite difference or finite volume schemes, usually will be solved using computers.

Analytical solutions are possible only for a limited number of cases (such as linear problems with simple geometry). Standard analytical techniques such as separation of variables can be found in basic textbooks on engineering mathematics, and will not be reproduced here. The student is encouraged to refer to textbooks on basic mathematics for an overview of the analytical solutions to heat diffusion problems. In the present lecture material, we will cover the graphical and numerical techniques, which are used quite conveniently by engineers for solving multi-dimensional heat conduction problems.

4.2 Graphical Method: Conduction Shape Factor

This approach applied to 2-D conduction involving two isothermal surfaces, with all other surfaces being adiabatic. The heat transfer from one surface (at a temperature T1) to the other surface (at T2) can be expressed as: q=Sk(T1-T2) where k is the thermal conductivity of the solid and S is the conduction shape factor.

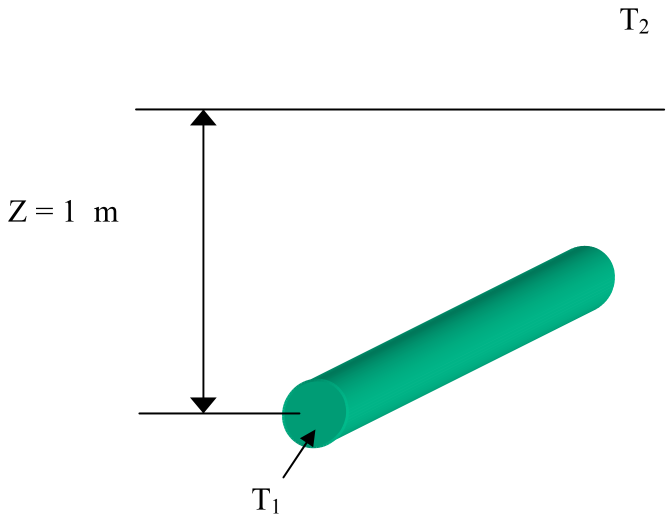
The shape factor can be related to the thermal resistance:

$$q=S.k.(T_1-T_2)=(T_1-T_2)/(1/kS)= (T_1-T_2)/R_t$$

where $R_t = 1/(kS)$ is the thermal resistance in 2D. Note that 1-D heat transfer can also use the concept of shape factor. For example, heat transfer inside a plane wall of thickness L is $q=kA(\Delta T/L)$, where the shape factor S=A/L. Common shape factors for selected configurations can be found in Table 4.1

Example: A 10 cm OD uninsulated pipe carries steam from the power plant across campus. Find the heat loss if the pipe is buried 1 m in the ground is the ground surface temperature is 50 ºC. Assume a thermal conductivity of the sandy soil as k = 0.52 w/m K.

Solution:



The shape factor for long cylinders is found in Table 4.1 as Case 2, with L >> D:

$$S = 2\cdot\pi\cdot L/\ln(4\cdot z/D)$$

Where z = depth at which pipe is buried.

$$S = 2\cdot\pi\cdot 1\cdot m/\ln(40) = 1.7\ m$$

Then

$$q' = (1.7\cdot m)(0.52\ W/m\cdot K)(100\,^{o}C - 50\,^{o}C)$$

$$q' = 44.2\ W$$

Table 4.1
Conduction shape factors for selected two-dimensional systems [$q = Sk(T_1 - T_2)$]

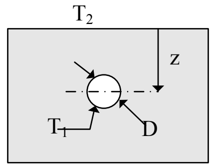
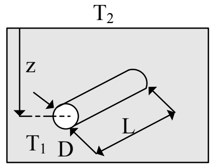
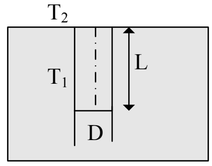
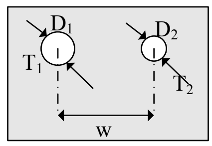
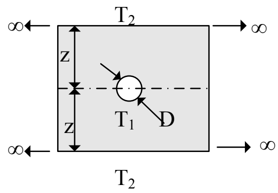
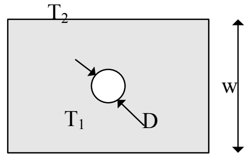
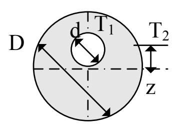
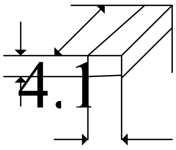
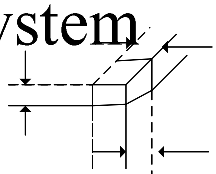
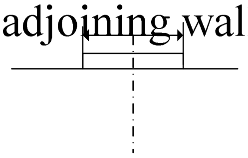
System	Schematic	Restrictions	Shape Factor
Isothermal sphere buried in as finite medium		z>D/2	$\frac{2\pi D}{1 - D/4z}$
Horizontal isothermal cylinder of length L buried in a semi finite medium		L>>D L>>D z>3D/2	$\frac{2\pi L}{\cosh^{-1}(2z/D)}$ $\frac{2\pi L}{\ln(4z/D)}$
Vertical cylinder in a semi finite medium		L>>D	$\frac{2\pi L}{\ln(4L/D)}$
Conduction between two cylinders of length L in infinite medium		L>>D1,D2 L>>w	$\frac{2\pi L}{\cosh^{-1}\left(\frac{4w^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$
Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width		z>>D/2 L>>2	$\frac{2\pi L}{\ln(8z/\pi D)}$
Circular cylinder of length L centered in a square solid of equal length		W>D L>>w	$\frac{2\pi L}{\ln(1.08w/D)}$
Eccentric circular cylinder of length L in a cylinder of equal length		D>d L>>D	$\frac{2\pi L}{\cosh^{-1}\left(\frac{D^2 + d^2 - 4z^2}{2Dd}\right)}$

Table 4.1 Continued

System	Schematic
Conduction through the edge of adjoining walls	
Conduction through corners of three walls with a temperature difference of ΔT_{1-2} across the walls	
Disk of diameter D and T1 on a semi-finite medium of thermal conductivity k and T_2	

4.3 Numerical Methods

Due to the increasing complexities encountered in the development of modern technology, analytical solutions usually are not available. For these problems, numerical solutions obtained using high-speed computer are very useful, especially when the geometry of the object of interest is irregular, or the boundary conditions are nonlinear. In numerical analysis, three different approaches are commonly used: the finite difference, the finite volume and the finite element methods. Brief descriptions of the three methods are as follows:

The Finite Difference Method (FDM)

This is the oldest method for numerical solution of PDEs, introduced by Euler in the 18th century. It's also the easiest method to use for simple geometries. The starting point is the conservation equation in differential form. The solution domain is covered by grid. At each grid point, the differential equation is approximated by replacing the partial derivatives by approximations in terms of the nodal values of the functions. The result is one algebraic equation per grid node, in which the variable value at that and a certain number of neighbor nodes appear as unknowns.

In principle, the FD method can be applied to any grid type. However, in all applications of the FD method known, it has been applied to structured grids. Taylor series expansion or polynomial fitting is used to obtain approximations to the first and second derivatives of the variables with respect to the coordinates. When necessary, these methods are also used to obtain variable values at locations other than grid nodes (interpolation).

On structured grids, the FD method is very simple and effective. It is especially easy to obtain higher-order schemes on regular grids. The disadvantage of FD methods is that the conservation is not enforced unless special care is taken. Also, the restriction to simple geometries is a significant disadvantage.

Finite Volume Method (FVM)

In this dissertation finite volume method is used. The FV method uses the integral form of the conservation equations as its starting point. The solution domain is subdivided into a finite number of contiguous control volumes (CVs), and the conservation equations are applied to each CV. At the centroid of each CV lies a computational node at which the variable values are to be calculated. Interpolation is used to express variable values at the CV surface in terms of the nodal (CV-center) values. As a result, one obtains an algebraic equation for each CV, in which a number of neighbor nodal values appear. The FVM method can accommodate any type of grid when compared to FDM, which is applied to only structured grids. The FVM approach is perhaps the simplest to understand and to program. All terms that need be approximated have physical meaning, which is why it is popular.

The disadvantage of FV methods compared to FD schemes is that methods of order higher than second are more difficult to develop in 3D. This is due to the fact that the FV approach requires two levels of approximation: interpolation and integration.

Finite Element Method (FEM)

The FE method is similar to the FV method in many ways. The domain is broken into a set of discrete volumes or finite elements that are generally unstructured; in 2D, they are usually triangles or quadrilaterals, while in 3D tetrahedra or hexahedra are most often used. The distinguishing feature of FE methods is that the equations are multiplied by a weight function before they are integrated over the entire domain. In the simplest FE methods, the solution is approximated by a linear shape function within each element in a way that guarantees continuity of the solution across element boundaries. Such a function can be constructed from its values at the corners of the elements. The weight function is usually of the same form.

This approximation is then substituted into the weighted integral of the conservation law and the equations to be solved are derived by requiring the derivative of the integral with respect to each nodal value to be zero; this corresponds to selecting the best solution within the set of allowed functions (the one with minimum residual). The result is a set of non-linear algebraic equations.

An important advantage of finite element methods is the ability to deal with arbitrary geometries. Finite element methods are relatively easy to analyze mathematically and can be shown to have optimality properties for certain types of equations. The principal drawback, which is shared by any method that uses unstructured grids, is that the matrices of the linearized equations are not as well structured as those for regular grids making it more difficult to find efficient solution methods.

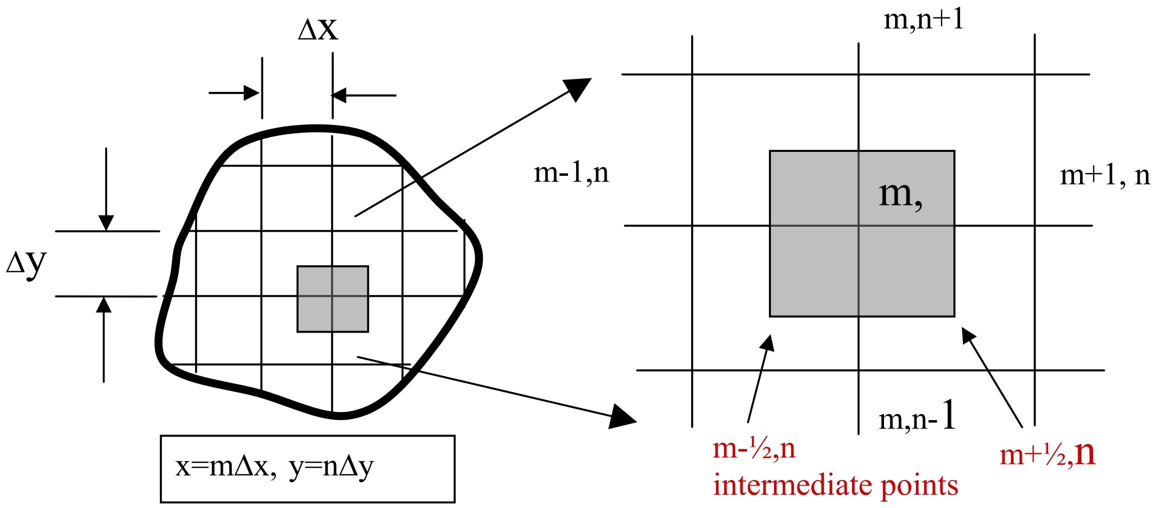
4.4 The Finite Difference Method Applied to Heat Transfer Problems:

In heat transfer problems, the finite difference method is used more often and will be discussed here in more detail. The finite difference method involves:

- Establish nodal networks
- Derive finite difference approximations for the governing equation at both interior and exterior nodal points
- Develop a system of simultaneous algebraic nodal equations
- Solve the system of equations using numerical schemes

<u>The Nodal Networks:</u>

The basic idea is to subdivide the area of interest into sub-volumes with the distance between adjacent nodes by Δx and Δy as shown. If the distance between points is small enough, the differential equation can be approximated locally by a set of finite difference equations. Each node now represents a small region where the nodal temperature is a measure of the average temperature of the region.
Example:



<u>Finite Difference Approximation:</u>

Heat Diffusion Equation: $\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$,

where $\alpha = \frac{k}{\rho C_P V}$ is the thermal diffusivity

No generation and steady state: $\dot{q}=0$ and $\frac{\partial}{\partial t} = 0, \Rightarrow \nabla^2 T = 0$

First, approximated the first order differentiation at intermediate points (m+1/2,n) & (m-1/2,n)

$$\left.\frac{\partial T}{\partial x}\right|_{(m+1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m+1/2,n)} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{(m-1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m-1/2,n)} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Next, approximate the second order differentiation at m,n

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\left.\partial T/\partial x\right|_{m+1/2,n} - \left.\partial T/\partial x\right|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Similarly, the approximation can be applied to the other dimension y

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)_{m,n} \approx \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}$$

To model the steady state, no generation heat equation: $\nabla^2 T = 0$

This approximation can be simplified by specify Δx=Δy

and the nodal equation can be obtained as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

This equation approximates the nodal temperature distribution based on the heat equation. This approximation is improved when the distance between the adjacent nodal points is decreased:

Since $\lim(\Delta x \to 0)\frac{\Delta T}{\Delta x}=\frac{\partial T}{\partial x}, \lim(\Delta y \to 0)\frac{\Delta T}{\Delta y}=\frac{\partial T}{\partial y}$

Table 4.2 provides a list of nodal finite difference equation for various configurations.

A System of Algebraic Equations

- The nodal equations derived previously are valid for all interior points satisfying the steady state, no generation heat equation. For each node, there is one such equation.

  For example: for nodal point m=3, n=4, the equation is

  $T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5} - 4T_{3,4} = 0$

  $T_{3,4} = (1/4)(T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5})$
- Nodal relation table for exterior nodes (boundary conditions) can be found in standard heat transfer textbooks.
- Derive one equation for each nodal point (including both interior and exterior points) in the system of interest. The result is a system of N algebraic equations for a total of N nodal points.

Matrix Form

The system of equations:

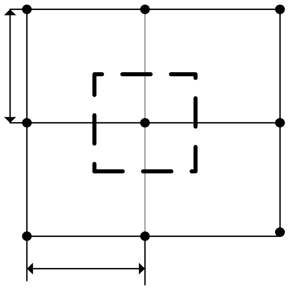
$$\begin{array}{l} a_{11}T_1 + a_{12}T_2 + \cdots + a_{1N}T_N = C_1 \\ a_{21}T_1 + a_{22}T_2 + \cdots + a_{2N}T_N = C_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{N1}T_1 + a_{N2}T_2 + \cdots + a_{NN}T_N = C_N \end{array}$$

A total of N algebraic equations for the N nodal points and the system can be expressed as a matrix formulation: [**A**][**T**]=[**C**] .

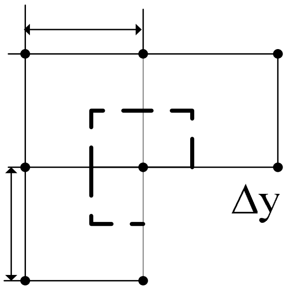
$$\text{where } A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, T=\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, C=\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}$$

Table 4.2 Summary of no

Configuration



$$T_{m,n+1}+T_{m,n-1}+T_{m+1,n}+T_{m-1,n}-4T_{m,n}=0$$



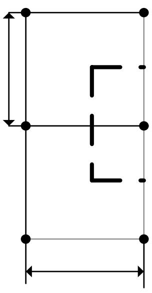
m,n+1

$$2(T_{m-1,n}+T_{m,n+1})+(T_{m+1,n}+T_{m,n-1})+2\frac{h\Delta x}{k}T_{\infty}-2\left(3+\frac{h\Delta x}{k}\right)T_{m,n}=0$$

Δy

m,n m+1,n

m-1,n



$$2(T_{m-1,n}+T_{m,n+1}+T_{m,n-1})+2\frac{h\Delta x}{k}T_{\infty}-2\left(\frac{h\Delta x}{k}+2\right)T_{m,n}=0$$

Δx m,n-1

Δx

m,n+1

m-1,n m,n

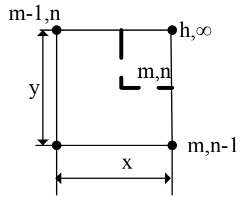
m+1,n

Δy h,∞

m,n-1

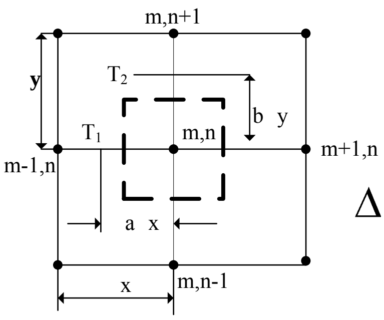
+1

Table 4.2 Summary of nodal finite-difference methods



$$2(T_{m-1,n}+T_{m,n+1}+T_{m,n-1})+2\frac{h\Delta x}{k}T_{\infty}-2\left(\frac{h\Delta x}{k}+1\right)T_{m,n}=0$$

Case 4. Node at an external corner with convection



$$\frac{2}{a+1}T_{m+1,n}+\frac{2}{b+1}T_{m,n-1}+\frac{2}{a(a+1)}T_1+\frac{2}{b)b+1)}T_2-\left(\frac{2}{a}+\frac{2}{b}\right)T_{m,n}=0$$

Case 5. Node near a curved surface maintained at a non uniform temperature

Numerical Solutions

Matrix form: [**A**][**T**]=[**C**].

From linear algebra: $[\mathbf{A}]^{-1}[\mathbf{A}][\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$, $[\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$

where [**A**]-1 is the inverse of matrix [**A**]. [**T**] is the solution vector.

- Matrix inversion requires cumbersome numerical computations and is not efficient if the order of the matrix is high (>10)
- Gauss elimination method and other matrix solvers are usually available in many numerical solution package. For example, "Numerical Recipes" by Cambridge University Press or their web source at www.nr.com.
- For high order matrix, iterative methods are usually more efficient. The famous Jacobi & Gauss-Seidel iteration methods will be introduced in the following.

Iteration

General algebraic equation for nodal point:

$$\sum_{j=1}^{i-1} a_{ij}T_j + a_{ii}T_i + \sum_{j=i+1}^{N} a_{ij}T_j = C_i,$$

(Example: $a_{31}T_1 + a_{32}T_2 + a_{33}T_3 + \cdots + a_{1N}T_N = C_1,\ i=3$)

Rewrite the equation of the form:

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1}\frac{a_{ij}}{a_{ii}}T_j^{(k)} - \sum_{j=i+1}^{N}\frac{a_{ij}}{a_{ii}}T_j^{(k-1)}$$

Replace (k) by (k-1) for the Jacobi iteration

- (k) - specify the level of the iteration, (k-1) means the present level and (k) represents the new level.
- An initial guess (k=0) is needed to start the iteration.
- • By substituting iterated values at (k-1) into the equation, the new values at iteration (k) can be estimated The iteration will be stopped when max$| T_i^{(k)}-T_i^{(k-1)} |\varepsilon\leq$, where ε specifies a predetermined value of acceptable error