Hirasugar Institute of Technology

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Department of Mechanical Engineering

Subject : Finite Element Method

Subject Code: 15ME61

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Semester : VI A

Module-I

Finite Element Methods

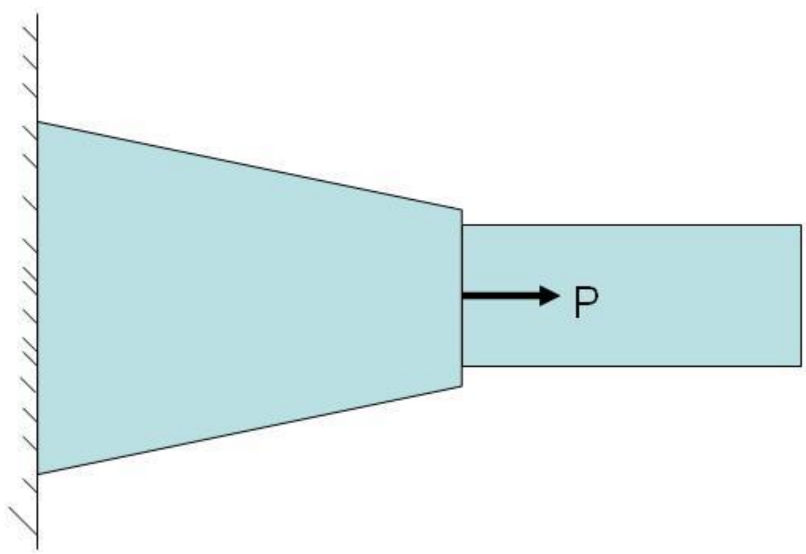
In the finite element method elements are grouped as 1D, 2D and 3D elements. Beams and plates are grouped as structural elements. One dimensional elements are the line segments which are used to model bars and truss. Higher order elements like linear, quadratic and cubic are also available. These elements are used when one of the dimension is very large compared to other two. 2D and 3D elements will be discussed in later chapters.

Seven basic steps in Finite Element Method

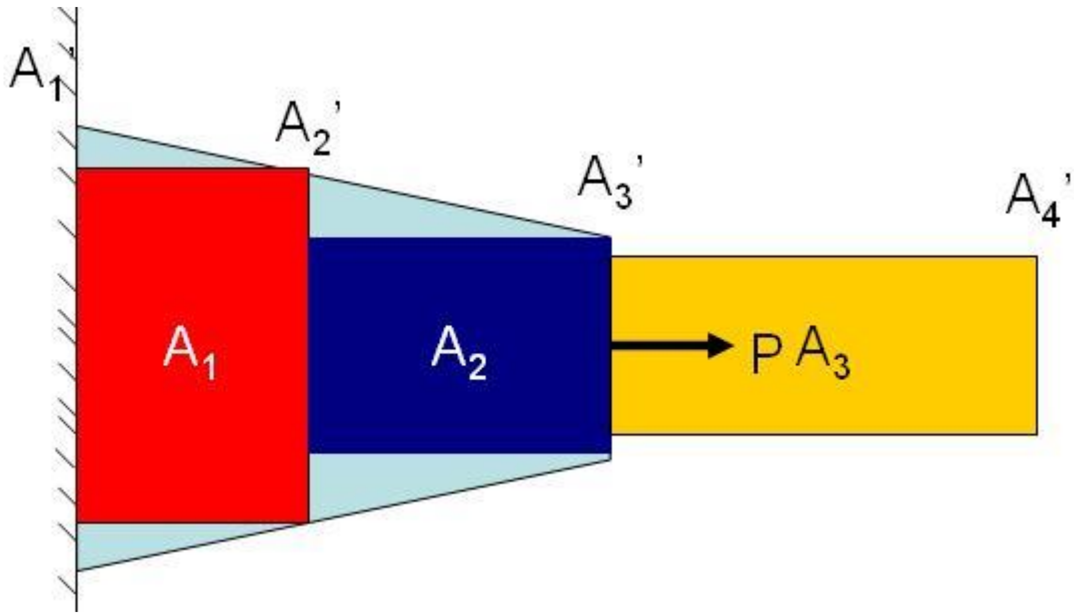
These seven steps include

- Modeling
- Discretization
- Stiffness Matrix
- Assembly
- Application of BC's
- Solution
- Results

Let's consider a bar subjected to the forces as shown

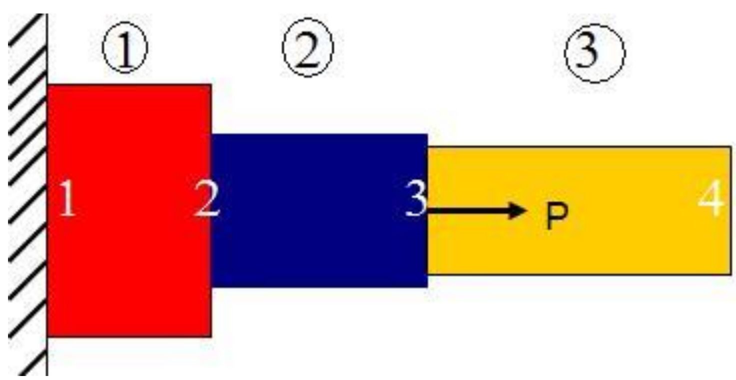


First step is the modeling lets us model it as a stepped shaft consisting of discrete number of elements each having a uniform cross section. Say using three finite elements as shown. Average c/s area within each region is evaluated and used to define elemental area with uniform cross-section.

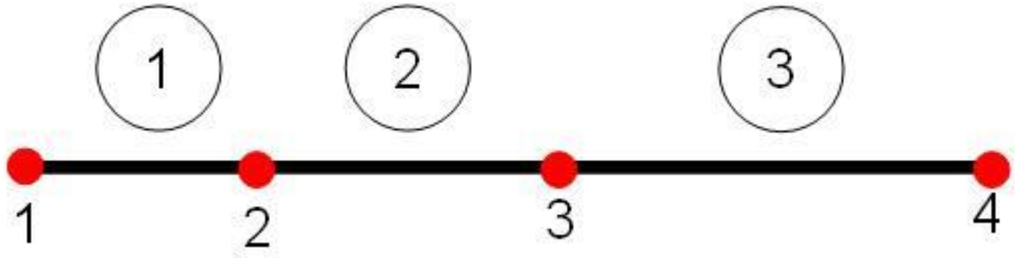


$A_1 = A_1' + A_2' / 2$ similarly A_2 and A_3 are evaluated

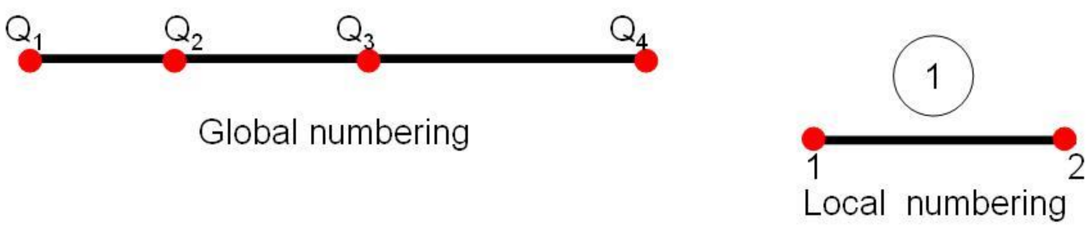
Second step is the Discretization that includes both node and element numbering, in this model every element connects two nodes, so to distinguish between node numbering and element numbering elements numbers are encircled as shown.



Above system can also be represented as a line segment as shown below.



Here in 1D every node is allowed to move only in one direction, hence each node as one degree of freedom. In the present case the model as four nodes it means four dof. Let Q1, Q2, Q3 and Q4 be the nodal displacements at node 1 to node 4 respectively, similarly F1, F2, F3, F4 be the nodal force vector from node 1 to node 4 as shown. When these parameters are represented for a entire structure use capitals which is called global numbering and for representing individual elements use small letters that is called local numbering as shown.



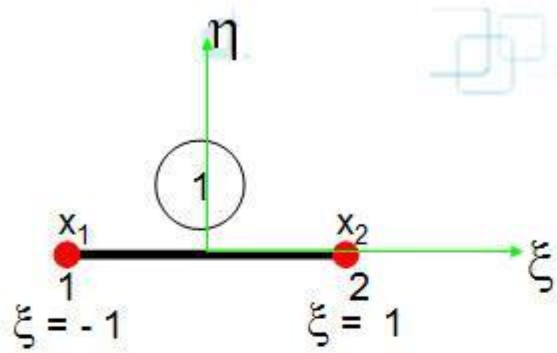
This local and global numbering correspondence is established using element connectivity element as shown

Elements e	Nodes 1 (Local)	Nodes 2 (Local)
1	1	2
2	2	3
3	3	4

(Rows 1–3 of the Nodes columns: Global)

Element Connectivity table

Now let's consider a single element in a natural coordinate system that varies in ξ and η, x_1 be the x coordinate of node 1 and x_2 be the x coordinate of node 2 as shown below.



Let us assume a polynomial

$$X = a_o + a_1 \xi$$

Now

$$@\ x = x_1 \qquad \xi = -1$$

$$@\ x = x_2 \qquad \xi = 1$$

After applying these conditions and solving for constants we have

$$x_1 = a_0 - a_1$$

$$x_2 = a_0 + a_1$$

$a_0 = x_1 + x_2/2$ $\qquad$ $a_1 = x_2 - x_1/2$

Substituting these constants in above equation we get

$$X = a_o + a_1 \xi$$

$$X = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2} \xi$$

$$X = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2$$

$$X = N_1 X_1 + N_2 X_2$$

$$N_1 = \frac{1-\xi}{2} \qquad N_2 = \frac{1+\xi}{2}$$

Where N_1 and N_2 are called shape functions also called as interpolation functions.

These shape functions can also be derived using nodal displacements say q1 and q2 which are nodal displacements at node1 and node 2 respectively, now assuming the displacement function and following the same procedure as that of nodal coordinate we get

$$U = \alpha_o + \alpha_1 \xi$$

$$U = \frac{1-\xi}{2} q_1 + \frac{1+\xi}{2} q_2$$

$$U = N_1 q_1 + N_2 q_2$$

$$= [N_1 \quad N_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$U = Nq$$

$$U = Nq$$

Where N is the shape function matrix and q is displacement matrix. Once the displacement is known its derivative gives strain and corresponding stress can be determined as follows.

$$U = N\,q$$

$$\varepsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx}$$

$$\varepsilon = \frac{q_2 - q_1}{2}\,\frac{2}{x_2 - x_1}$$

$$\varepsilon = \frac{q_2 - q_1}{L} \qquad \text{where } L = x_2 - x_1$$

$$\varepsilon = \frac{1}{L}\,[-1 \quad 1]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\varepsilon = B\,q$$

where $B = \frac{1}{L}\,[-1 \quad 1]$ element strain displacement matrix

$$\sigma = E\,\varepsilon = B\,q\,E$$

From the potential approach we have the expression of Π as

From the potential energy concept

$$\pi = \tfrac{1}{2}\int_V \sigma^T\varepsilon\, dv - \int_V u^T f_b\, dv - \int_S u^T T\, ds - \sum_{i=1}^{n} u_i p_i$$

Since body is divide

$$\pi_e = \int_e u_e - w_e \, dv$$

$$\pi = \frac{1}{2} \int B^T q^T E B q \, dv - \sum_{i=1}^{n} u_i p_i$$

Now total potential energy

$$\pi = \sum \pi_e = \frac{1}{2} Q^T \left(\int B^T EBAL \right) Q - \sum Qi^T Fi$$

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

To extremise the potential energy

$$\frac{d\pi}{dQ^T} = 0 = KQ - F$$

Third step in FEM is finding out stiffness matrix from the above equation we have the value of K as

$$K = \int_V B^T E B \, dv \qquad \text{where } B = \frac{1}{L} [\, -1 \quad 1 \,]$$

For an element

$$K = \int_e B^T E B A \, dx$$

But

$$\frac{dx}{d\xi} = L/2$$

Therefore now substituting the limits as -1 to +1 because the value of ξ varies between -1 & 1 we have

$$K = \int_{-1}^{+1} B^T E\, B\, A \frac{L}{2} d\xi$$

Integration of above equations gives K which is given as

$$K = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

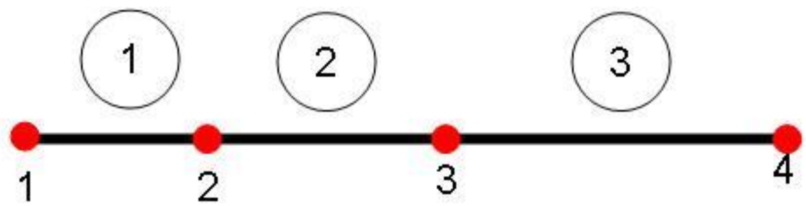
Fourth step is assembly and the size of the assembly matrix is given by number of nodes X degrees of freedom, for the present example that has four nodes and one degree of freedom at each node hence size of the assembly matrix is 4 X 4. At first determine the stiffness matrix of each element say k_1, k_2 and k_3 as

$$K_1 = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{A_1E_1}{L_1} & -\frac{A_1E_1}{L_1} \\ -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1} \end{bmatrix}$$

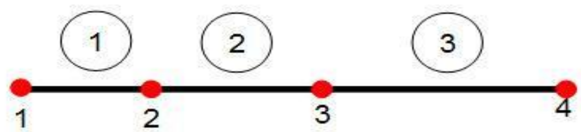
Similarly determine k_2 and k_3

$$K_2 = \begin{bmatrix} \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{bmatrix} \quad K_3 = \begin{bmatrix} \frac{A_3E_3}{L_3} & -\frac{A_3E_3}{L_3} \\ -\frac{A_3E_3}{L_3} & \frac{A_3E_3}{L_3} \end{bmatrix}$$

The given system is modeled as three elements and four nodes we have three stiffness matrices.



Since node 2 is connected between element 1 and element 2, the elements of second stiffness matrix (k_2) gets added to second row second element as shown below similarly for node 3 it gets added to third row third element



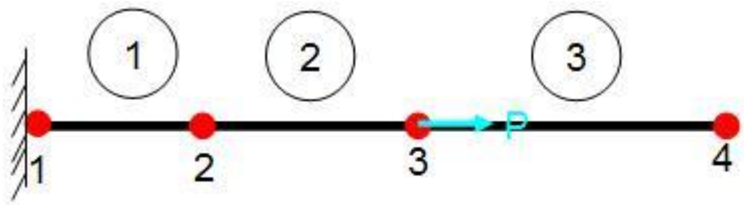
$$\begin{array}{c} \\ \\ \end{array} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix}$$

$$\begin{bmatrix} \frac{A_1E_1}{L_1} & -\frac{A_1E_1}{L_1} & 0 & 0 \\ -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} & 0 \\ 0 & -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} + \frac{A_3E_3}{L_3} & -\frac{A_3E_3}{L_3} \\ 0 & 0 & -\frac{A_3E_3}{L_3} & \frac{A_3E_3}{L_3} \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Fifth step is applying the boundary conditions for a given system. We have the equation of equilibrium KQ=F

K = global stiffness matrix
Q = displacement matrix
F= global force vector

Let Q1, Q2, Q3, and Q4 be the nodal displacements at node 1 to node 4 respectively. And F1, F2, F3, F4 be the nodal load vector acting at node 1 to node 4 respectively.



Given system is fixed at one end and force is applied at other end. Since node 1 is fixed displacement at node 1 will be zero, so set q1 =0. And node 2, node 3 and node 4 are free to move hence there will be displacement that has to be determined. But in the load vector because of fixed node 1 there will reaction force say R1. Now replace F1 to R1 and also at node 3 force P is applied hence replace F3 to P. Rest of the terms are zero.

After applying BC,s

$$\begin{pmatrix} \frac{A_1E_1}{L_1} & -\frac{A_1E_1}{L_1} & 0 & 0 \\ -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1}+\frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} & 0 \\ 0 & -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2}+\frac{A_3E_3}{L_3} & -\frac{A_3E_3}{L_3} \\ 0 & 0 & -\frac{A_3E_3}{L_3} & \frac{A_3E_3}{L_3} \end{pmatrix} \begin{pmatrix} 0 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \\ P \\ 0 \end{pmatrix}$$

Sixth step is solving the above matrix to determine the displacements which can be solved either by

- Elimination method
- Penalty approach method

Details of these two methods will be seen in later sections.

Last step is the presentation of results, finding the parameters like displacements, stresses and other required parameters.

Rayleigh-Ritz Method

As discussed, one can solve axially loaded bars of arbitrary cross-section and material composition along the length using the lumped mass-spring model. As shown in Figure 12 of Exercise 2.4, one can approach the exact solution very closely by dividing the bar into more elements. One of the disadvantages of the lumped models is that we can only compute the deflection at the locations of the lumped masses (we call these points *nodes*), and we know nothing about what happens within the element. Consequently, if we want to get the smooth shape of the deflection curve, we need to take a very large number of elements. The Raleigh-Ritz method offers an alternative method to overcome these problems. This method also uses the MPE principle.

Referring back to the tapering beam problem, what we were able to do with the lumped model is essentially solving the governing differential equation that represents the deflection of axially loaded bars. Our method of solution was of course numerical. It is worthwhile to study the differential equation that we just solved numerically in Chapter 2.

Thus, the objectives of this Chapter are: (i) Derive the differential equation of an axially loaded bar using the force-balance method (ii) Derive the same equation using the MPE principle (iii) Discuss the Rayleigh-Ritz method.

3.1 Derivation of the governing differential equation of an axially loaded bar using the force-balance method

Let $A(x)$, the cross-section area of the bar at x, be given. There is a body-force (gravity-like force), $f(x)$, per unit volume of the bar. $\sigma(x)$, the axial stress and $u(x)$, the axial deflection, are two unknown functions. We would like to derive a differential equation that describes the axially loaded bar so that we can solve for $\sigma(x)$ and $u(x)$.

Consider a differential element of length dx at some x. The stress and area at the left end of the differential element are $\sigma(x)$ and $A(x)$. At $(x+dx)$, the right end, the same quantities can be approximated as $\sigma(x) + \dfrac{d\sigma(x)}{dx}dx$ and $A(x) + \dfrac{dA(x)}{dx}dx$. The free-body-diagram of the infinitesimally small differential element shows that the internal forces (stresses multiplied by areas of cross-section) balance the body-force acting to the right. The body force acting on the differential element is given by $f(x)\, A(x)dx$. Let us now expand and simplify the internal force acting to the right.

$$\left(\sigma(x) + \frac{d\sigma(x)}{dx} dx\right)\left(A(x) + \frac{dA(x)}{dx} dx\right)$$

$$= \sigma(x)\, A(x) + \sigma(x)\, \frac{dA(x)}{dx} dx + A(x) \frac{d\sigma(x)}{dx} dx + \underbrace{\frac{d\sigma(x)}{dx}\frac{dA(x)}{dx} dx^2}_{\to 0} \quad (1)$$

The last term in the above expression is a small second-order term and hence it can be ignored as shown stricken by an arrow in Equation (1). The first term balances the internal force acting on the left end of the differential element. So, the second and third terms and the body-force term should sum to zero for equilibrium

$$\sigma(x)\, \frac{dA(x)}{dx} dx + A(x)\, \frac{d\sigma(x)}{dx} dx + f(x)\, A(x)dx = 0 \quad (2a)$$

You can easily check that after canceling dx although in the above equation, the two terms on the left hand side can be collapsed as one term as shown below.

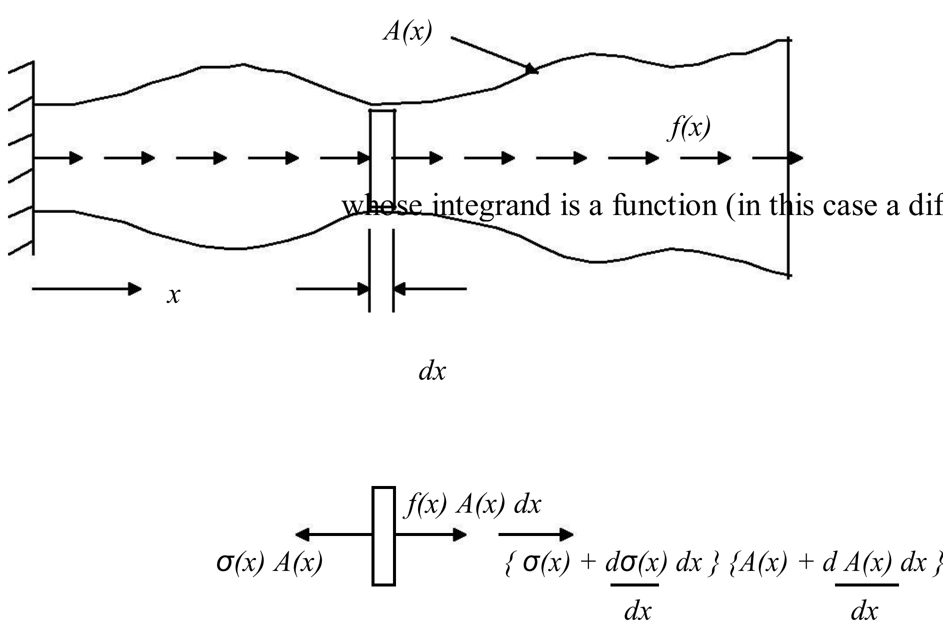
$$\frac{d\left(\sigma(x)\, A(x)\right)}{dx} + f(x)\, A(x)dx = 0 \quad (2)$$

This leads to the following differential equation in $\sigma(x)$.

$$\frac{d}{dx}\left(\sigma(x)\, A(x)\right) + f(x)\, A(x) = 0 \quad (3)$$

Next, we would like to express *u(x)* in terms of $\sigma(x)$ so that we can get the governing differential equation in *u(x)*. From the definition of axial strain (change in length divide by the original length), we get the following expression for strain, $\varepsilon(x) = \frac{du(x)}{dx}$, where *du(x)* is the deflection of the differential element of length *dx*. We also know the relationship between stress and strain: $\sigma(x) = E\, \varepsilon(x)$ where *E* is the Young's modulus. By substituting these relationships into Equation (3), we get the governing differential equation:

$$\frac{d}{dx}\left(E\,A(x)\frac{du(x)}{dx}\right) + f(x)\,A(x) = 0 \tag{4}$$



whose integrand is a function (in this case a differential

Figure 1 Force balance of a differential element in an axially loaded bar

We had observed in Chapter 2 that the equilibrium equations could be written using the force balance method as well as the MPE principle. For the continuous model of an axially loaded bar, we just derived the equilibrium differential equation using the force-balance method. We will obtain the same equation using the MPE principle now.

3.2 Derivation of the governing equation using the *MPE* principle

In this method, first we need to write down the *PE* of the system. Since this is a continuous model, both *SE* and *WP* are integrals over the length of the bar. Note that

$$SE = \int_{dV} (strain\ energy\ density)\ dV = \int_{dV} \frac{1}{2} (stress)\ (strain)\ dV$$

$$= \int_0^L \frac{1}{2} E \frac{du(x)}{dx} \frac{du(x)}{dx} A(x)dx \quad (5)$$

$$WP = -\int_0^L f(x)\, A(x)\, u(x)\, dx \quad (6)$$

By denoting $\frac{du(x)}{dx}$ by u', from Equations (5) and (6), the *PE* can be written as the sum of *SE* and *WP*.

$$PE = SE + WP = \int_0^L \frac{1}{2} A(x)Eu'^2\, dx - \int_0^L f(x)A(x)u(x)dx \quad (7)$$

As before, we have to minimize *PE* with respect to the deformation variables. Here, the deflection variable, *u(x)* is a continuous function, and the PE is an integral. In fact, *PE* in Equation (7) is called a functional in this case an integral relation) of some function *u(x)*.

Next we will show that if *PE* is minimized with respect to all kinematically admissible displacement *u(x)*, then that *u(x)* satisfies the differential equation (4). To show this, consider the kinematically admissible displacement $\tilde{u}(x) = u(x) + \alpha\delta u(x)$ where the variation from the exact solution *u(x)* is given by the function $\delta u(x)$ times the parameter α. Since $\tilde{u}(x)$ must satisfy the same kinematical boundary conditions as *u(x)*, it follows that $\delta u(x = 0) = 0$. With $\tilde{u}(x)$ substituted in the place of *u(x)* in the *PE* expression in Equation (7), for a given $\delta u(x)$, we can regard the potential energy to be a function of the parameter α, i.e., $PE(\alpha)$. Then, minimizing $PE(\alpha)$ with respect to α and setting $\alpha = 0$ gives the desired governing differential equation:

$$PE(\alpha) = \int_0^L \frac{1}{2} EA(x)(u' + \alpha\delta u')^2\, dx - \int_0^L f(x)A(x)\big(u + \alpha\, u\big)dx$$

$$\frac{d(PE)}{d\alpha} = \int_0^L EA(x)(u' + \alpha\delta u')\, \delta u' dx - \int_0^L f(x)A(x)\big(\delta\, u\big)dx = 0$$

By substituting $\alpha = 0$, we get

$$\left.\frac{d(PE)}{d\alpha}\right|_{\alpha=0} = \int_0^L EA(x)(u')\, \delta u' dx - \int_0^L f(x)A(x)\big(\delta\, u\big)dx = 0$$

3.5

Integrating the expression in the last equation by parts and using the boundary conditions on $\delta u(x)$, we arrive at (note: we substitute $u' = \frac{du(x)}{dx}$ to get back to our original notation)

$$\int_0^L \left[\frac{d}{dx}\left(EA(x) \frac{du(x)}{dx} \right) + f(x)\,A(x) \right] \delta u\, dx = 0 \qquad (8)$$

Since this last integral must vanish for <u>all</u> kinematically admissible δu when the potential energy of the deformed beam is minimized, it follows that the integrand itself must vanish, i.e.:

$$\frac{d}{dx}\left(EA(x) \frac{du(x)}{dx} \right) + f(x)\,A(x) = 0 \qquad (9)$$

which is the same as Equation (4).

We have demonstrated above that the MPE principle can be applied to continuous elastic systems as well. In fact, in doing so, we have utilized a fundamental mathematical approach in the *calculus of variations*. We could also have derived Equation (9) by applying what is known as Euler-Lagrange equation of calculus of variations. The Euler-Lagrange equation helps us minimize a functional (the *PE* expression in Equation (7) in our case) with respect to a function (in our case *u(x)*). It is given by

$$\frac{d}{dx}\frac{\partial(PE)}{\partial u'} - \frac{\partial(PE)}{\partial u} = 0 \qquad (10)$$

You should verify that Equation (10) also leads to Equation (9).

Once again, the *MPE* principle gave us the solution with less work and more systematically as compared to the force-balance method. It is systematic in the following sense. If you were to derive the governing equilibrium differential equation for a beam, all you need is its *PE*, as opposed to the force-balance method where you need to know much more about the internal forces. Much of the theoretical basis for the finite element method is rooted in the method we used above. In particular, Equation (10) is a fundamental equation in calculus of variations – an important mathematical tool in FEM formulations. Refer to any book on calculus of variations for more details. References to two books are given in the bibliography at the end.

3.3 Rayleigh-Ritz method

In Chapter 2, we solved a problem numerically the differential equation of which we derived in this chapter. We noted that the lumped-model method gives us deflections at only some discrete points (nodes), and we know nothing in between the nodes. Rayleigh-Ritz method is an alternative numerical method to solve the same equation in a simple way to know what happens in between as well.

There is one more thing to bear in mind. The lumped-model method gave us a nice set of linear equations, which we can easily solve. Also, we reduced a continuous system to a discretized system so that we can easily implement it on the computer. We don't want to lose these advantages in the Rayleigh-Ritz method. Thus, the Rayleigh-Ritz method is another way to discretize the continuous model.

Let us refer to Equation (7). We need to minimize *PE* to find *u(x)*. If *u(x)* were to be a scalar variable, we could have minimized *PE* very easily as we did several times in Chapter 2. So, we have to employ a trick to get *u(x)* to become scalar variables somehow. We can do that as follows.

Note from Figure 12 of Chapter 2 that as we increased the number of elements, the deflection curve converged to a continuous shape. And that shape looks like a parabola. So, the unknown function *u(x)* can be assumed to be a quadratic equation of the form shown below.

$$u(x) = a_0 + a_1 x + a_2 x^2 \qquad (10)$$

But, what we don't know are three scalars viz. a_0, a_1, and a_2. That is perfectly agreeable to us, because we can substitute for *u(x)* from Equation (10) into the expression for *PE* given in Equation (7). Then, we get *PE* in terms of scalar quantities as we wanted. Now invoke the *MPE* principle.

$$\textit{Extremize}\ \ PE(a_0, a_1, a_2)\ \ \textit{with respect to}\ a_0, a_1, \&\ a_2 \qquad (11)$$

The conditions for solving the above are:

$$\frac{\partial(PE)}{\partial a_i} = 0 \qquad i = 0, 1, 2 \qquad (12)$$

Equations (12) result in three linear equations in a_0, a_1, and a_2, which can easily be solved. In fact, you would note at once that $a_0 = 0$ as $u(x=0) = 0$. That is our assumed function for *u(x)* should satisfy the boundary condition. Or in other words, it should be a kinematically admissible deformation. If you didn't appreciate kinematic admissibility in Chapter 2, here is the second chance!

Exercise 3.1

For the same tapered bar problem considered in Chapter 1, use the Rayleigh-Ritz method. That is, write Equations (7), and (12) to solve for a_0, a_1, and a_2.

- Work it out by hand so that you can understand more.
- Try it out with Maple also so that you can solve more interesting and larger problems.
- Check the Rayleigh-Ritz solution with the lumped-model solution with a large number of elements.

Exercise 3.2

Consider the overhanging simply supported beam shown below in Figure 2. In order to use the Rayleigh-Ritz method, we would like to approximate the deflected profile, $v(x)$ as $a \cos\frac{2\pi x}{L}$ where L is the length of the beam. Use the minimum potential energy principle to compute the unknown constant, a .

- Draw the assumed deflected profile. Is it a kinematically admissible function?
- Write down the expression for the strain energy of the beam.
- What is the work potential due to each force (use $y_{x=0}$, $y_{x=40}$, and $y_{x=80}$)?
- Compute the expression for the total potential energy in terms of a .
- Compute the value of a .

If a single assumed function is not adequate to represent the deformation, one can use more than one function for different parts of the structure. Each of these functions will have unknown coefficients which can be determined by minimizing *PE*. If more than one function is used, one needs to ensure continuity of the functions at points where they connect with each other. The following exercise uses this technique.

Exercise 3.3

Repeat the tapered bar problem if the area of cross-section varies as follows. Area at the top is the same as before (i.e., A_0). The cross-section area remains constant up to the middle of the bar ($x=0.5$), and then increases parabolically to become three times A_0 at the bottom.

$$A_1(x) = A_0 \qquad \text{for} \qquad 0 \le x \le 0.5$$

$$A_2(x) = A_0(3 - 8 + 8x^2) \qquad \text{for} \qquad 0.5 \le x \le 1$$

Use two different polynomials for the ranges ($0 \le x \le 0.5$) and ($0.5 \le x \le 1$) to approximate $u(x)$ with two piece-wise continuous polynomials. Note that you should ensure continuity at $x = 0.5$ so that $u(x)$ and its derivative are continuous.

Exercise 3.4

Comfy Beds, Inc. is considering a new design for the box-spring system. It consists of top and bottom grids of thin strips of metal connected by linear helical springs. A portion of this new box-spring system is shown in the figure. Use Rayleigh-Ritz method to determine the maximum deflections of the top and bottom beams. (see Figure 3).

Use

$$y_1 = a_1 x_1 (x_1 - l_1)$$
$$y_2 = -a_2 x_2^2$$

as the basis functions where y_1 and y_2 are the deformations of the top and bottom beams respectively. x_1 and x_2 are zero at the left end of each beam.

(a) Do the above basis functions satisfy the kinematic admissibility conditions? Explain how.

(b) The strain energy for a beam is given by $\int_0^L \frac{EI}{2}\left(\frac{d^2y}{dx^2}\right)^2 dx$. Write the total strain energy stored in the two beams and the spring in terms of a_1 and a_2.

+ What is the work potential due to the applied force, F of 5 lb? (again in terms of a_1 and a_2).

+ Use the principle of the minimum potential energy to find the equilibrium values of a_1 and a_2.

Both beams have rectangular cross-section of thickness 0.1 in and a width of 1 in. The Young's modulus is 30E6 psi, and the spring constant, k is 10 lb/in. The applied force F is 5 lb. l_1 and l_2 are respectively 40 in and 30 in.

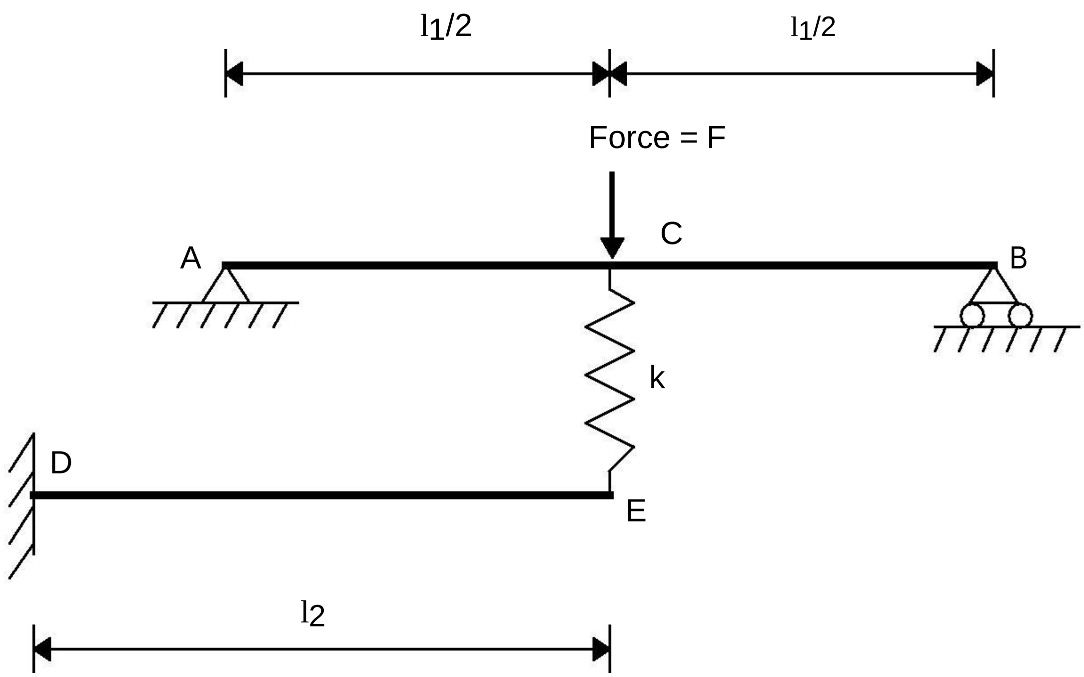


Figure 3 The schematic of the springs used by Comfy Beds, Inc.

The Rayleigh-Ritz method is a powerful method to use if we know *a priori,* the nature of the function for the deformation. However, we may not be able to guess such a function or several piece-wise functions for any given problem. The FEM enables us to come up with such functions systematically. Those functions are called shape functions. They serve the following purpose.

- Approximate the continuous deformation using piece-wise functions defined over elements.
- Shape functions depend on some scalar quantities and those scalar quantities are nothing but the value of the deformation at the nodes.
- *Interpolation*, i.e., knowing what happens within the element is readily available through shape functions.

The following Table summarizes the basic concepts we laid out in Chapters 2 and 3. In the next chapter, we will study the shape functions and apply this concept to the axially loaded bars once again. This is the real beginning of our FEM discussion.

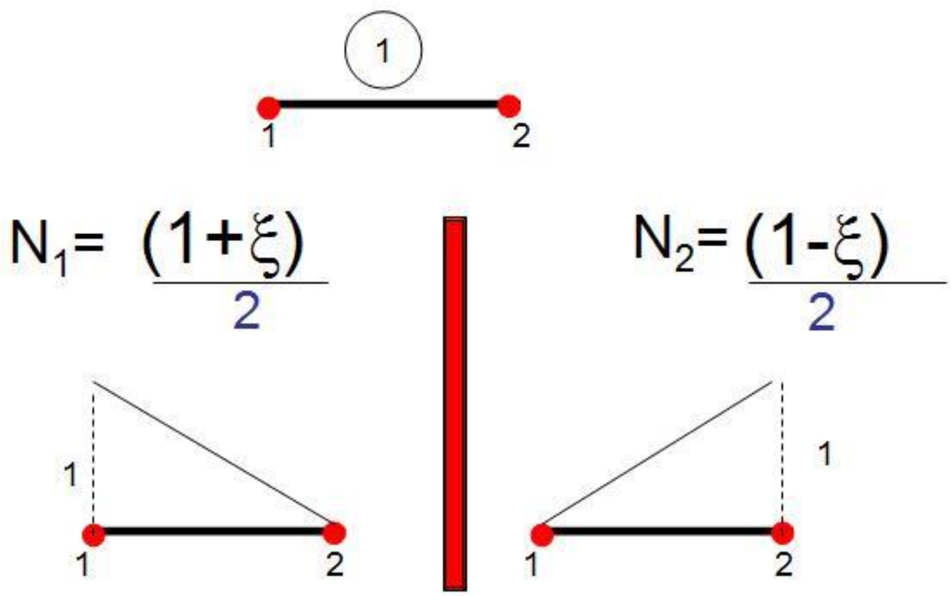
Table 1 Comparison of three approaches to deformation analysis

	Lumped-model	Rayleigh-Ritz	FEM
Discretization	Divide into segments ("element"). The value of the deformation at the discrete points ("nodes") are the unknown scalar quantities to be determined using the MPE principle.	Discretization concept is different. You do convert a continuous problem into a discrete problem. But, the discrete (scalar) unknowns are coefficients of the assumed polynomials (basis functions).	In principle, it is the same as the lumped model, i.e., the discretization is physical.
Interpolation	Not possible.	You need to know the nature of the function so that you can approximate the deformation curve with one or more trial (guess) functions globally. The procedure is not systematic.	The procedure is systematic. Shape functions are used for interpolation locally for small elements.

Module 2

Body force distribution for 2 noded bar element

We derived shape functions for 1D bar, variation of these shape functions is shown below .As a property of shape function the value of N_1 should be equal to 1 at node 1 and zero at rest other nodes (node 2).



From the potential energy of an elastic body we have the expression of work done by body force as

$$\int_V u^T f_b \, dv$$

$$U = N_1 q_1 + N_2 q_2$$

For an element

$$\int_e u^T f_b A \, dx$$

Where f_b is the body acting on the system. We know the displacement function $U = N_1q_1 + N_2q_2$ substitute this U in the above equation we get

$$= A f_b \int_e (N_1q_1 + N_2q_2)\, dx$$

$$= A f_b \int_e [N_1 \ \ N_2] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} dx$$

$$= A f_b \int_e \underbrace{[q_1 \ \ q_2]}_{qT} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$$

$$= A f_b \ qT \int_e \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$$

$$= qT \begin{bmatrix} A f_b \int_e N_1 \, dx \\ A f_b \int_e N_2 \, dx \end{bmatrix}$$

Now

$$\int_e N_1 \, dx = \int_e \frac{1-\xi}{2} \, dx$$

but $\frac{dx}{d\xi} = l_{e/2}$

$$= \int_{-1}^{+1} \frac{1-\xi}{2} \frac{l_e}{2} \, d\xi = \frac{l_e}{2}$$

Similarly

$$\int_e N_2 \, dx = \frac{l_e}{2}$$

Therefore

$$\int u^T f_b A \, dx = qT \underbrace{A f_b \frac{l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{f_e}$$

This amount of body force will be distributed at 2 nodes hence the expression as 2 in the denominator.

Surface force distribution for 2 noded bar element

Now again taking the expression of work done by surface force from potential energy concept and following the same procedure as that of body we can derive the expression of surface force as

$$\int_s u^T T \, ds = \int_e u^T T \, dx$$

$$= q^T \underbrace{\frac{l_e T}{2}\begin{Bmatrix}1\\1\end{Bmatrix}}_{T_e}$$

Where T_e is element surface force distribution.

Methods of handling boundary conditions

We have two methods of handling boundary conditions namely Elimination method and penalty approach method. Applying BC's is one of the vital role in FEM improper specification of boundary conditions leads to erroneous results. Hence BC's need to be accurately modeled.

Elimination Method: let us consider the single boundary conditions say $Q_1 = a_1$.Extremising Π results in equilibrium equation.

$Q = [Q_1, Q_2, Q_3 \ldots\ldots\ldots Q_N]^T$ be the displacement vector and

$F = [F_1, F_2, F_3 \ldots\ldots\ldots\ldots F_N]^T$ be load vector

Say we have a global stiffness matrix as

$$K = \begin{bmatrix} K_{11} & K_{12} & \ldots\ldots\ldots\ldots & K_{1N} \\ K_{21} & K_{22} & \ldots\ldots\ldots\ldots & K_{2N} \\ . & & & \\ . & & & \\ . & & & \\ K_{N1} & K_{N2} & \ldots\ldots\ldots\ldots & K_{NN} \end{bmatrix}$$

Now potential energy of the form $\Pi = ½\, Q^T K Q - Q^T F$ can written as

$$
\begin{aligned}
\Pi = ½\,(& Q_1K_{11}Q_1 + Q_1K_{12}Q_2 + \ldots + Q_1K_{1N}Q_N \\
& + Q_2K_{21}Q_1 + Q_2K_{22}Q_2 + \ldots + Q_2K_{2N}Q_N \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + Q_NK_{N1}Q_1 + Q_NK_{N2}Q_2 + \ldots + Q_NK_{NN}Q_N) \\
& - (Q_1F_1 + Q_2F_2 + \ldots + Q_NF_N)
\end{aligned}
$$

Substituting $Q_1 = a_1$ we have

$$
\begin{aligned}
\Pi = ½\,(& a_1K_{11}a_1 + a_1K_{12}Q_2 + \ldots + a_1K_{1N}Q_N \\
& + Q_2K_{21}a_1 + Q_2K_{22}Q_2 + \ldots + Q_2K_{2N}Q_N \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + Q_NK_{N1}a_1 + Q_NK_{N2}Q_2 + \ldots + Q_NK_{NN}Q_N) \\
& - (a_1F_1 + Q_2F_2 + \ldots + Q_NF_N)
\end{aligned}
$$

Extremizing the potential energy

ie $d\Pi/dQi = 0$ gives

Where $i = 2, 3 \ldots N$

$$
\begin{aligned}
& K_{22}Q_2 + K_{23}Q_3 + \ldots + K_{2N}Q_N = F_2 - K_{21}a_1 \\
& K_{32}Q_2 + K_{33}Q_3 + \ldots + K_{3N}Q_N = F_3 - K_{31}a_1 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& K_{N2}Q_2 + K_{N3}Q_3 + \ldots + K_{NN}Q_N = F_N - K_{N1}a_1
\end{aligned}
$$

Writing the above equation in the matrix form we get

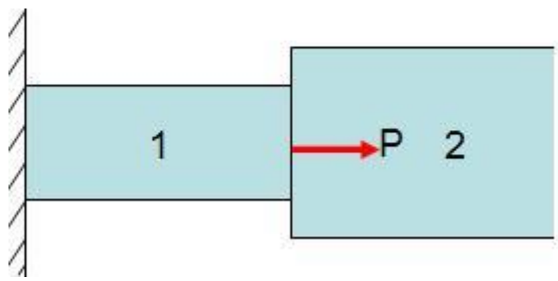
$$
\begin{pmatrix}
K_{22} & K_{23} \ldots\ldots & K_{2N} \\
K_{32} & K_{33} \ldots\ldots & K_{2N} \\
\cdot & & \\
\cdot & & \\
\cdot & & \\
K_{N2} & K_{N3} \ldots\ldots & K_{NN}
\end{pmatrix}
\begin{pmatrix}
Q_2 \\ Q_3 \\ \cdot \\ \cdot \\ \\ Q_N
\end{pmatrix}
=
\begin{pmatrix}
F_2 - K_{21}a_1 \\ F_3 - K_{31}a_1 \\ \\ \\ \\ F_N - K_{N1}a_1
\end{pmatrix}
$$

Now the N X N matrix reduces to N-1 x N-1 matrix as we know $Q_1=a_1$ ie first row and first column are eliminated because of known Q_1. Solving above matrix gives displacement components. Knowing the displacement field corresponding stress can be calculated using the relation $\sigma = \varepsilon Bq$.

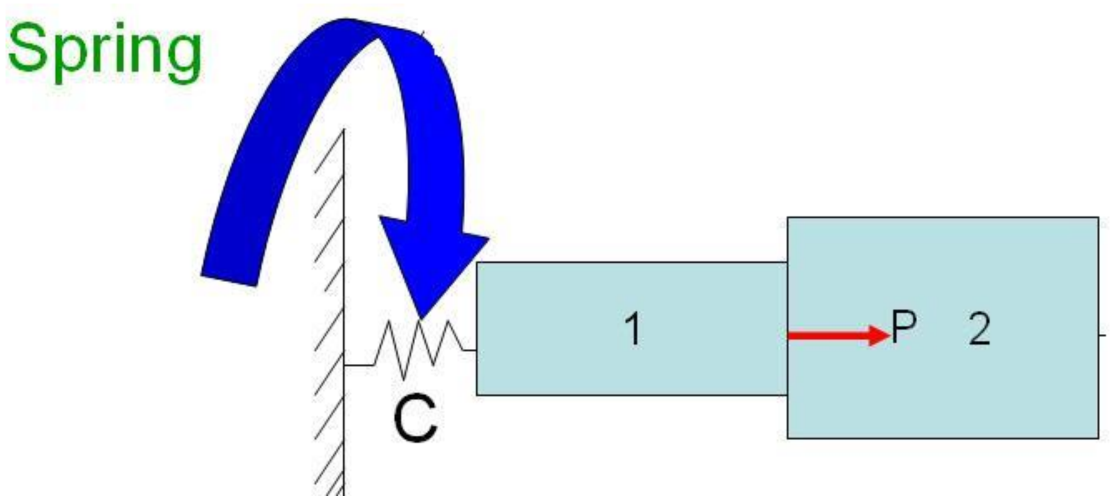
Reaction forces at fixed end say at node1 is evaluated using the relation

$$R_1 = K_{11}Q_1 + K_{12}Q_2 + \ldots\ldots\ldots\ldots\ldots + K_{1N}Q_N - F_1$$

Penalty approach method: let us consider a system that is fixed at both the ends as shown



In penalty approach method the same system is modeled as a spring wherever there is a support and that spring has large stiffness value as shown.



Let a_1 be the displacement of one end of the spring at node 1 and a_3 be displacement at node 3. The displacement Q_1 at node 1 will be approximately equal to a_1, owing to the relatively small resistance offered by the structure. Because of the spring addition at the support the strain energy also comes into the picture of Π equation .Therefore equation Π becomes

$$\Pi = \tfrac{1}{2} Q^T K Q + \tfrac{1}{2} C (Q_1 - a_1)^2 - Q^T F$$

The choice of C can be done from stiffness matrix as

$$C = \max [K_{ij}] \times 10^4$$

We may also choose 10^5 & 10^6 but 10^4 found more satisfactory on most of the computers.

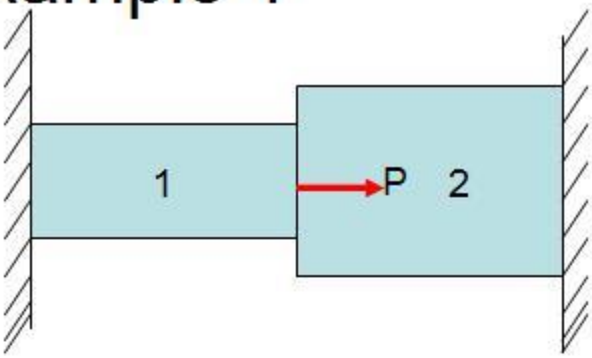
Because of the spring the stiffness matrix has to be modified ie the large number c gets added to the first diagonal element of K and Ca_1 gets added to F_1 term on load vector. That results in.

$$\begin{pmatrix} K_{11} + C & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} = \begin{pmatrix} F1 + C\,a1 \\ F2 \\ F3 \end{pmatrix}$$

A reaction force at node 1 equals the force exerted by the spring on the system which is given by

$$\text{Reaction forces} = -C(Q_1 - a_1)$$

Example 1



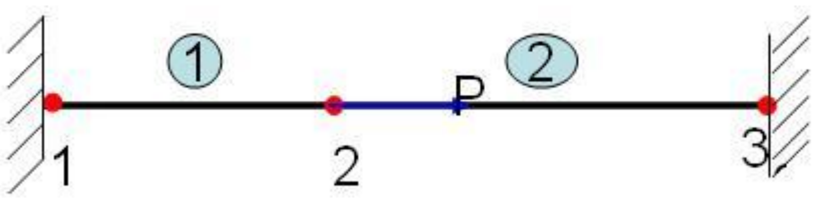
$A_1 = 900 mm^2$ $\quad$ $A_2 = 1200 mm^2$

$E_1 = 70 \times 10^9 N/m^2$ $\quad$ $E_2 = 200 \times 10^9 N/m^2$

$L_1 = 200mm$ $\quad$ $L_2 = 300mm$

$P = 300 KN$

To solve the system again the seven steps of FEM has to be followed, first 2 steps contain modeling and discretization. this result in



Third step is finding stiffness matrix of individual elements

$$K_1 = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{900 \times 0.75 \times 10^5}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 3.15 & -3.15 \\ -3.15 & 3.15 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

Similarly

$$K_2 = \frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{matrix} 2 & \;\;3 \end{matrix} \\ \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} \end{array} \begin{matrix} 2 \\ 3 \end{matrix}$$

Next step is assembly which gives global stiffness matrix

$$K = \begin{array}{c} \begin{matrix} 1 & \qquad 2 & \qquad 3 \end{matrix} \\ \begin{bmatrix} 3.15 & -3.15 & 0 \\ -3.15 & 3.15+8 & -8 \\ 0 & -8 & 8 \end{bmatrix} \end{array} 10^5 \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

Now determine global load vector

$$F = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 300 \times 10^3 \\ R_3 \end{Bmatrix}$$

We have the equilibrium condition KQ=F

$$10^5 \begin{bmatrix} 3.15 & -3.15 & 0 \\ -3.15 & 3.15+8 & -8 \\ 0 & -8 & 8 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 300 \times 10^3 \\ R_3 \end{Bmatrix} \begin{matrix} \\ -(-3.15 \times 10^5 \times Q1) \\ -(0 \times Q1) \end{matrix}$$

After applying elimination method we have Q2 = 0.26mm

Once displacements are known stress components are calculated as follows

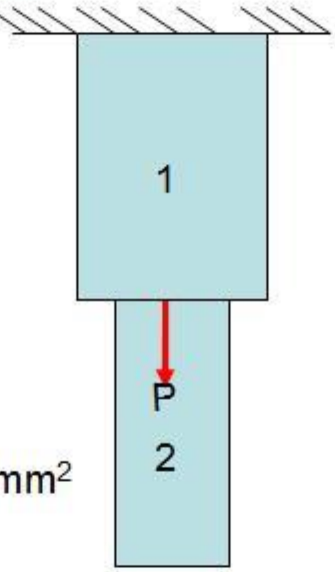
For element 1

$$\sigma_1 = E_1 \frac{1}{L_1} [-1 \quad 1] \begin{Bmatrix} Q1 \\ Q2 \end{Bmatrix} = 94.17 \text{ N/mm}^2$$

For element 2

$$\sigma_2 = E_2 \frac{1}{L_2} [-1 \quad 1] \begin{Bmatrix} Q2 \\ Q3 \end{Bmatrix} = -179.34 \text{ N/mm}^2$$

Example 2



$E_1 = 2.06 \times 10^5$ MPa

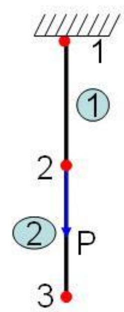
$A_1 = 3387.09mm^2$ $A_2 = 2419.35mm^2$

$L_1 = L_2 = 304.8mm$

$P = 444.8N$

Body force = $f_b = 7.69 \times 10^{-5} N/mm^3$

Solution:



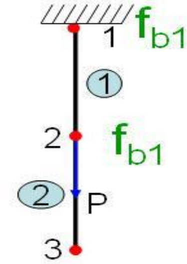
$$K_1 = \frac{A_1E_1}{L_1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 10^6 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 2.28 & -2.28 \\ -2.28 & 2.28 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

$$K_2 = \frac{A_2E_2}{L_2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 10^6 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{pmatrix} 1.63 & -1.63 \\ -1.63 & 1.63 \end{pmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

$$K = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 2.28 & -2.28 & 0 \\ -2.28 & 2.28+1.63 & -1.63 \\ 0 & -1.63 & 1.63 \end{pmatrix} \end{array} 10^6 \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Body force terms

Element 1



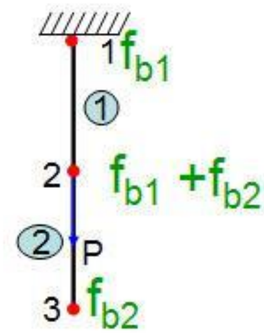
$$f_{b1} = \frac{A_1 f_b L_1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$= \frac{3387.09 \times 7.69 \times 10^{-5} \times 304.8}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$= \begin{pmatrix} 39.69 \\ 39.69 \end{pmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

Body force terms

Element 2

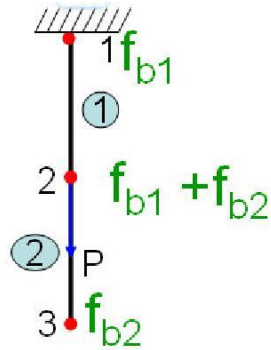


$$f_{b2} = \frac{A_2 f_b L_2}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

$$= \frac{2419.35 \times 7.69 \times 10^{-5} \times 304.8}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

$$= \begin{pmatrix} 28.3 \\ 28.3 \end{pmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

Global load vector:



$$F=\begin{pmatrix}F_1\\ F_2\\ F_3\end{pmatrix}=\begin{pmatrix}f_{b1}\\ p+f_{b1}+f_{b2}\\ fb2\end{pmatrix}=\begin{pmatrix}39.69\\ 512.8\\ 28.3\end{pmatrix}$$

We have the equilibrium condition KQ=F

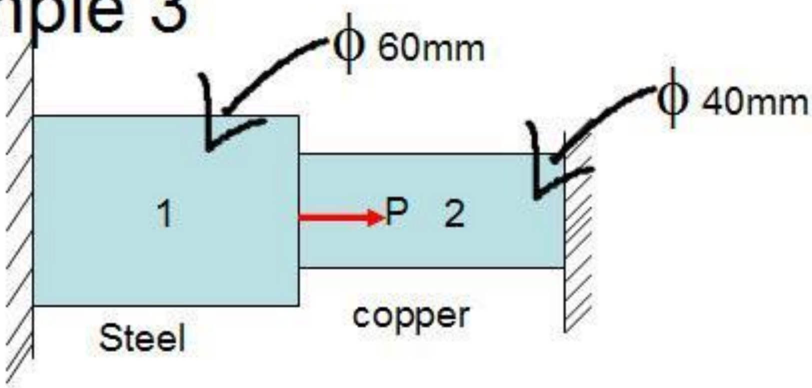
$$10^6\begin{pmatrix}2.28 & -2.28 & 0\\ -2.28 & 6.92 & -16.3\\ 0 & -1.63 & 1.63\end{pmatrix}\begin{pmatrix}Q1\\ Q2\\ Q3\end{pmatrix}=\begin{pmatrix}39.69+R1\\ 512.8\\ 28.3\end{pmatrix}$$

$Q2 = 0.23 \times 10^{-3}$mm

$Q3 = 2.5 \times 10^{-4}$mm

After applying elimination method and solving matrices we have the value of displacements as $Q2 = 0.23 \times 10^{-3}$mm & $Q3 = 2.5 \times 10^{-4}$mm

Example 3

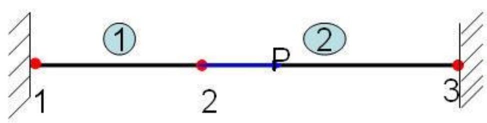


$E_1 = 2 \times 10^5$ MPa $\qquad$ $E_2 = 1 \times 10^5$ MPa

$L_1 = 800$ mm $\qquad$ $L_2 = 500$ mm

$P = 100$ KN

Solution:



$$A_1 = \pi/4\,(60)^2 = 2827.43\,\text{mm}^2$$

$$A_2 = \pi/4\,(40)^2 = 1256.63\,\text{mm}^2$$

$$K_1 = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2827.43 \times 2 \times 10^5}{800}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 7.06 & -7.06 \\ -7.06 & 7.06 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

(column indices: 1, 2)

$$K_2 = \frac{A_2 E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 2.51 & -2.51 \\ -2.51 & 2.51 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

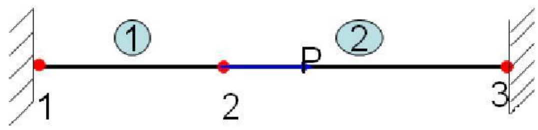
(column indices: 2, 3)

Global stiffness matrix

$$K = \begin{pmatrix} 7.07 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 \end{pmatrix} 10^5$$

(columns and rows numbered 1, 2, 3)

Global load vector:



$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \times 10^3 \\ 0 \end{pmatrix}$$

Equilibrium Equation

$$K\,Q = F$$

$$K = \begin{pmatrix} 7.07 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 \end{pmatrix} 10^5 \begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \times 10^3 \\ 0 \end{pmatrix}$$

$$C = \max [\, K_{ij} \,] \times 10^4 = 9.583 \times 10^5 \times 10^4$$

Modification required

$$\begin{bmatrix} 7.07 + C & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 + C \end{bmatrix} 10^5 \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} = \begin{Bmatrix} 0 + C\,a1 \\ 100 \times 10^3 \\ 0 + C\,a3 \end{Bmatrix}$$

After Modification

$$\begin{bmatrix} 9.583 \times 10^4 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 9.583 \times 10^4 \end{bmatrix} 10^5 \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \times 10^3 \\ 0 \end{Bmatrix}$$

Solving the matrix we have

$Q1 = 7.698 \times 10^{-6}$mm, $Q2 = 0.104$mm, $Q3 = 2.736 \times 10^{-6}$mm

Reaction forces

@ node 1

$$R_1 = C(Q1 - a1) = -73597.44\text{N}$$

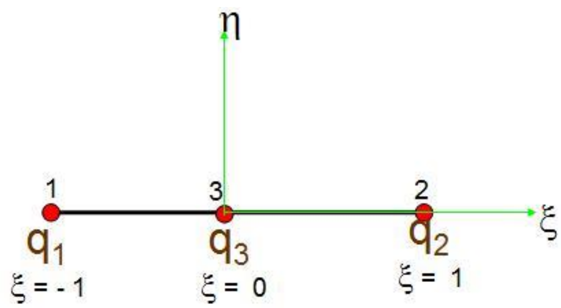
@ node 3

$$R_3 = C(Q3 - a3) = -26219.08\text{N}$$

Quadratic 1D bar element

In the previous sections we have seen the formulation of 1D linear bar element , now lets move a head with quadratic 1D bar element which leads to for more accurate results . linear element has two end nodes while quadratic has 3 equally spaced nodes ie we are introducing one more node at the middle of 2 noded bar element.

Consider a quadratic element as shown and the numbering scheme will be followed as left end node as 1, right end node as 2 and middle node as 3.



Let's assume a polynomial as

$$U = \alpha_o + \alpha_1 \xi + \alpha_2 \xi^2$$

Now applying the conditions as

@ node 1	$u = q_1$	$\xi = -1$
@ node 2	$u = q_2$	$\xi = 1$
@ node 3	$u = q_3$	$\xi = 0$

ie

$$q_1 = \alpha_o - \alpha_1 + \alpha_2$$

$$q_2 = \alpha_o + \alpha_1 + \alpha_2$$

$$q_3 = \alpha_o$$

Solving the above equations we have the values of constants

$$\alpha_1 = \frac{q_2 - q_1}{2} \qquad \alpha_2 = \frac{q_1 + q_2 - 2q_3}{2}$$

And substituting these in polynomial we get

$$U = \alpha_o + \alpha_1 \xi + \alpha_2 \xi^2$$

$$= q_3 + \left(\frac{q_2 - q_1}{2}\right)\xi + \left(\frac{q_1 + q_2 - 2q_3}{2}\right)\xi^2$$

$$= \frac{\xi(\xi-1)}{2} q_1 + \frac{\xi(\xi+1)}{2} q_2 + (1-\xi^2) q_3$$

Or

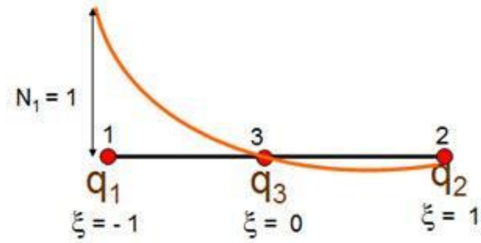
$$U = N_1 q_1 + N_2 q_2 + N_3 q_3$$

Where N1 N_2 N_3 are the shape functions of quadratic element

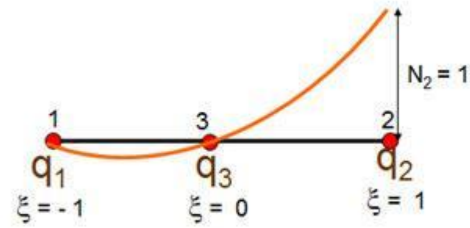
$$N_1 = \frac{\xi(\xi-1)}{2}$$

$$N_2 = \frac{\xi(\xi+1)}{2}$$

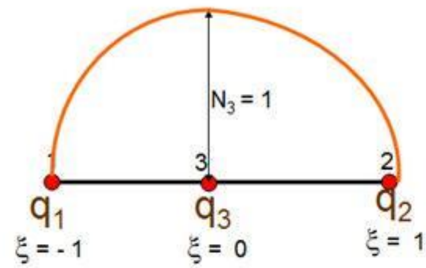
$$N_3 = (1-\xi^2)$$



$$N_1 = \frac{\xi(\xi-1)}{2}$$



$$N_2 = \frac{\xi(\xi+1)}{2}$$



$$N_3 = (1-\xi^2)$$

Graphs show the variation of shape functions within the element .The shape function N_1 is equal to 1 at node 1 and zero at rest other nodes (2 and 3). N_2 equal to 1 at node 2 and zero at rest other nodes(1 and 3) and N_3 equal to 1 at node 3 and zero at rest other nodes(1 and 2)

Element strain displacement matrix If the displacement field is known its derivative gives strain and corresponding stress can be determined as follows

WKT

$$U = N_1 q_1 + N_2 q_2 + N_3 q_3$$

$$\varepsilon = \frac{du}{dx}$$

$$= \frac{du}{d\xi} \frac{d\xi}{dx} \quad \text{By chain rule}$$

Now

$$\frac{du}{d\xi} = \frac{d[N_1 q_1 + N_2 q_2 + N_3 q_3]}{d\xi}$$

Splitting the above equation into the matrix form we have

$$\frac{du}{d\xi} = \frac{d[N_1\ N_2\ N_3]}{d\xi} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$\frac{du}{d\xi} = \begin{pmatrix} \frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi \end{pmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Therefore

$$\varepsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx}$$

$$= \left[\frac{(2\xi-1)}{2} \quad \frac{(2\xi-1)}{2} \quad -2\xi \right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \frac{d\xi}{dx}$$

$$= \frac{2}{le} \left[\frac{(2\xi-1)}{2} \quad \frac{(2\xi+1)}{2} \quad -2\xi \right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$\varepsilon = Bq$$

B is element strain displacement matrix for 3 noded bar element

Stiffness matrix:

We know the stiffness matrix equation

$$K = \int_V B^T E\, B\, dv$$

For an element

$$K = \int_e B^T E\, B\, A\, dx$$

$$= \int_e B^T E\, B\, \frac{A\, L_e}{2}\, d\xi$$

Taking the constants outside the integral we get

$$K = \frac{E\,A\,L_e}{2} \int_e B^T B \, d\xi$$

Where

$$B = \frac{2}{le} \begin{bmatrix} \frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi \end{bmatrix}$$

and B^T

$$B^T = \frac{2}{le} \begin{bmatrix} \frac{(2\xi-1)}{2} \\ \frac{(2\xi+1)}{2} \\ -2\xi \end{bmatrix}$$

Now taking the product of B^T X B and integrating for the limits -1 to +1 we get

$$K = \frac{E\,A\,L_e}{2} \int_e B^T B \, d\xi$$

$$= \frac{E\,A\,L_e}{2} \int_{-1}^{+1} \frac{4}{L_e^2} \begin{bmatrix} \frac{1}{4}(2\xi-1)^2 & \frac{1}{4}(2\xi-1)(2\xi+1) & -(2\xi-1)\xi \\ \frac{1}{4}(2\xi-1)(2\xi+1) & \frac{1}{4}(2\xi+1)^2 & -(2\xi+1)\xi \\ -(2\xi-1)\xi & -(2\xi+1)\xi & 4\xi^2 \end{bmatrix}$$

Integration of a matrix results in

$$K = \frac{EA}{3L_e}\begin{pmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{pmatrix}$$

Body force term & surface force term can be derived as same as 2 noded bar element and for quadratic element we have

Body force:

$$f_e = A f_b l_e \begin{pmatrix} 1/6 \\ 1/6 \\ 2/3 \end{pmatrix}$$

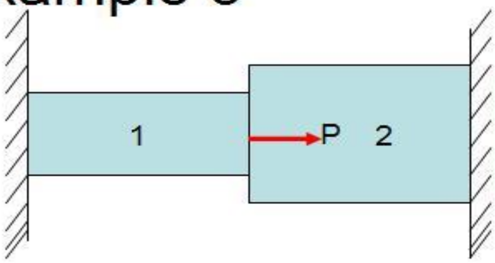
Surface force term:

$$T_e = T \ l_e \begin{pmatrix} 1/6 \\ 1/6 \\ 2/3 \end{pmatrix}$$

This amount of body force and surface force will be distributed at three nodes as the element as 3 equally spaced nodes.

Problems on quadratic element

Example 5



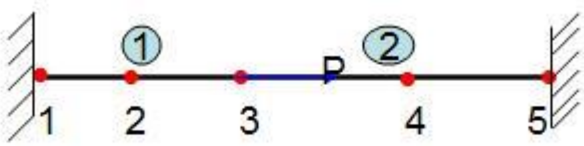
$A_1 = 600mm^2$ $A_2 = 800mm^2$

$E= 2 \times 10^5 N/mm^2$

$L_1 = 150mm$ $L_2 = 220mm$

$P = 30 KN$

Solution:



Column order: 1, 3, 2; row order: 1, 3, 2

$$K_1 = 10^5 \begin{pmatrix} 18.6 & 2.6 & -21.3 \\ 2.6 & 18.6 & -21.3 \\ -21.3 & -21.3 & 42.6 \end{pmatrix}$$

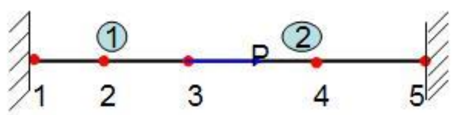
Column order: 3, 5, 4; row order: 3, 5, 4

$$K_2 = 10^5 \begin{pmatrix} 16.9 & 2.42 & -19.3 \\ 2.42 & 16.9 & -19.3 \\ -19.3 & -19.3 & 38.7 \end{pmatrix}$$

Global stiffness matrix

$$
K = 10^5 \begin{array}{c} \\ \end{array}
\begin{array}{ccccc|c}
1 & 2 & 3 & 4 & 5 & \\
\hline
18.6 & -21.3 & 2.6 & 0 & 0 & 1 \\
-21.3 & 42.6 & -21.3 & 0 & 0 & 2 \\
2.6 & -21.3 & 35.5 & -19.3 & 2.4 & 3 \\
0 & 0 & -19.3 & 38.7 & -19.3 & 4 \\
0 & 0 & 2.4 & 19.3 & 16.9 & 5
\end{array}
$$

Global load vector



$$
F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \\ P \\ 0 \\ R_5 \end{pmatrix}
$$

By the equilibrium equation KQ=F, solving the matrix we have Q2, Q3 and Q4 values

$$10^5\begin{bmatrix} 18.6 & -21.3 & 2.6 & 0 & 0 \\ -21.3 & 42.6 & -21.3 & 0 & 0 \\ 2.6 & -21.3 & 35.5 & -19.3 & 2.4 \\ 0 & 0 & -19.3 & 38.1 & -19.3 \\ 0 & 0 & 2.4 & 19.3 & -16.9 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ P \\ 0 \\ R_5 \end{Bmatrix}$$

$Q_2 = 1.25 \times 10^{-7}$ mm

$Q_3 = 2.14 \times 10^{-3}$ mm

$Q_5 = 5.13 \times 10^{-3}$ mm

Stress components in each element

For element 1 @ node 1

$$\sigma_{1/1} = \frac{2}{l_1}\begin{bmatrix} \frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi \end{bmatrix}\begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} E$$

$$\sigma_{1/1} = \frac{2}{150}\begin{bmatrix} -3/2 & -½ & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.01 \\ 0.02 \end{Bmatrix} 2 \times 10^5$$

$= 93.1$ N/mm^2

For element 1 @ node 2

$$\sigma_{1/2} = \frac{2}{l_1}\begin{bmatrix} \frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi \end{bmatrix}\begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} E$$

$$\sigma_{1/2} = \frac{2}{150}\begin{bmatrix} -½ & ½ & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.01 \\ 0.02 \end{Bmatrix} 2 \times 10^5$$

$= 13.33$ N/mm^2

For element 1 @ node 3

$$\sigma_{1/3} = \frac{2}{l_1}\begin{pmatrix}\frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi\end{pmatrix}\begin{pmatrix}Q1 \\ Q2 \\ Q3\end{pmatrix} E$$

$$\sigma_{1/3} = \frac{2}{150}\begin{pmatrix}½ & 3/2 & -2\end{pmatrix}\begin{pmatrix}0 \\ 0.01 \\ 0.02\end{pmatrix} 2 \times 10^5$$

$= -66.5\ N/mm^2$

For element 2 @ node 3

$$\sigma_{2/3} = \frac{2}{l_2}\begin{pmatrix}\frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi\end{pmatrix}\begin{pmatrix}Q3 \\ Q4 \\ Q5\end{pmatrix} E$$

$$\sigma_{2/3} = \frac{2}{220}\begin{pmatrix}-3/2 & -½ & 2\end{pmatrix}\begin{pmatrix}0.02 \\ 0.01 \\ 0\end{pmatrix} 2 \times 10^5$$

$= -63.63\ N/mm^2$

For element 2 @ node 4

$$\sigma_{2/4} = \frac{2}{l_2}\begin{pmatrix}\frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi\end{pmatrix}\begin{pmatrix}Q3 \\ Q4 \\ Q5\end{pmatrix} E$$

$$\sigma_{2/4} = \frac{2}{220}\begin{pmatrix}-½ & ½ & 0\end{pmatrix}\begin{pmatrix}0.02 \\ 0.01 \\ 0\end{pmatrix} 2 \times 10^5$$

$= -9.09\ N/mm^2$

For element 2 @ node 5

$$\sigma_{2/5} = \frac{2}{l_1}\begin{bmatrix}\frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi\end{bmatrix}\begin{Bmatrix}Q3\\Q4\\Q5\end{Bmatrix} E$$

$$\sigma_{2/5} = \frac{2}{150}\begin{bmatrix}½ & 3/2 & -2\end{bmatrix}\begin{Bmatrix}0.02\\0.01\\0\end{Bmatrix} 2 \times 10^5$$

$$= 45.45 \text{ N/mm}^2$$

Solution to Simultaneous Algebraic Equations – Gauss Elimination Method:

Consider n simultaneous equations ,

$a_{11} x_1 + a_{12} x_2 +, a_{13} x_3$ ………… $+ a_{1n} x_n$ b_1

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3$ …………$+ a_{2n} x_n$ b_2

$a_{31} x_1 + a_{32} x_2 + a_{33} x_3$ ………… $+ a_{3n} x_n$ b_3

$a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3$ ……… $+ a_{nn} x_n$ · b_n

write the given set of equations in matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ . & . & . & \dots & \\ . & . & . & \dots & \\ . & . & . & \dots & \\ . & . & . & \dots & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ \dots \\ \dots \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ \dots \\ \dots \\ \dots \\ b_n \end{bmatrix}$$

In Gauss elimination method the variables x_2, … ……….. …… x_{n-1}, will be successively eliminated using **Row Operations**. This step is called **forward elimination** . The given matrix will be converted to into an upper triangular matrix, Lower triangular elements become zeros.

After forward elimination the n^{th} equation (last equation) become simple , it as an equation with one variable x_n , determine x_n. Now using $(n-1)^{th}$ equation x_{n-1} can be determined. Similarly using $(n-2)^{nd}$ equation x_{n-2} can be determined. Using $(n-3)^{rd}$ equation x_{n-3} can be determined. Continue up to first equation until all the unknowns are determined. This is called **backward substitution.**

Forward Elimination

Step 1 : a_{11} becomes pivot, eliminate x_1 from row2 , row3, row4, row n etc

Row2 $a_{21} = a_{21} - (a_{21} / a_{11})\, a_{11}$ a_{21} becomes 0

$a_{22} = a_{22} - (a_{21} / a_{11})\, a_{12}$ a_{22} changes

$a_{23} = a_{23} - (a_{21} / a_{11})\, a_{13}$ a_{23} changes

etc up to $a_{1n} = a_{2n} - (a_{21} / a_{11})\, a_{1n}$ a_{2n} changes

$b_2 = b_2 - (a_{21} / a_{11})\, b_1$ b_2 changes

whatever we did to make $a_{21} = 0$ applied the same to other elements of that row

Row3 $a_{31} = a_{31} - (a_{31} / a_{11})\, a_{11}$ a_{31} becomes 0

$a_{32} = a_{32} - (a_{31} / a_{11})\, a_{12}$ a_{32} changes

$a_{33} = a_{33} - (a_{31} / a_{11})\, a_{13}$ a_{33} changes

etc up to $a_{3n} = a_{3n} - (a_{31} / a_{11})\, a_{1n}$ a_{3n} changes

$b_3 = b_3 - (a_{31} / a_{11})\, b_1$ b_3 changes

whatever we did to make $a_{31} = 0$ applied the same to other elements of that row

……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………

Row n $a_{n1} = a_{n1} - (a_{n1} / a_{11})\, a_{11}$ a_{n1} becomes 0

$a_{n2} = a_{n2} - (a_{n1} / a_{11})\, a_{12}$ a_{n2} changes

$a_{n3} = a_{n3} - (a_{n1} / a_{11})\, a_{13}$ a_{n3} changes

etc up to $a_{nn} = a_{nn} - (a_{n1} / a_{11})\, a_{1n}$ a_{nn} changes

$b_3 = b_3 - (a_{31} / a_{11})\, b_1$ b_3 changes

whatever we did to make $a_{n1} = 0$ applied the same to other elements of that row.

Now, re- write the whole matrix equation. First row remains same, elements of other rows will be different.

Step2 : : a_{22} becomes pivot, eliminate x_2 from row3 , row4, row5, etc., row n following the same method

Now, re-write the whole matrix equation. First row , Second row remains same, elements of other rows will be different

Step3 : : a_{33} becomes pivot, eliminate x_3 from row4 , row5, row6, etc., row n following the same method

Now, re-write the whole matrix equation. First row , Second row , Third row remains same, elements of other rows will be different

Continue until the variables x_2, x_3, x_4 … ………………………… x_{n-1} will be successively eliminated and all the lower triangular elements becomes zero.

Backward substitution.
After forward elimination the n^{th} equation (last equation) become simple , it as an equation with one variable x_n , determine x_n. Now , using $(n-1)^{th}$ equation x_{n-1} can be determined. Similarly using $(n-2)^{nd}$ equation x_{n-2} can be determined. Using $(n-3)^{rd}$ equation x_{n-3} can be determined. Continue up to first equation until all the unknowns are determined. The method is best understood by solving problems.

Different Methods used to Solve Set of Simultaneous Equations in FEM.

Method of Matrix Inversion
Gauss elimination method
Cholesky Decomposition Technique
Gauss-Seidal Iteration Technique
Relaxation Method

Numerical examples illustrating Gauss elimination method :

Problem 1. Solve the following set of equation by Gaussian elimination technique.

5x1 + 3x2 + 2x3 + x4 = 4
4x1 + 3x2 - 3x3 - 2x4 = 5
x1 + 2x2 - 2x3 + 3x4 = 6 -
4x1 + 3x2 – 5x3 + 2x4 = 7

Solution : Write the given equations in Matrix Form

5	3	2	1		x1		4
4	3	-3	-2		x2		5
1	2	-2	3	*	x3	=	6
-4	3	-5	2		x4		7

[CO] [X] = [CONS]
[a] [x] = [b]

Step 1 : $a_{ij} = a_{ij} - (a_{i1} / a_{11})\, a_{1j}$ $\quad b_i = b_i - (a_{i1} / a_{11})\, b_1$
$i = 2, \quad j = 1,2,3,4$

Row 2 i = 2 j = 1,2,3,4
4 – (4/5) 5 = 4 – (4) = 0.0 3 – (4/5) 3 = 3 – 2.4 = 0.6
-3 – (4/5)2 = -3 –1.6 = -4.6 -2 – (4/5) 1 = -2 – 0.8 = -2.8

Row 3 i = 3 j = 1,2,3,4
1 – (1/5)5 = 1 – 1 = 0.0 2 – (1/5)3 = 2 -0 .6 = 1.4

-2 – (1/5) 2 = -2– 0.4= -2.4 3 – (1/5) 1 = 3 – 0.2 = 2.8

Row 4 : i = 4 j = 1,2,3,4
-4 – (-4 / 5)5 = 4 - 4 = 0 3 – (-4 / 5)3 = 3+2.4 = 5.4
-5 – (-4 /5) 2 = -5 + 1.6 = -3 .4 2 – (-4 /5) 1 = 3 – 0.2 = 2.8

b **i** = b **i** – (a **i1**/ a**11**) b **1**
i = 2 b2 = 5 – (4/5)4 = 1.8
i = 3 b3 = 6 – (1/5)4 = 5.2
i = 4 b4 = 7 – (-4/5)4 = 10.2

The modified matrix equation , after eliminating x1 from 2nd , 3rd and 4th equations.

5	3	2	1		x1		4
0	0.6	-4.6	-2.8		x2		1.8
0	1.4	-2.4	2.8	*	x3	=	5.2
0	5.4	-3.4	2.8		x4		10.2

Step 2 : To eliminate x2 from Row 3 and Row 4

$a_{ij} = a_{ij} – (a_{i2} / a_{22})\, a_{2j} \quad b_i = b_i – (a_{i2} / a_{22})\, b_2 \quad i = 3, j = 2,3,4$

Row 3   i =3   j = 2,3,4

1.4 – (1.4/0.6) 0.6 =1.4 – 1.4 = 0.0   j = 2

-2.4 – (1.4/0.6) (-4.6) = – 2.4+10.73 =8.33 j = 3

2.8 – (1.4/0.6)(-2.8) = -2.8 – 6.53 = 9.33   j = 4

Row 4   i = 4   j = 2,3,4

5.4– (5.4/0.6)0.6 = 5.4-5.4 = 0.0   j = 2

-3.4 – (5.4/0.6)(-4.6) = -3.4+41.4 = 38 j = 3

2.8 – (5.4/0.6)2.8 = 2.8 +25.2 = 28   j = 4

b i = b i – (a i2/ a22) b 2

i = 3 b3 = b3 – (a32/a22)b2

5.2 - (1.4/0.6) 1.8 = 5.2 – 4.2 = 1

i = 4 b4 = b4 – (a42/a22)b2

10.2 - (5.4/0.6)1.8 = 10.2 -16.2 = -6

The modified matrix after step 2 eliminating x2 from 3rd and 4th equations.

5	3	2	1		x1		4
0	0.6	-4.6	-2.8		x2		1.8
0	0	8.33	9.32	*	x3	=	1
0	0	38	28		x4		-6

Step 3 : To eliminate x3 from Row 4

$a_{ij} = a_{ij} – (a_{i3} / a_{33})\, a_{3j} \quad b_i = b_i – (a_{i3} / a_{33})\, b_3 \quad i = 4, j = 3,4$

a43 = a43 – (a43/ a**33)** a**33** = 38 – (38 / 8.33) 8.33 = 38 – 38 = 0.0

$a44 = a44 - (a43 / a\mathbf{33})\ a34 = \quad 28 - (38 / 8.33)\ 9.32$
$= 28 – 42.52 = -14.52$

$b\ i = b\ i – (a\ \mathbf{i3} / a\mathbf{33})\ b\ \mathbf{3} \quad i = 4,$
$b4 = b4 - (a43 / a\mathbf{33})\ b3 \qquad b4 \quad = -6 - (38 / 8.33)\ 1 = - 6 \ - 4.56 = - 10.56$

The modified matrix, after step 3 , eliminating x3 from 4th equation.

5	3	2	1		x1		4
0	0.6	-4.6	-2.8		x2		1.8
0	0	8.33	9.32	*	x3	=	1
0	0	0	-14.52		x4		-10.56

Back Substitution:

The modified equations are

$5\ x1 + 3\ x2 + 2\ x3 + x4 = 4$
$0.6\ x2 – 4.6\ x3 - 2.8\ x4 = 1.8$
$8.33\ x3 + 9.32\ x4 = 1$
$-14.52\ x4 = -10.56$
$x4 = (-10.56 / -14.52) = \mathbf{0.727}$

$8.33\ x3 + 9.32\ (0.727) = 1$
$x3 = (1 - 6.776) / 8.33 = \mathbf{- 0.693}$

$0.6\ x2 – 4.6(- 0.693) -2.8\ (0.727) =$
$1.8\ 0.6\ x2 = (1.8 - 3.1878 + 2.0356)$
$x2 = (1.8 - 3.1878 + 2.0356) / 0.6 = \mathbf{1.079}$

$5x1 + 3\ (1.079) + 2\ (- 0.693) + 0.727 = 4$
$x1 = (4 - 3\ (1.079) + 2\ (0.693) - 0.727) / 5 = \mathbf{0.155}$

$x1= 0.155 \quad x2=1.079 \quad x3= - 0.693 \quad x4= 0.727$

Prob2 : Solve using gauss elimination method

x1- 2x2 + 6 x3 = 0   2x1 +2x2 + 3x3 = 3   -x 1 + 3x2 = 2

$$\begin{bmatrix} 1 & -2 & 6 \\ 2 & 2 & 3 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Step 1 : $a_{ij} = a_{ij} - (a_{i1} / a_{11})\, a_{1j}$ I = 2,3 j = 2,3

2- (2/1)* 1 = 0   2 – (2/1) (-2) = 6   3 – (2/1) (6) = -9
-1- (-1/1) 1 = 0   3 – (-1/1) (-2) = 1   0 – (-1/1) 6 = 6

$$b_i = b_i - (a_{i1} / a_{11})\, b_1$$

3 – (2/1) *0 = 3   2 – (-1/1) *0 = 2

Modified Matrix After step1

$$\begin{bmatrix} 1 & -2 & 6 \\ 0 & 6 & -9 \\ 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Step 2 : $a_{ij} = a_{ij} - (a_{i2} / a_{22})\, a_{2j}$ i = 3 j = 2,3

1 – (1/6)* (6) = 0 6 – (1/6) (-9) = 7.5

$$b_i = b_i - (a_{i2} / a_{22})\, b_2$$

2 – (1/6) 3 = 1.5

Modified Matrix After step2

1	-2	6		x_1	=	0
0	6	-9		x_2		3
0	0	7.5		x_3		1.5

Back Substitution :

$7.5\,x_3 = 1.5$ $x_3 = 1.5 / 7.5 = 0.2$ $x_3 = 0.2$

$6\,x_2 - 9x_3 = 3$ $x_2=(3 + 9(0.2))/6 =0.8$ $x_2 = 0.8$

$x_1 - 2x_2 +6x_3 = 0$ $x_1 = 2(0.8)-6(0.2) = 0.4$ $x_1 = 0.4$

Prob 3 :Solve using gauss elimination method

$4x_1 +6x_2 +8x_3 = 2$ $8x_1 +4x_2 + 6x_3 = 4$ $6x_1 + 2x_2+4x_3= 6$

Solution : writing the equations in matrix form

4	6	8	=	x_1		2
8	4	6		x_2		4
6	2	4		x_3		6

Step 1 : $a_{ij} = a_{ij} - (a_{i1} / a_{11})\, a_{1j}$ $i = 2,3$ $j = 2,3$

$8 - (8/4)* 4 = 0$ $4-(8/4)(6) = -8$ $6 -(8/4)8 = -10$

$6 -(6/4) 4 = 0$ $2-(6/4 (6) = -7$ $4 - (6/4) 8 = -8$

$b_i = b_i - (a_{i1} / a_{11})\, b_1$

$4 - (8/4) 2 = 0$ $6 - (6/4) 2 = 3$

Modified matrix equation after step 1

4	6	8	=	X1		2
0	-8	-10		X2		0
0	-7	-8		X3		3

Step 2 : $a_{ij} = a_{ij} - (a_{i2} / a_{22})\, a_{2j}$ i = 3 j = 2,3

-7 – (-7/-8) (-8) = 0

-8 -(-7 / -8)(-10) = -8 +70/8 = -8 + 8.75 = 0.75

$b_i = b_i - (a_{i2} / a_{22})\, b_2$

3 – (-7/-8) 0 = 3

Modified matrix equation after step 2 :

4	6	8	X1		2
0	-8	-10	X2		0
0	0	0.75	x3		3

0.75 x3 = 3   x3 = 3/0.75 = 4   x3 = 4

-8 x2 -10 x3 = 0   -8x2 -10 (4) = 0   -8x2 = 40 x2 = -5

4x1+6x2 + 8x3 = 2   4x1+ 6(-5) +8(4) = 2

x1 = (2+30 -32) / 4 =0

x1 = 0 x2 = -5   x3= 4

Prob 4 : Solve using gauss elimination method

3x1 -3x2 -2x3 = 5   2x1+2x2+3x3= 6   3x1 - 5x2+2x3= 7

3	-3	-2		5
2	2	3		6
3	-5	2		7

Step 1 :

$a_{ij} = a_{ij} – (a_{i1} / a_{11})\, a_{1j}$ i = 2,3 j = 2,3

2 - (2/3)3 = 0   2 - (2/3)(-3) = 4 3   -(2/3)(-2) = 4.33

3 -(3/3) 3 = 0   -5 – (3/3 (-3) = -2   2 – (3/3)(-2) = 4

$b_i = b_i - (a_{i1} / a_{11})\, b_1$

6 – (2/3) 5 = 6-10/3 = 2.667 7 – (3/3) 5 = 2

Modified matrix is

3	-3	-2		5
0	4	4.33		2.667
0	-2	4		2

Step 2 :

$a_{ij} = a_{ij} – (a_{i2} / a_{22})\, a_{2j}$ i = 3 j = 2,3

-2 - (-2/4)4 = 0   4 - (-2/4)(4.333) = 4+ 2.166 = 6.166

$b_i = b_i - (a_{i2} / a_{22})\, b_2$

2 – (-2/4) 2.667 = 2+(2.667/2) = 2+1.333 = 3.333

3	-3	-2		5
0	4	4.33		2.667
0	0	6.166		3.333

6.166 x3 = 3.333 x3 = (3.333/6.166) = 0.504

4x2 + 4.333x3 = 2.667 4x2 = 2.667 - 4.333 (0.504) x2 = 0.483 /4 = 0.120

3x1 -3x2 – 2x3 = 5   3x1 -3(0.120) -2(0.504) = 5

x1 = (5 + 0.360 +1.008) / 3 = 2.122

x1 = 2.122 x2 = 0.120   x3 = 0.504

HIGHER ORDER ELEMENTS

Many engineering structures and mechanical components are subjected to loading in two directions. Shafts, gears, couplings, mechanical joints, plates, bearings, are few examples. Analysis of many three dimensional systems reduces to two dimensional, based on whether the loading is plane stress or plane strain type. Triangular elements or Quadrilateral elements are used in the analysis of such components and systems. The various load vectors, displacement vectors, stress vectors and strain vectors used in the analysis are as written below,

the displacement vector **u** = $[u, v]^T$,
u is the displacement along x direction, v is the displacement along y direction,

the body force vector **f** = $[f_x , f_y]^T$
f_x , is the component of body force along x direction, f_y is the component of body force along y direction

the traction force vector **T** = $[T_x , T_y]^T$
T_x , is the component of body force along x direction, T_y is the component of body force along y direction

Two dimensional stress strain equations

From theory of elasticity for a two dimensional body subjected to general loading the equations of equilibrium are given by

$[\ _x / x] + [\ _{yx} /\ y] + F_x = 0$

$[\ _{xy} / x] + [\ _y / y] + F_y = 0$

Also $_{xy} = \ _{yx}$

The strain displacement relations are given by

$_x = u / x,\ _y = v / y,\ _{xy} = u / y + v / x = [u / x, v / y,$
$(u / y + v / x)]^T$

The stress strain relationship for plane stress and plane strain conditions are given by the matrices shown in the next page. $_x$ $_y$ $_{xyx}$ $_y$ $_{xy}$ are usual stress strain components, v is the poisons ratio. E is young’s modulus. Please note the differences in [D] matrix .

Two dimensional elements

Triangular elements and **Quadrilateral elements** are called two dimensional elements. A simple triangular element has straight edges and corner nodes. This is also a linear element. It can have constant thickness or variable thickness.

The stress strain relationship for plane stress loading is given by

x			1	V	0		z
y	=	E / (1-v ^{2})	V	1	0	*	y
xy			0	0	1-v / 2		yz

[ ] = [**D**] [ ]

The stress strain relationship for plane strain loading is give by

x			1-v	V	0		z
y			V	1-v	0		y
xy	=	E / (1+v)(1-2v)	0	0	½ -v	*	yz

[ ] = [**D**] [ ]

The element having mid side nodes along with corner nodes is a higher order element. Element having curved sides is also a higher order element.

A simple quadrilateral element has straight edges and corner nodes. This is also a linear element. It can have constant thickness or variable thickness. The quadrilateral having mid side nodes along with corner nodes is a higher order element. Element having curved sides is also a higher order element.

The given two dimensional component is divided in to number of triangular elements or quadrilateral elements. If the component has curved boundaries certain small region at the boundary is left uncovered by the elements. This leads to some error in the solution.

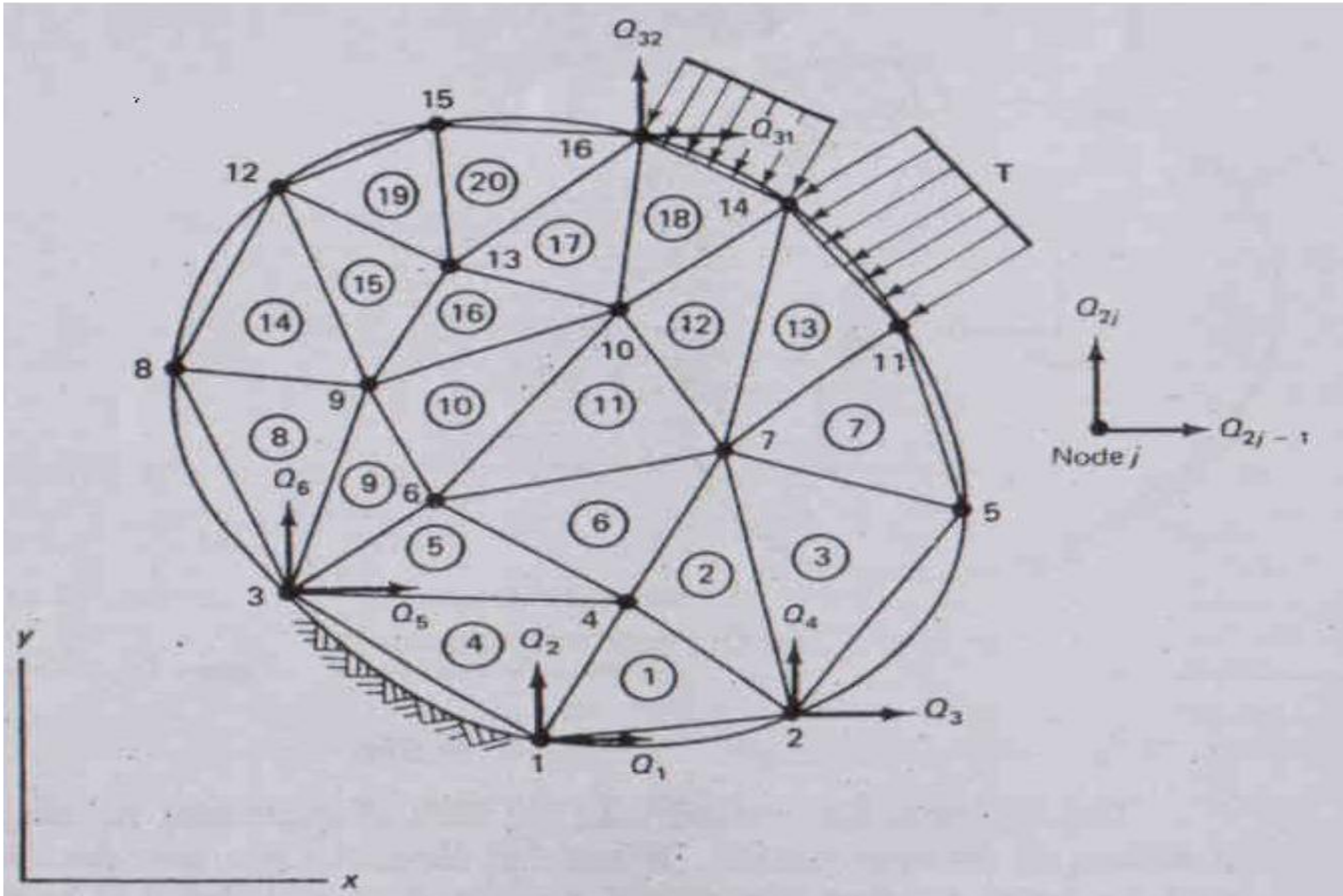
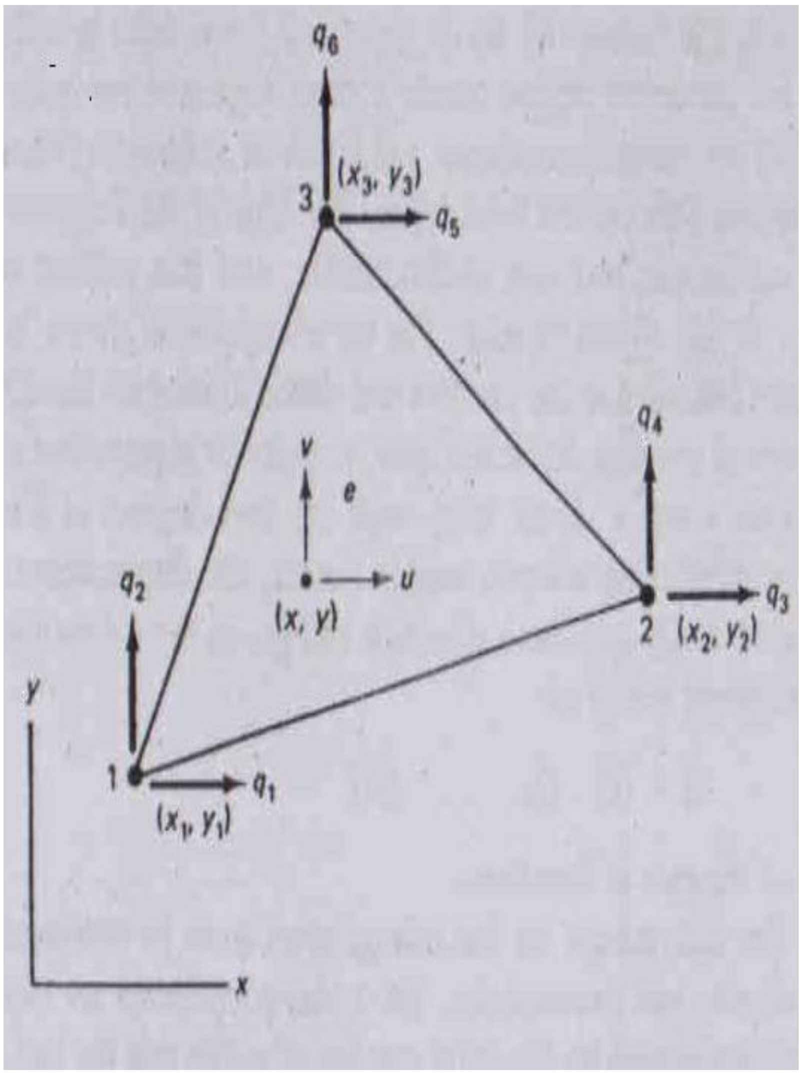
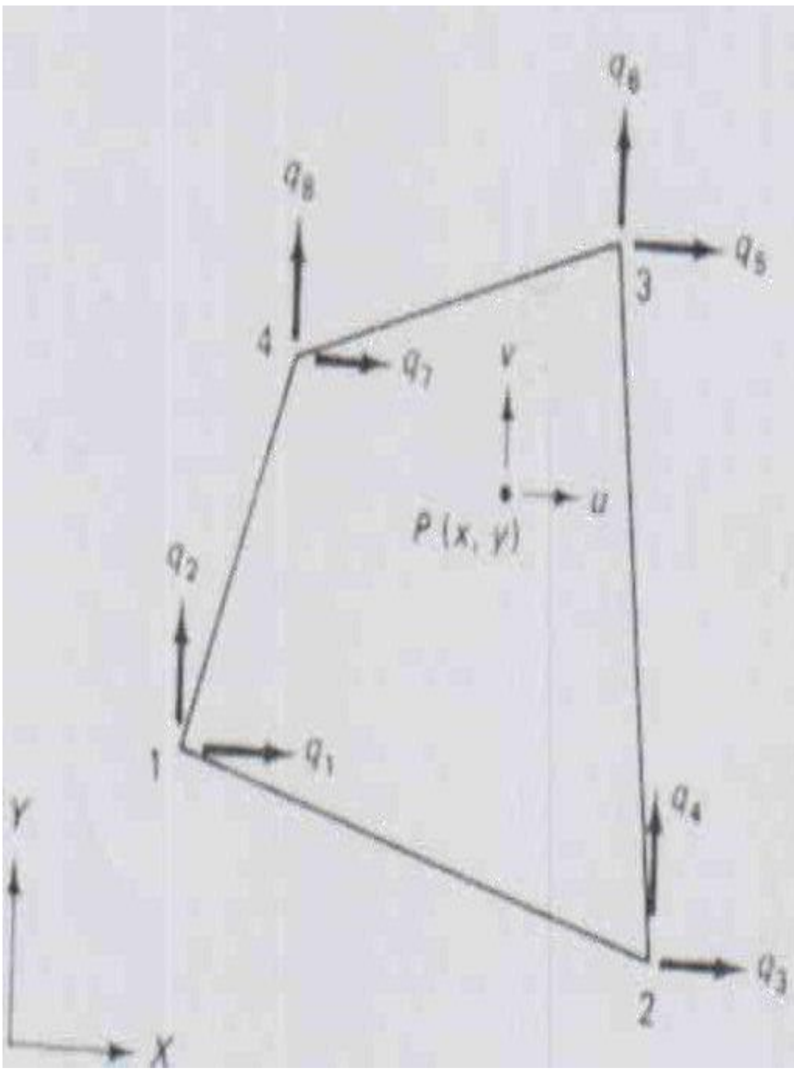


Figure 5.2 Finite element discretization.



Constant Strain Triangle



Quadrilateral

Constant Strain Triangle

It is a triangular element having three straight sides joined at three corners. and imagined to have a node at each corner. Thus it has three nodes, and each node is permitted to displace in the two directions, along x and y of the Cartesian coordinate system. The loads are applied at nodes. Direction of load will also be along x direction and y direction, +ve or –ve etc. Each node is said to have two degrees of freedom. The nodal displacement vector for each element is given by,

$\mathbf{q} = [q_1, q_2, q_3, q_4, q_5, q_6]$

q_1, q_3, q_5 are nodal displacements along x direction of node1, node2 and node3 simply called horizontal displacement components.

q_2, q_4, q_6 are nodal displacements along y direction of node1, node2 and node3 simply called vertical displacement components. q_{2j-1} is the displacement component in x direction and q_{2j} is the displacement component in y direction.

Similarly the nodal load vector has to be considered for each element. Point loads will be acting at various nodes along x and y …………………………

(x_1, y_1), (x_2, y_2), (x_3, y_3) are cartesian coordinates.of node 1 node 2 and node 3.

In the discretized model of the continuum the node numbers are progressive, like 1,2,3,4,5,6,7,8……….etc and the corresponding displacements are Q_1, Q_2, Q_3, Q_4, Q_5 Q_6, Q_7, Q_8, Q_9, Q_{10}….. Q_{16}, two displacement components at each node.

Q_{2j-1} is the displacement component in x direction and Q_{2j} is the displacement component in y direction. Let j = 10, ie 10^{th} node, $Q_{2j-1} = Q_{19}$ $Q_{2j} = Q_{20}$

The element connectivity table shown establishes correspondence of local and global node numbers and the corresponding degrees of freedom. Also the (x_1, y_1), (x_2, y_2) and (x_3, y_3) have the global correspondence established through the table.

Element Connectivity Table Showing
Local – Global Node Numbers

Element Number	Local Nodes Numbers 1	2	3	
1	1	2	4	Corres-
2 3	4	2	7	-ponding-
..	..	..	..	Global-
11	6	7	10	Node-
..	..	..	..	
20	13	16	15	Numbers

Nodal Shape Functions: u nder the action of the given load the nodes are assumed to deform linearly. element has to deform elastically and the deformation has to become zero as soon as the loads are zero. It is required to define the magnitude of deformation

and nature of deformation for the element Shape functions or Interpolation functions are used to model the magnitude of displacement and nature of displacement.

The Triangular element has three nodes. Three shape functions N1 , N2 , N3 are used at nodes 1,2 and 3 to define the displacements. Any linear combination of these shape functions also represents a plane surface.

N1 = , N2 = , N3 = 1 – – (1.8)

The value of N1 is unity at node 1 and linearly reduces to 0 at node 2 and 3. It defines a plane surface as shown in the shaded fig. N2 and N3 are represented by similar surfaces having values of unity at nodes 2 and 3 respectively and dropping to 0 at the opposite edges. In particular N1 + N2 + N3 represents a plane at a height of 1 at nodes 1 , 2 and 3 The plane is thus parallel to triangle 1 2 3.

Shape Functions N_1 , N_2 , N_3

For every N1 , N2 and N3 , N1 + N2 + N3 = 1 N1 N2 and N3 are therefore not linearly independent.

N1 =N2 =N3 = 1 – – , where and are natural coordinates
The displacements inside the element are given by,
u = N1 q1 + N2 q3 + N3 q5
v = N1 q2 + N2 q4 + N3 q6 writing these in the matrix form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} * \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

[u] = [N] [q]

Iso Paramatric Formulation :

The shape functions N1, N2, N3 are also used to define the geometry of the element apart from variations of displacement. This is called Iso-Parametric formulation

- u = N1 q1 + N2 q3 + N3 q5
  v = N1 q2 + N2 q4 + N3 q6 , defining variation of displacement.

- x = N1 x1 + N2 x2 + N3 x3
- y = N1 y1 + N2 y2 + N3 y3 , defining geometry.

Potential Energy :

Total Potential Energy of an Elastic body subjected to general loading is given by
= Elastic Strain Energy + Work Potential

$= \frac{1}{2}\ {}^{T}\ dv - u^{T} f\ dv - u^{T} T\ ds - u^{T} i\ Pi$

For the 2- D body under consideration P.E. is given by

$$= \frac{1}{2}\ {}^{T}D\ \ te\ dA - u^{T} f\ t\ dA - u^{T} T\ t\ dl - u^{T} i\ Pi$$

This expression is utilised in deriving the elemental properties such as Element stiffness matrix **[K] ,** load vetors **f^{e} , T^{e} ,** etc .

Derivation of Strain Displacement Equation and Stiffness Matrix for CST (derivation of [B] and [K]) :

Consider the equations

u = N1 q1 + N2 q3 + N3 q5 v = N1 q2 + N2 q4 + N3 q6

x = N1 x1 + N2 x2 + N3 x3 y = N1 y1 + N2 y2 + N3 y3 Eq (1)

We Know that u and v are functions of x and y and they in turn are functions ofand .

u = u (x (,) , y (,)) v = v (x (,) , y (,))

taking partial derivatives for u , using chain rule, we have equation (A) given by

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

Eq (A)

Similarly, taking partial derivatives for v using chain rule, we have equation (B) given by

$$\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta}$$

Eq (B)

now consider equation (A), writing it in matrix form

$$\begin{matrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{matrix} = \begin{matrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{matrix} \quad \begin{matrix} +u \\ -x \\ +u \\ -y \end{matrix}$$

$$\begin{matrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{matrix}$$ Is called **JACOBIAN [J]**

Jacobian is used in determining the strain components, now we can get

$$\begin{matrix} \bullet u \\ x \\ \bullet u \\ y \end{matrix} = [\mathbf{J}]^{-1} \begin{matrix} \partial u \\ \partial \xi \\ \\ \partial u \\ \partial \eta \end{matrix}$$

In the Left vector u / x = x , is the strain component along x-dirction.

Similarly writing equation (B) in matrix form and considering [J] we get ,

$$\begin{matrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{matrix} = [\mathbf{J}]^{-1} \begin{matrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{matrix}$$

In the left vector v / y = y , is the strain component along y-direction..

u/ x = x , v/ y = y , xy = u/ y + v/ x

We have to determine [J] , $[J]^{-1}$ which is same for both the equations.

First we will take up the determination u/ x = x and u/ y using J and J^{-1} , Consider the equations

u = N1 q1 + N2 q3 + N3 q5 v = N1 q2 + N2 q4 + N3 q6

Substituting for N1 , N2 and N3 , in the above equations we get

u = q1 + q3 + (1 – –) q5 = (q1 - q5) + (q3 - q5) + q5

= q_{15} + q_{35} + q_{5}

u / = q15 u / = q 35

v = q2 + q4 + (1 – –) q6 = (q2 - q6) + (q4 - q6) + q6

= q_{26} + q_{46} + q_{6}

v / = q 26 v / = q 46

Consider x = N1 x1 + N2 x2 + N3 x3

y = N1 y1 + N2 y2 + N3 y3

Substituting for N1 , N2 and N3 , in the above equations we get

x = x1 + x2 + (1 – –) x3

x = (x1 - x3) + (x2 - x3) + x3 = x_{13} + x_{23} + x3

x / = x13 x / = q 23

y = y1 + y2 + (1 – –) y3

y = (y1 - y3) + (y2 - y3) + y3 = y_{13} + y_{23} + y3

y / = y13 y / = y23

To determine [J] , [J] $^{-1}$

u / = q15 u / = q 35 v / = q 26 v / = q 46 y

x / = x13 x / = y23 y / = y13 / = y23

[J] =x / y / [J] = x13 , y13 x1 - x 3 , y1 - y 3

x / y / x23 , y23 x2 - x 3 , y2 - y3

To determine [J] $^{-1}$: find out co factors [J]

co-factors of x ij = $(-1)^{i+j}$ | |

co-factors [co] = (y2 - y3), -(x2 - x 3) y23 , x32

-(y1 - y3), (x1 - x 3) y31, x13

Adj [J] = $[co]^T$ = y23 y31
x32 x13

$[J]^{-1}$ = Adj [J] / | J |

$[J]^{-1}$ = (1/ |J|) y23 y31
x32 x13

Also we have
u / = q15 = q1- q5 u / = q 35 = q3 –q5

u / x = $[J]^{-1}$ u /
u / y u /

u / x = (1/ |J|) y23 y31 q1- q5
u / y x32 x13 q3 –q5

u / x = (1/ |J|) y23 q1- q5 + y31 q3 –q5
u / y x32 q1- q5 + x13 q3 –q5

u / x = (1/ |J|) y23 q1- y23 q5 + y31 q3 – y31q5
u / y x32 q1- x32 q5 + x13 q3 – x13q5

u / x = (1/ |J|) y23q1 + y31 q3 - y23 q5– y31q5
u / y x32 q1 + x13 q3 - x32 q5– x13q5

u / x = (1/ |J|) y23q1 + y31 q3 - q5 (y2 - y3+ y3 -y1)
u / y x32 q1 + x13 q3 - q5 (x3 - x2+x1 - x3)

u / x = (1/ |J|) y23q1 + y31 q3 - q5 (y2 -y1)
u / y x32 q1 + x13 q3 - q5 (- x2+x1)

u / x = (1/ |J|) y23q1 + y31 q3 + q5 (y1- y2)
u / y x32 q1 + x13 q3 + q5 (x2 -x1)

u / x = (1/ |J|) y23q1 + y31 q3 + y12 q5
u / y x32 q1 + x13 q3 + x21 q5

Writing the R.H.S of above equation in Matrix form

u / x = 1/ |J| y23 0 y31 0 y12 0 q1
u / y x32 0 x13 0 x21 0 q2
q3
q4
q5
q6

…… eq (6)

Similarly Considering equation (B) we get

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = [\mathbf{J}]^{-1} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

[J] = x / y / =x13 , y13 x1 - x 3 , y1 - y 3
x /y / x23 , y23 x2 - x 3 , y2 - y3

[J] -1 = 1/ |J| y23 y31
x32 x13

consider v = N1 q2 + N2 q4 + N3 q6

v = q2 + q4 + (1 – –) q6
v = (q2 - q6) + (q4 - q6) + q6
= q26 + q46 + q6

v / = q26 v
/ = q46

v / x = [J] $^{-1}$ v / v
v / y /

$$\begin{matrix} v/x \\ v/y \end{matrix} = (1/|J|) \begin{matrix} y23 & y31 \\ x32 & x13 \end{matrix} \quad \begin{matrix} q2-q6 \\ q4-q6 \end{matrix}$$

$$\begin{matrix} v/x \\ v/y \end{matrix} = (1/|J|) \begin{matrix} y23\ (q2-q6) + y31\ (q4-q6) \\ x32\ (q2-q6) + x13\ (q4-q6) \end{matrix}$$

$$\begin{matrix} v/x \\ v/y \end{matrix} = (1/|J|) \begin{matrix} y23\ q2 - y23\ q6 + y31\ q4 - y31q6 \\ x32\ q2 - x32q6 + x13\ q4 - x13q6 \end{matrix}$$

$$\begin{matrix} v/x \\ v/y \end{matrix} = (1/|J|) \begin{matrix} y23q2 + y31\ q4 - y23q6 - y31q6 \\ x32\ q2 + x13\ q4 - x32q6 - x13q6 \end{matrix}$$

$$\begin{matrix} v/x \\ v/y \end{matrix} = (1/|J|) \begin{matrix} y23q2 + y31q4 - q6(y2 - y3 + y3 - y1) \\ x32\ q2 + x13\ q4 - q6(x3 - x2 + x1 - x3) \end{matrix}$$

canceling y3 and x3 , we get

$$\begin{matrix} v/x \\ v/y \end{matrix} = (1/|J|) \begin{matrix} y23q2 + y31q4 - q6(y2 - y1) \\ x32\ q2 + x13\ q4 - q6(-x2 + x1) \end{matrix}$$

$$\begin{matrix} v/x \\ v/y \end{matrix} = (1/|J|) \begin{matrix} y23q2 + y31q4 + q6(y1 + y2) \\ x32\ q2 + x13\ q4 + q6(x2 + x1) \end{matrix}$$

$$\begin{matrix} v/x \\ v/y \end{matrix} = (1/|J|) \begin{matrix} y23q2 + y31q4 + y12\ q6 \\ x32\ q2 + x13\ q4 + x21q \end{matrix}$$

Writing in matrix form

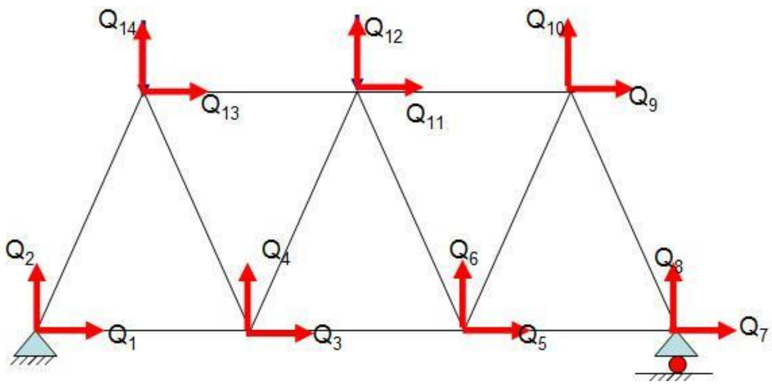
$$\begin{matrix} v/x \\ v/y \end{matrix} = 1/|J| \begin{matrix} 0 & y23 & 0 & y31 & 0 & y12 \\ 0 & x32 & 0 & x13 & 0 & x21 \end{matrix} \quad \begin{matrix} q1 \\ q2 \\ q3 \\ q4 \\ q5 \\ q6 \end{matrix}$$

ANALYSIS OF TRUSSES

A Truss is a two force members made up of bars that are connected at the ends by joints. Every stress element is in either tension or compression. Trusses can be classified as plane truss and space truss.

- Plane truss is one where the plane of the structure remain in plane even after the application of loads
- While space truss plane will not be in a same plane

Fig shows 2d truss structure and each node has two degrees of freedom. The only difference between bar element and truss element is that in bars both local and global coordinate systems are same where in truss these are different.



There are always assumptions associated with every finite element analysis. If all the assumptions below are all valid for a given situation, then truss element will yield an exact solution. Some of the assumptions are:

Truss element is only a prismatic member ie cross sectional area is uniform along its length
It should be a isotropic material
Constant load ie load is independent of time Homogenous material

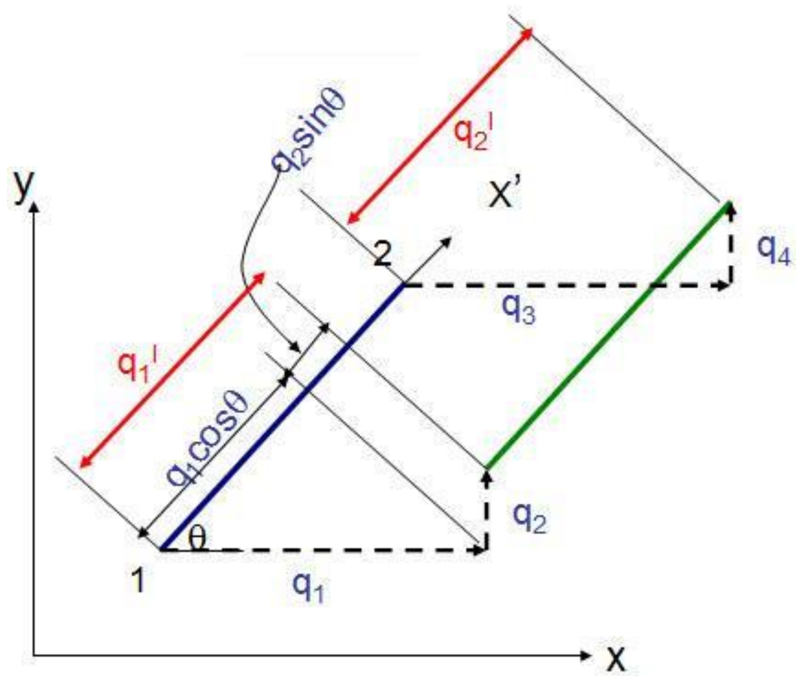
A load on a truss can only be applied at the joints (nodes)

Due to the load applied each bar of a truss is either induced with tensile/compressive forces

The joints in a truss are assumed to be frictionless pin joints

Self weight of the bars are neglected

Consider one truss element as shown that has nodes 1 and 2 .The coordinate system that passes along the element (x^l axis) is called local coordinate and X-Y system is called as global coordinate system. After the loads applied let the element takes new position say locally node 1 has displaced by an amount q_1^l and node2 has moved by an amount equal to q_2^l.As each node has 2 dof in global coordinate system .let node 1 has displacements q_1 and q_2 along x and y axis respectively similarly q_3 and q_4 at node 2.



Resolving the components q_1, q_2, q_3 and q_4 along the bar we get two equations as

$$q_1^l = q_1 \cos\theta + q_2 \sin\theta$$

$$q_2^l = q_3 \cos\theta + q_4 \sin\theta$$

Or

$$q_1^l = q_1 \ell + q_2 m$$

$$q_2^l = q_3 \ell + q_4 m$$

Writing the same equation into the matrix form

$$\begin{Bmatrix} q_1^l \\ q_2^l \end{Bmatrix} = \begin{pmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{pmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$q^l = L\, q$$

Where L is called transformation matrix that is used for local –global correspondence.

Strain energy for a bar element we have

$$U = \tfrac{1}{2}\, q^T K q$$

For a truss element we can write

$$U = \tfrac{1}{2}\, q^{lT} K\, q^l$$

Where $q^l = L\, q$ and $q^{lT} = L^T q^T$

Therefore

$$U = \tfrac{1}{2} q^{lT} K q^{l}$$

$$= \tfrac{1}{2} L^T q^T K L q$$

$$= \tfrac{1}{2} q^T (L^T K L) q$$

$$= \tfrac{1}{2} q^T K_T q$$

Where K_T is the stiffness matrix of truss element

$$K_T = L^T K L$$

$$L = \begin{pmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{pmatrix} \qquad L^T = \begin{pmatrix} \ell & 0 \\ m & 0 \\ 0 & \ell \\ 0 & m \end{pmatrix}$$

$$K = \frac{AE}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Taking the product of all these matrix we have stiffness matrix for truss element which is given as

$$K_T = \frac{AE}{L} \begin{pmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{pmatrix}$$

Stress component for truss element

The stress σ in a truss element is given by

$$\sigma = \varepsilon E$$

But strain $\varepsilon = B\,q^1$ and $q^1 = T\,q$

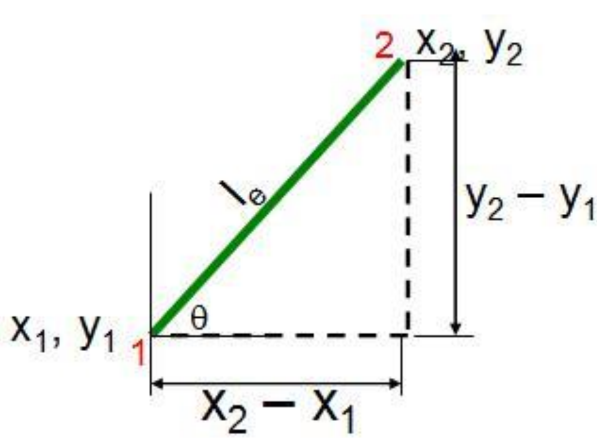
$$\text{where } B = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

Therefore

$$\sigma = \frac{E}{L_e}\begin{pmatrix} -\ell & -m & \ell & m \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

How to calculate direction cosines

Consider a element that has node 1 and node 2 inclined by an angle θ as shown .let (x1, y1) be the coordinate of node 1 and (x2,y2) be the coordinates at node 2.



When orientation of an element is know we use this angle to calculate and m as:

$\cos\theta$ m = cos (90 - θ sinθ

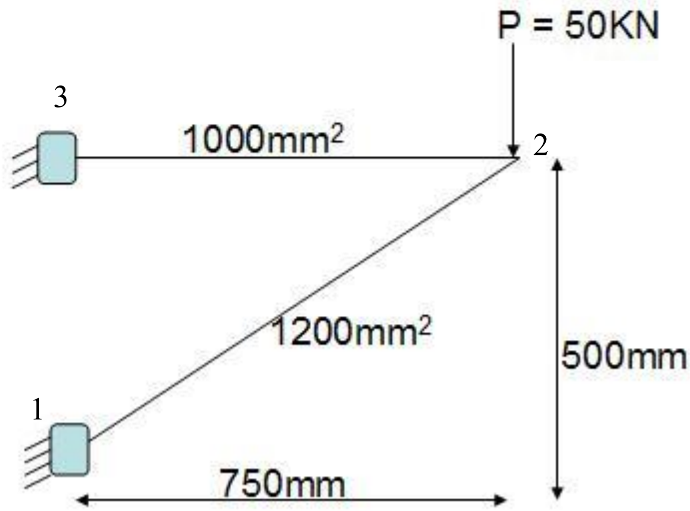
and by using nodal coordinates we can calculate using the relation

$$\ell = \frac{x_2 - x_1}{l_e} \quad m = \frac{y_2 - y_1}{l_e}$$

We can calculate length of the element as

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 6



Solution: For given structure if node numbering is not given we have to number them which depend on user. Each node has 2 dof say q1 q2 be the displacement at node 1, q3 & q4 be displacement at node 2, q5 &q6 at node 3.

Tabulate the following parameters as shown

Element	θ	L	$\ell = \cos\theta$	m=sin θ
1	33.6	901.3	0.832	0.554
2	0	750	1	0

For element 1 θ can be calculate by using $\tan\theta = 500/700$ ie $\theta = 33.6$, length of the element is

$$l_e = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$= 901.3$ mm

Similarly calculate all the parameters for element 2 and tabulate

Calculate stiffness matrix for both the elements

$$K_T = \frac{AE}{L}\begin{bmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{bmatrix}$$

$$K_1 = 10^5 \begin{bmatrix} 1.84 & 1.22 & -1.84 & -1.22 \\ 1.22 & 0.816 & -1.22 & -0.816 \\ -1.84 & -1.22 & 1.84 & 1.22 \\ -1.22 & -0.816 & 1.22 & 0.816 \end{bmatrix} \quad \text{(rows and columns numbered 1, 2, 3, 4)}$$

$$K_2 = 10^5 \begin{bmatrix} 2.66 & 0 & -2.66 & 0 \\ 0 & 0 & 0 & 0 \\ -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(rows and columns numbered 3, 4, 5, 6)}$$

Element 1 has displacements q1, q2, q3, q4. Hence numbering scheme for the first stiffness matrix (K1) as 1 2 3 4 similarly for K_2 3 4 5 & 6 as shown above.

Global stiffness matrix: the structure has 3 nodes at each node 3 dof hence size of global stiffness matrix will be 3 X 2 = 6

ie 6 X 6

$$K = 10^5 \begin{bmatrix} 1.84 & 1.22 & -1.84 & -1.22 & 0 & 0 \\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0 \\ -1.84 & -1.22 & 4.5 & 1.22 & -2.66 & 0 \\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(rows and columns numbered 1 to 6)}$$

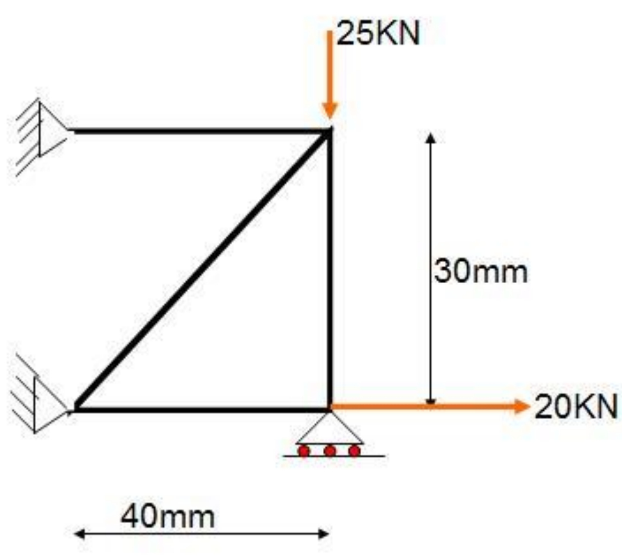
From the equation KQ = F we have the following matrix. Since node 1 is fixed q1=q2=0 and also at node 3 q5 = q6 = 0 .At node 2 q3 & q4 are free hence has displacements.

In the load vector applied force is at node 2 ie F4 = 50KN rest other forces zero.

$$10^5 \begin{bmatrix} 1.84 & 1.22 & -1.84 & -1.22 & 0 & 0 \\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0 \\ -1.84 & -1.22 & 4.5 & 1.22 & -2.66 & 0 \\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -50 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$$

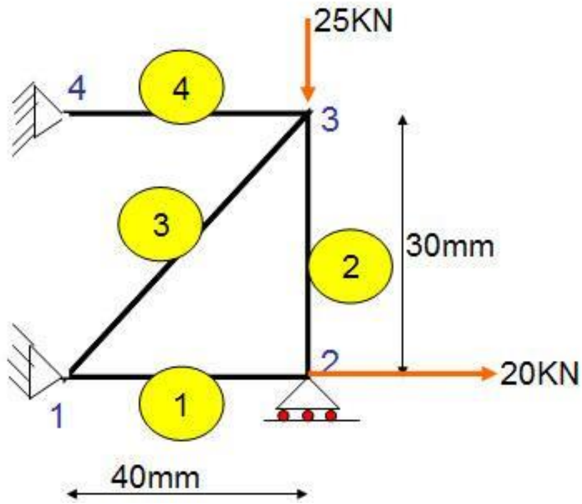
By elimination method the matrix reduces to 2 X 2 and solving we get Q3= 0.28mm and Q4 = -1.03mm. With these displacements we calculate stresses in each element.

Example 7



$E = 29.5 \times 10^6\ N/mm^2 \qquad A = 1mm^2$

Solution: Node numbering and element numbering is followed for the given structure if not specified, as shown below



Let Q1, Q2 …..Q8 be displacements from node 1 to node 4
and F1, F2……F8 be load vector from node 1 to node 4.

Tabulate the following parameters

Element	θ	L	$\ell = \cos\theta$	$m = \sin\theta$
1	0	40	1	0
2	90	30	0	1
3	36.8	50	0.8	0.6
4	0	40	1	0

Determine the stiffness matrix for all the elements

$$K_1 = 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$K_2 = 10^5 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 6.66 & 0 & -6.66 \\ 0 & 0 & 0 & 0 \\ 0 & -6.66 & 0 & 6.66 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$K_3 = 10^5 \begin{array}{c} \\ \end{array}\begin{array}{cccc|c} 5 & 6 & 1 & 2 & \\ \hline 2.56 & 1.92 & -2.56 & -1.92 & 5 \\ 1.92 & 1.44 & -1.92 & -1.44 & 6 \\ -2.56 & -1.92 & 2.56 & 1.92 & 1 \\ -1.92 & -1.44 & 1.92 & 1.44 & 2 \end{array}$$

$$K_4 = 10^5 \begin{array}{cccc|c} 5 & 6 & 7 & 8 & \\ \hline 5 & 0 & -5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \\ -5 & 0 & 5 & 0 & 7 \\ 0 & 0 & 0 & 0 & 8 \end{array}$$

Global stiffness matrix: the structure has 4 nodes at each node 3 dof hence size of global stiffness matrix will be 4 X 2 = 8

ie 8 X 8

$$\begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ \hline 7.56 & 1.92 & -5 & 0 & -2.56 & -1.92 & 0 & 0 & 1 \\ 1.92 & 1.44 & 0 & 0 & -1.92 & -1.44 & 0 & 0 & 2 \\ -5 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 6.66 & 0 & -6.66 & 0 & 0 & 4 \\ -2.56 & -1.92 & 0 & 0 & 7.56 & 1.92 & -5 & 0 & 5 \\ -1.92 & -1.44 & 0 & -6.66 & 1.92 & 8.11 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & -5 & 0 & 5 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{array}$$

From the equation KQ = F we have the following matrix. Since node 1 is fixed q1=q2=0 and also at node 4 q7 = q8 = 0 .At node 2 because of roller support q3=0 & q4 is free hence has displacements. q5 and q6 also have displacement as they are free to move.

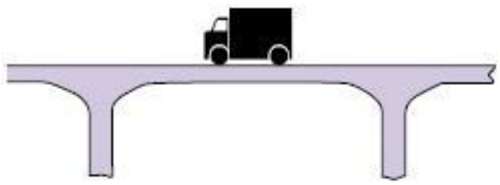
In the load vector applied force is at node 2 ie F3 = 20KN and at node 3 F6 = 25KN, rest other forces zero.

$$10^5 \begin{pmatrix} 7.56 & 1.92 & -5 & 0 & -2.56 & -1.92 & 0 & 0 \\ 1.92 & 1.44 & 0 & 0 & -1.92 & -1.44 & 0 & 0 \\ -5 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.66 & 0 & -6.66 & 0 & 0 \\ -2.56 & -1.92 & 0 & 0 & 7.56 & 1.92 & -5 & 0 \\ -1.92 & -1.44 & 0 & -6.66 & 1.92 & 8.11 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \\ Q7 \\ Q8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20 \times 10^3 \\ 0 \\ 0 \\ -25 \times 10^3 \\ 0 \\ 0 \end{pmatrix}$$

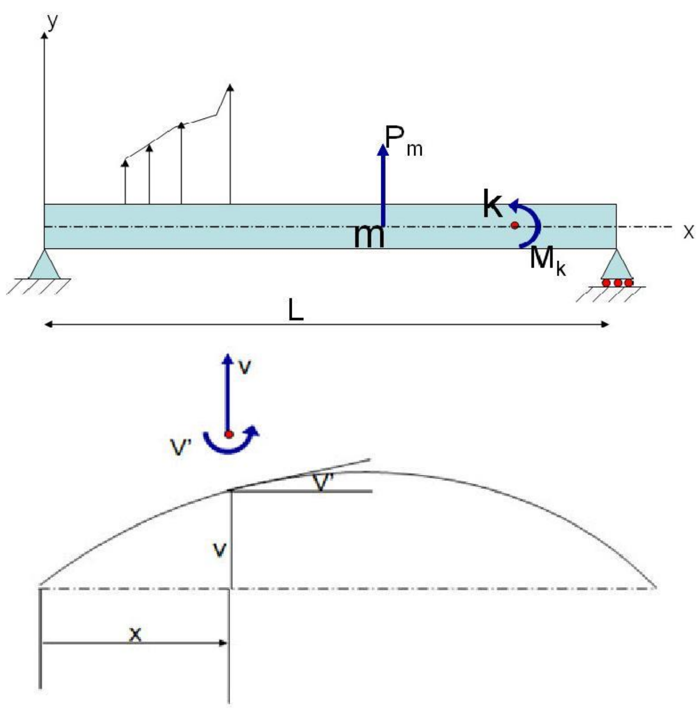
Solving the matrix gives the value of q3, q5 and q6.

Beam element

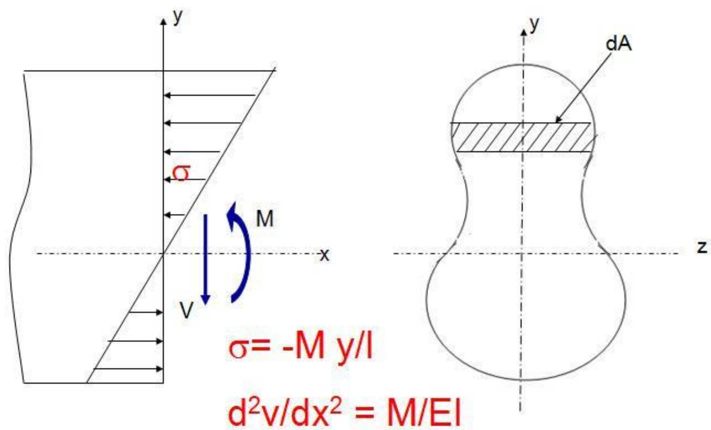
Beam is a structural member which is acted upon by a system of external loads perpendicular to axis which causes bending that is deformation of bar produced by perpendicular load as well as force couples acting in a plane. Beams are the most common type of structural component, particularly in Civil and Mechanical Engineering. A *beam* is a bar-like structural member whose primary function is to support *transverse loading* and carry it to the supports



A truss and a bar undergoes only axial deformation and it is assumed that the entire cross section undergoes the same displacement, but beam on other hand undergoes transverse deflection denoted by v. Fig shows a beam subjected to system of forces and the deformation of the neutral axis



We assume that cross section is doubly symmetric and bending take place in a plane of symmetry. From the strength of materials we observe the distribution of stress as shown.



Where M is bending moment and I is the moment of inertia.
According to the Euler Bernoulli theory. The entire c/s has the same transverse deflection V as the neutral axis, sections originally perpendicular to neutral axis remain plane even after bending

Deflections are small & we assume that rotation of each section is the same as the slope of the deflection curve at that point (dv/dx). Now we can call beam element as simple line segment representing the neutral axis of the beam. To ensure the continuity of deformation at any point, we have to ensure that V & dv/dx are continuous by taking 2 dof @ each node V & θ(dv/dx). If no slope dof then we have only transverse dof. A prescribed value of moment load can readily taken into account with the rotational dof θ .

Potential energy approach

Strain energy in an element for a length dx is given by

$$= \frac{1}{2} \int_A \sigma \, \varepsilon \, dA \, dx$$

$$= \frac{1}{2} \int_A \sigma \, \sigma/E \, dA \, dx$$

$$= \frac{1}{2} \int_A \sigma^2/E \, dA \, dx$$

But we know $\sigma = M\,y / I$ substituting this in above equation we get.

$$= \frac{1}{2} \int_A \frac{M^2}{EI^2}\, y^2 dA\, dx$$

$$= \frac{1}{2} \frac{M^2}{EI^2} \left[\int_A y^2 dA \right] dx$$

$$= \frac{1}{2} \frac{M^2}{EI}\, dx$$

But

$$M = EI\, d^2v/dx^2$$

Therefore strain energy for an element is given by

$$= \frac{1}{2} \int_0^L EI\, (d^2v/dx^2)^2\, dx$$

Now the potential energy for a beam element can be written as

$$\Pi = \frac{1}{2} \int_0^L EI \left(\frac{d^2v}{dx^2} \right)^2 dx - \int_0^L p\, v\, dx - \sum_m P_m V_m - \sum_k M_k V'_k$$

P ---- distribution load per unit length

P_m----- point load @ point m

V_m----- deflection @ point m

M_k----- momentum of couple applied at point k

V'_k----- slope @ point k

Hermite shape functions:

1D linear beam element has two end nodes and at each node 2 dof which are denoted as Q_{2i-1} and Q_{2i} at node i. Here Q_{2i-1} represents transverse deflection where as Q_{2i} is slope or rotation. Consider a beam element has node 1 and 2 having dof as shown.

The shape functions of beam element are called as Hermite shape functions as they contain both nodal value and nodal slope which is satisfied by taking polynomial of cubic order

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

that must satisfy the following conditions

ξ	H_1	H_1'	H_2	H_2'	H_3	H_3'	H_4	H_4'
$\xi = -1$	1	0	0	1	0	0	0	0
$\xi = 1$	0	0	0	0	1	0	0	1

Applying these conditions determine values of constants as

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

@ node 1

$$H_1 = 1,\ \ H_1' = 0\ ,\ \ \xi = -1$$

$$1 = a_1 - b_1 + c_1 - d_1 \longrightarrow (1)$$

$$H_1' = \frac{dH_1}{d\xi} = 0 = b_1 - 2c_1 + 3d_1 \longrightarrow (2)$$

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

@ node 2

$$H_1 = 1,\ \ H_1' = 0\ ,\ \ \xi = 1$$

$$0 = a_1 + b_1 + c_1 + d_1 \longrightarrow (3)$$

$$H_1' = \frac{dH_1}{d\xi} = 0 = b_1 + 2c_1 + 3d_1 \longrightarrow (4)$$

Solving above 4 equations we have the values of constants

$$a_1 = \tfrac{1}{2}\ ,\ b_1 = -\tfrac{3}{4}\ ,\ \ c_1 = 0,\ \ d_1 = \tfrac{1}{4}$$

Therefore

$$H_1 = \tfrac{1}{4}\,(2 - 3\xi + \xi^3)$$

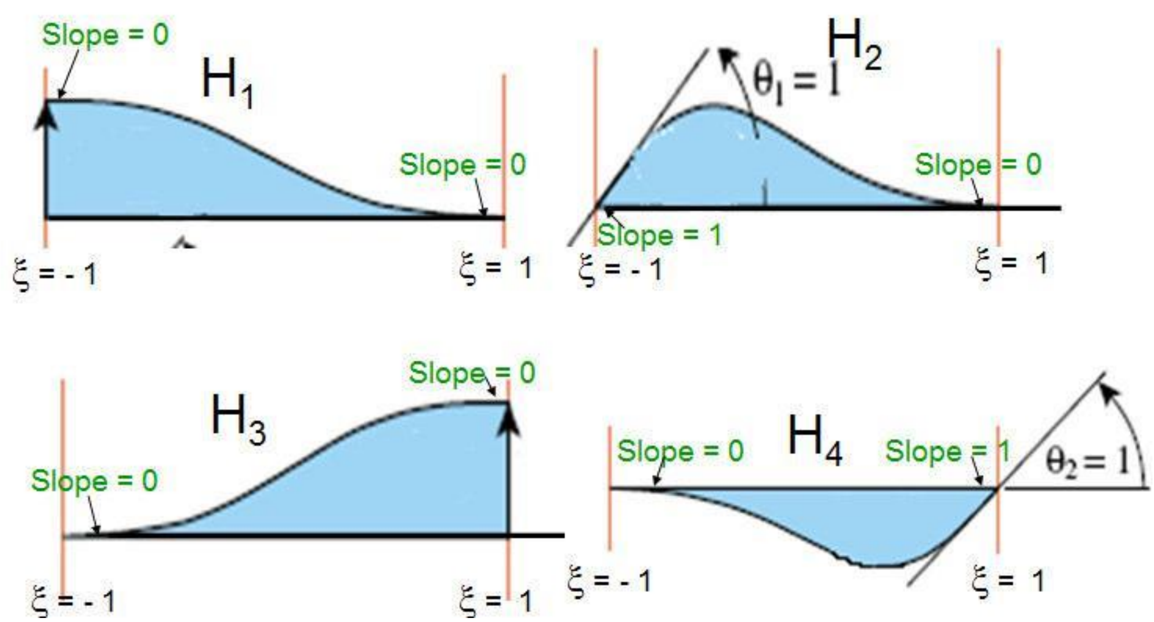
Similarly we can derive

$$H_2 = \tfrac{1}{4}\,(1 - \xi - \xi^2 + \xi^3)$$

$$H_3 = \tfrac{1}{4}\,(2 + 3\xi - \xi^3)$$

$$H_4 = \tfrac{1}{4}\,(-1 - \xi + \xi^2 + \xi^3)$$

Following graph shows the variations of Hermite shape functions



Stiffness matrix:

Once the shape functions are derived we can write the equation of the form

$$V(\xi) = H_1V_1 + H_2\left(\frac{dv}{d\xi}\right)_1 + H_3V_3 + H_4\left(\frac{dv}{d\xi}\right)_2$$

But

$$\frac{dv}{d\xi} = \frac{dv}{dx}\,\frac{dx}{d\xi}$$

$$= \frac{dv}{dx}\frac{Le}{2}$$

ie

$$V(\xi) = H_1V_1 + H_2\left(\frac{dv}{dx}\frac{Le}{2}\right) + H_3V_3 + H_4\left(\frac{dv}{dx}\frac{Le}{2}\right)$$

$$V(\xi) = H_1q_1 + H_2q_2\frac{L_e}{2} + H_3q_3 + H_4q_4\frac{L_e}{2}$$

We know $V = H\,q$

where

$$H = \left(H_1 \quad H_2\frac{L_e}{2} \quad H_3 \quad H_4\frac{L_e}{2}\right)$$

Strain energy in the beam element we have

$$= \tfrac{1}{2}\int_0^L EI\,(d^2v/dx^2)^2\,dx$$

$$\frac{d^2v}{dx^2} = \frac{d}{dx}\left(\frac{dv}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{2}{L_e}\frac{dv}{d\xi}\right)$$

$$= \frac{2}{L_e}\frac{d}{dx}\left(\frac{dv}{d\xi}\right)$$

$$= \frac{2}{L_e}\frac{d}{dx}(m) \qquad \text{Where } m = \frac{dv}{d\xi}$$

$$= \frac{2}{L_e}\left(\frac{2}{L_e}\frac{dm}{d\xi}\right)$$

$$\frac{d^2v}{dx^2} = \frac{4}{L_e^2}\left(\frac{d^2v}{d\xi^2}\right) \qquad \left(\frac{d^2v}{dx^2}\right)^2 = \frac{16}{L_e^4}\left(\frac{d^2v}{d\xi^2}\right)^2$$

$$V = H\, q$$

$$\left(\frac{d^2v}{dx^2}\right)^2 = q^T \frac{16}{L_e^4}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right) q$$

Where

$$\left(\frac{d^2H}{d\xi^2}\right) = \left[\frac{3\xi}{2}\ ,\ \left(\frac{-1+3\xi}{2}\right)\frac{l_e}{2}\ ,\ \frac{-3\xi}{2}\ ,\ \left(\frac{1+3\xi}{2}\right)\frac{l_e}{2}\right]$$

Therefore total strain energy in a beam is

$$= \tfrac{1}{2}\int_e EI\,(d^2v/dx^2)^2\,dx$$

$$= \tfrac{1}{2}\int_e EI\,(d^2v/dx^2)^2\,l_e/2\ d\xi$$

$$= \frac{EI}{2}\frac{L_e}{2}\int_e q^T \frac{16}{L_e^4}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right) q\ d\xi$$

$$= \tfrac{1}{2}\, q^T\left\{\frac{8EI}{L_e^3}\int_{-1}^{+1}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right)\right\} q\, d\xi$$

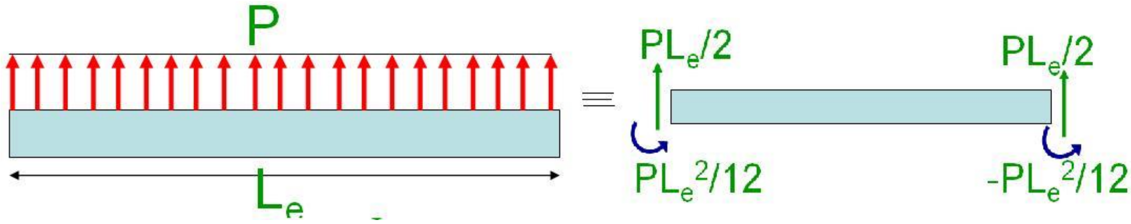
$$= \tfrac{1}{2}\, q^T K\, q$$

Now taking the K component and integrating for limits -1 to +1 we get

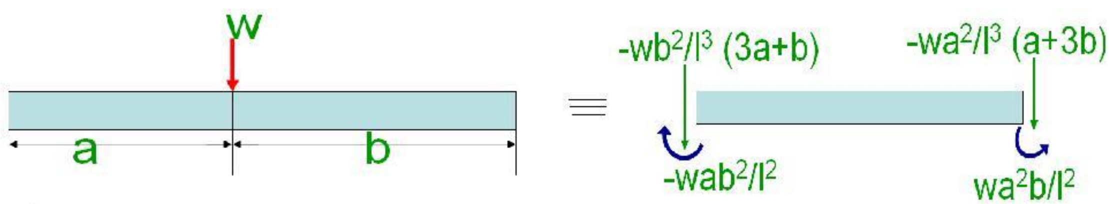
$$K = \frac{EI}{Le^3}\begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}$$

Beam element forces with its equivalent loads

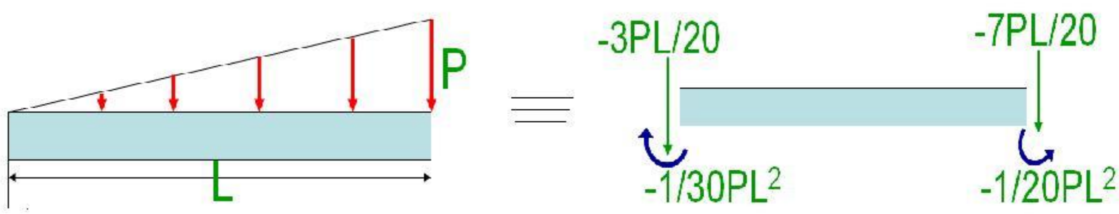
Uniformly distributed load



Point load on the element



Varying load



Bending moment and shear force

We know

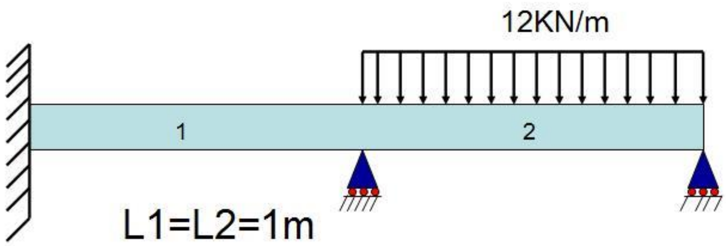
$$M=EI\left(\frac{d^2v}{dx^2}\right) \quad V=\left(\frac{dM}{dx}\right) \quad V = Hq$$

Using these relations we have

$$M=\frac{EI}{l_e^2}\left[6\xi q_1 +(3\xi -1)l_e q_2 -6\xi q_3+(3\xi +1)l_e q_4\right]$$

$$V =\frac{6EI}{l_e^3}\left[2q_1 +l_e q_2 -2q_3+l_e q_4\right]$$

Example 8



L1=L2=1m

E = 200GPa

I = 4 X 10^6N/mm^4

Solution:

Let's model the given system as 2 elements 3 nodes finite element model each node having 2 dof. For each element determine stiffness matrix.

$$K_1 = 8 \times 10^5 \begin{array}{c} \\ \left(\begin{array}{cccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 \\ 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 4 & -6 & 4 \end{array}\right) \end{array} \begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \\ \scriptstyle 4 \end{array} \quad K_2 = 8 \times 10^5 \left(\begin{array}{cccc} \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 \\ 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 4 & -6 & 4 \end{array}\right) \begin{array}{c} \\ \scriptstyle 3 \\ \scriptstyle 4 \\ \scriptstyle 5 \\ \scriptstyle 6 \end{array}$$

Global stiffness matrix

$$K = 8 \times 10^5 \left(\begin{array}{cccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 \\ 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{array}\right) \begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \\ \scriptstyle 4 \\ \scriptstyle 5 \\ \scriptstyle 6 \end{array}$$

Load vector because of UDL

Element 1 do not contain any UDL hence all the force term for element 1 will be zero.

ie

$$F_1 = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For element 2 that has UDL its equivalent load and moment are represented as

ie

$$F_2 = \begin{pmatrix} F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} -6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

Global load vector:

$$F = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

From KQ=F we write

$$8 \times 10^5 \begin{pmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

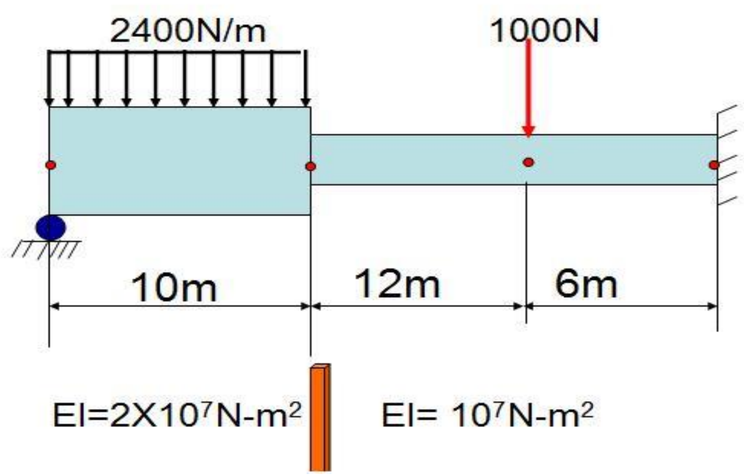
At node 1 since its fixed both q1=q2=0
node 2 because of roller q3=0
node 3 again roller ie q5= 0
By elimination method the matrix reduces to 2 X 2 solving this we have Q4= -2.679×10^{-4}mm and Q6 = 4.464×10^{-4}mm

To determine the deflection at the middle of element 2 we can write the displacement function as

$$V(\xi) = H_1 q_3 + H_2 q_4 \frac{L_e}{2} + H_3 q_5 + H_4 q_6 \frac{L_e}{2}$$

= -0.089mm

Example 9



Solution: Let's model the given system as 3 elements 4 nodes finite element model each node having 2 dof. For each element determine stiffness matrix. Q1, Q2……Q8 be nodal displacements for the entire system and F1……F8 be nodal forces.

$$K_1 = \frac{2X\,10^7}{10^3}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & 60 & -12 & 60 \\ 60 & 400 & -60 & 200 \\ -12 & -60 & 12 & -60 \\ 60 & 200 & -60 & 400 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$K_2 = \frac{10^7}{12^3}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 72 & -12 & 72 \\ 72 & 576 & -72 & 288 \\ -12 & -72 & 12 & -72 \\ 72 & 288 & -72 & 576 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

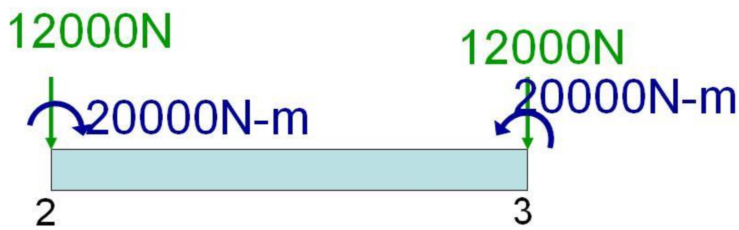
$$K_3 = \frac{10^7}{6^3}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 12 & 36 & -12 & 36 \\ 36 & 14 & -36 & 72 \\ -12 & -36 & 12 & -36 \\ 36 & 72 & -36 & 144 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Global stiffness matrix:

$$K = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} \\ \\ & & & 8\ X\ 8 & & & \\ \\ \\ \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Load vector because of UDL:

For element 1 that is subjected to UDL we have load vector as



ie

$$F_1 = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \end{pmatrix} = \begin{pmatrix} -12000 \\ -20000 \\ -12000 \\ 20000 \end{pmatrix}$$

Element 2 and 3 does not contain UDL hence

$$F_2 = \begin{pmatrix} F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad F_3 = \begin{pmatrix} F5 \\ F6 \\ F7 \\ F8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

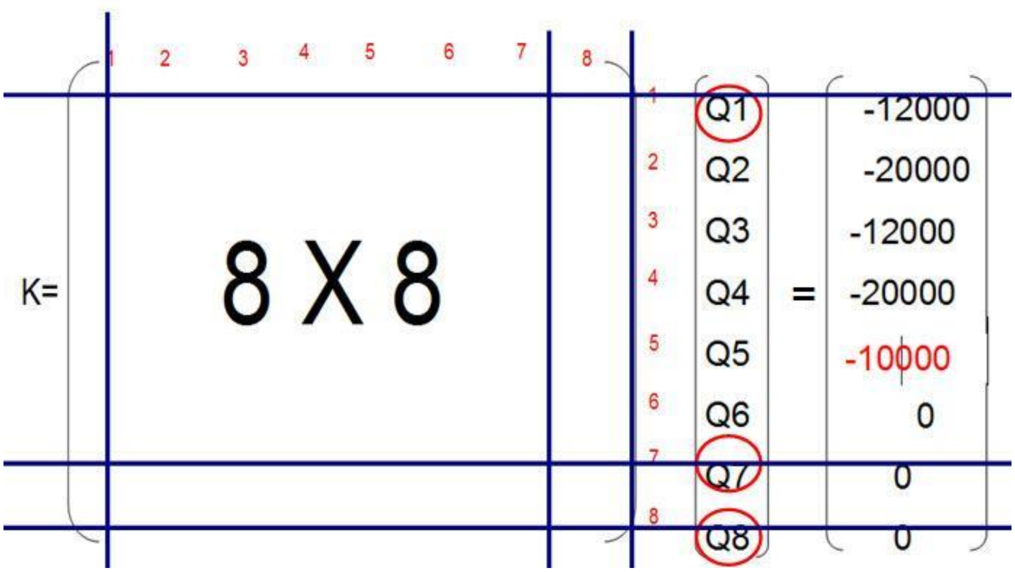
Global load vector:

$$F = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \\ F7 \\ F8 \end{pmatrix} = \begin{pmatrix} -12000 \\ -20000 \\ -12000 \\ -20000 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

And also we have external point load applied at node 3, it gets added to F5 term with negative sign since it is acting downwards. Now F becomes,

$$F=\begin{pmatrix}F1\\F2\\F3\\F4\\F5\\F6\\F7\\F8\end{pmatrix}=\begin{pmatrix}-12000\\-20000\\-12000\\-20000\\0\ \ -10000\\0\\0\\0\end{pmatrix}$$

From KQ=F



At node 1 because of roller support

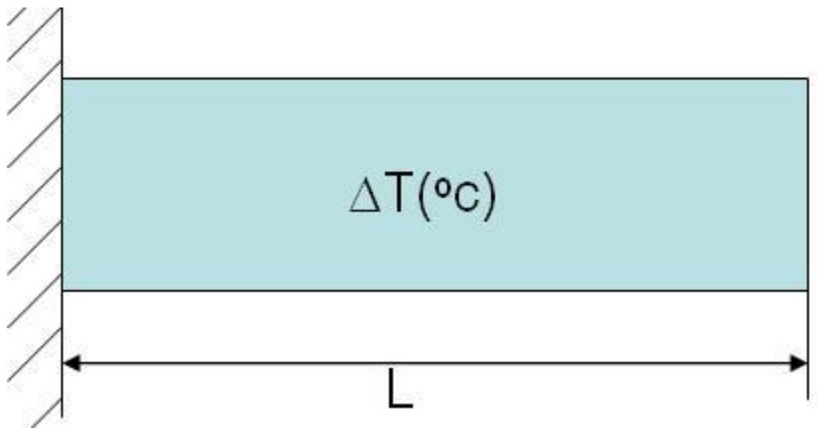
q1=0 Node 4 since fixed q7=q8=0

After applying elimination and solving the matrix we determine the values of q2, q3, q4, q5 and q6.

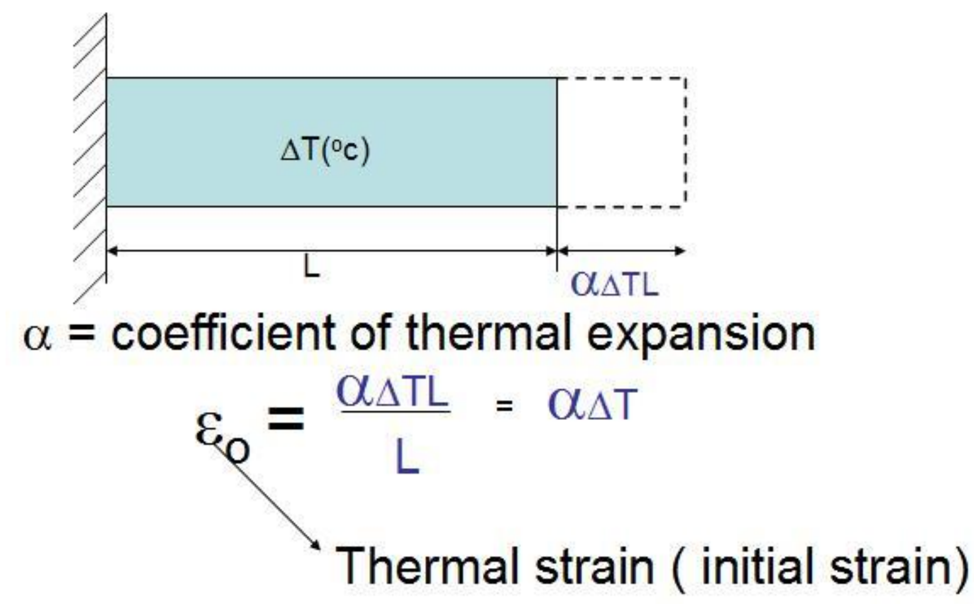
Module 4

Temperature effect on 1D bar element

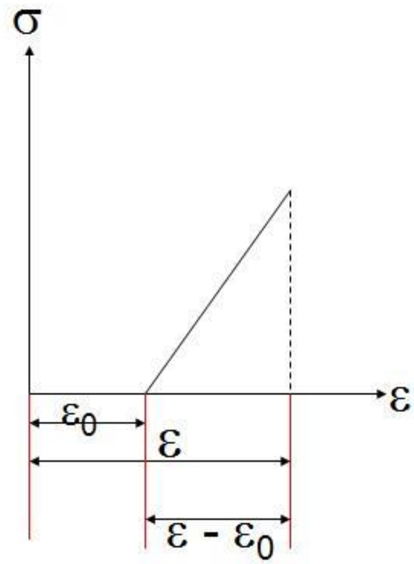
Lets us consider a bar of length L fixed at one end whose temperature is increased to ΔT as shown.



Because of this increase in temperature stress induced are called as thermal stress and the bar gets expands by a amount equal to αΔTL as shown. The resulting strain is called as thermal strain or initial strain



In the presence of this initial strain variation of stress strain graph is as shown below



Hooke's law

$$\frac{\text{Stress}}{\text{Strain}} = \frac{\sigma}{\varepsilon - \varepsilon_0} = E$$

$$\sigma = (\varepsilon - \varepsilon_0)\, E$$

We know that

Strain energy in a bar

$$U = \tfrac{1}{2} \int \sigma^T \varepsilon \, dv$$

For an element

$$U = \tfrac{1}{2} \int_e \sigma^T \varepsilon \, A \, dx$$

Therefore

$$U = \tfrac{1}{2} \int_e E \, (\varepsilon - \varepsilon_0)^T (\varepsilon - \varepsilon_0) \, A \, dx$$

$$U = \tfrac{1}{2} \int_e E \, (Bq - \varepsilon_0)^T (Bq - \varepsilon_0) \, A \, dx$$

$$U = \tfrac{1}{2} \int_e E\,(Bq - \varepsilon_0)^T (Bq - \varepsilon_0)\,A\,dx$$

But $dx/d\xi = L_e/2$

$$U = \tfrac{1}{2} EA \int_e (Bq - \varepsilon_0)^T (Bq - \varepsilon_0)\,Le/2\,d\xi$$

$$U = \tfrac{1}{2} EA/2 \int_e (Bq - \varepsilon_0)^T (Bq - \varepsilon_0)\,Le\,d\xi$$

$$U = \tfrac{1}{2} EA/2 \int_e (B^T q^T - \varepsilon_0)(Bq - \varepsilon_0)\,Le\,d\xi$$

$$U = \tfrac{1}{2} Le\,EA/2 \int_e [B^T q^T B q - B^T q^T \varepsilon_0 - Bq\varepsilon_0 + \varepsilon_0^2]\,d\xi$$

$$U = \tfrac{1}{2} Le\,EA/2 \int_e [B^T q^T B q - \varepsilon_0 (B^T q^T + Bq) + \varepsilon_0^2]\,d\xi$$

Therefore

$$U = \tfrac{1}{2} Le\,EA/2 \int_e [B^T q^T B q - \varepsilon_0\, 2B^T q^T + \varepsilon_0^2]\,d\xi$$

Integrating individual terms

$$U = \frac{1}{2} q^T EA \frac{Le}{2} \int_e [B^T B\, d\xi]\, q$$

Stiffness matrix

$$- \frac{1}{2} q^T EA \frac{Le}{2}\, \varepsilon_0 \int_e 2B^T\, d\xi$$

Thermal load vector

$$+ \frac{1}{2} EA \frac{Le}{2} \int_e \varepsilon_0^2\, d\xi$$

0

Extremizing the potential energy first term yields stiffness matrix, second term results in thermal load vector and last term eliminates that do not contain displacement filed

Thermal load vector

From the above expression taking the thermal load vector lets derive what is the effect of thermal load.

$$\theta_e = \frac{1}{2} \, EA \frac{Le}{2} \varepsilon_0 \int_e 2B^T \, d\xi$$

$$= \frac{1}{} \, EA \frac{Le}{2} \varepsilon_0 \int_e B^T \, d\xi$$

We know that $B^T = \frac{1}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\theta_e = \frac{EA}{2} \varepsilon_0 \int_{-1}^{1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} d\xi$$

$$= \frac{EA}{2} \varepsilon_0 \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$= EA \, \varepsilon_0 . \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\theta = EA\alpha\Delta T \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Stress component because of thermal load

$$\sigma = (\varepsilon - \varepsilon_0)\, E$$

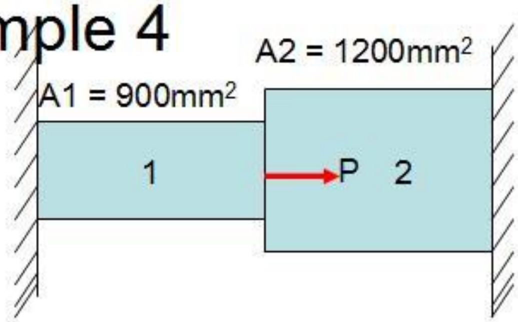
We know $\varepsilon = Bq$ and $\varepsilon_o = \alpha\Delta T$ substituting these in above equation we get

$$= (Bq - \alpha\Delta T)\, E$$

$$= E\, Bq - E\, \alpha\Delta T$$

$$\sigma = E\, \frac{1}{L} [-1 \quad 1] q - E\, \alpha\Delta T$$

Example 4



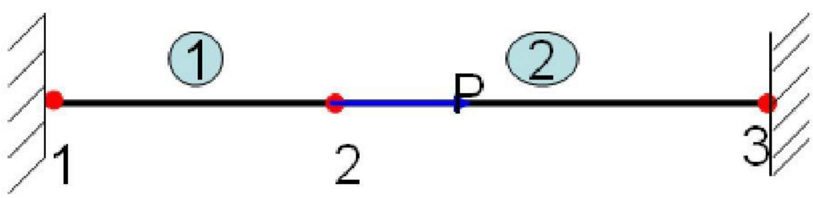
$\alpha_1 = 23 \times 10^{-6}$ Per ^{0}C $\qquad \alpha_2 = 11.7 \times 10^{-6}$ Per ^{0}C

$E_1 = 70 \times 10^9 \ N/m^2 \qquad E_2 = 200 \times 10^9 \ N/m^2$

$L_1 = 200mm \qquad L_2 = 300mm$

P = 300 KN is applied at 20^0c ,the temperature is then raised to 60^0c

Solution:



$$K_1 = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{900x70x\,10^3}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3 \begin{bmatrix} 315 & -315 \\ -315 & 315 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix}$$

(column labels: 1, 2)

$$K_2 = \frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3 \begin{bmatrix} 800 & -800 \\ -800 & 800 \end{bmatrix}\begin{matrix} 2 \\ 3 \end{matrix}$$

(column labels: 2, 3)

Global stiffness matrix:

$$K = \begin{pmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{pmatrix} 10^3$$

(columns 1, 2, 3; rows 1, 2, 3)

Thermal load vector:

We have the expression of thermal load vector given by

$$\theta = EA\alpha\Delta T \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Element 1

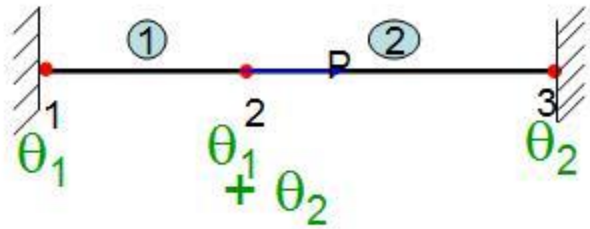
$$\theta_1 = 70 \times 10^3 \times 900 \times 23 \times 10^{-6} \times 40 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$\theta_1 = 10^3 \begin{pmatrix} -57.96 \\ 57.96 \end{pmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

Similarly calculate thermal load distribution for second element

$$\theta_2 = 10^3 \begin{pmatrix} -112.32 \\ 112.32 \end{pmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

Global load vector:



$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ P + \theta_2 + \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -57.96 \\ 245.64 \\ 112.32 \end{pmatrix} 10^3$$

From the equation KQ=F we have

$$\begin{pmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{pmatrix} 10^3 \begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} = \begin{pmatrix} -57.96 \\ 245.64 \\ 112.32 \end{pmatrix} 10^3 \quad -(-315 \times 10^3)Q1 - (-8 \times 10^5)Q3 \quad -(0)Q1$$

After applying elimination method and solving the matrix we have Q2= 0.22mm

Stress in each element:

For element 1

$$\sigma_1 = E_1 \frac{1}{L_1} [-1 \quad 1] \begin{pmatrix} Q1 \\ Q2 \end{pmatrix} - E_1 \alpha_1 \Delta T$$

$= 12.60\text{MPa}$

For element 2

$$\sigma_2 = E_2 \frac{1}{L_2} [-1 \quad 1] \begin{pmatrix} Q2 \\ Q3 \end{pmatrix} - E_2 \alpha_2 \Delta T$$

$= -240.27\text{MPa}$

Subject: Finite Element Methods

Sub. Code: 15ME61

Topic: Axisymmetric Elements

Presented by: S A Goudadi

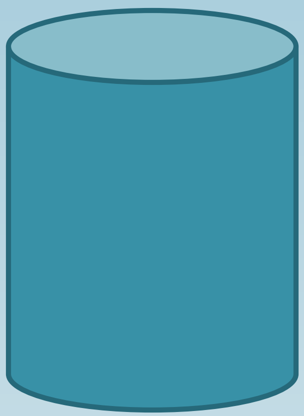
Class: VI A

Introduction

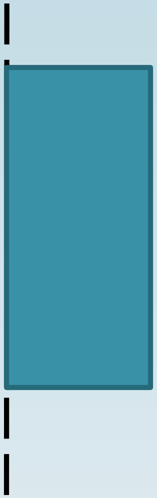
- Axisymmetric elements are 2-D elements that can be used to model axisymmetric geometries with axisymmetric loads
- These convert a 3-D problem to a 2-D problem
  - Smaller models
  - Faster execution
  - Easier postprocessing
- We only model the cross section, and ANSYS accounts for the fact that it is really a 3-D, axisymmetric structure (no need to change coord. Systems)

Modeling

- To model this:



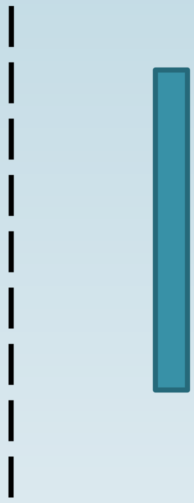
- We just need this:



Modeling

- To model this:

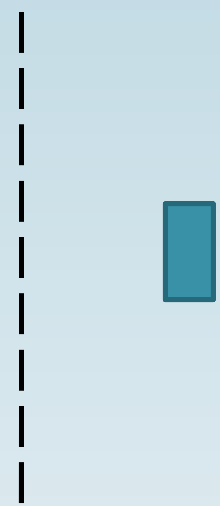
- We just need this:



Modeling

- To model this:

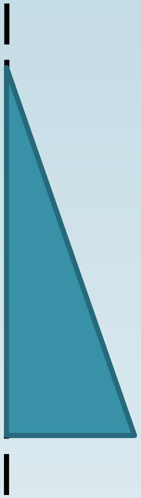
- We just need this:



Modeling

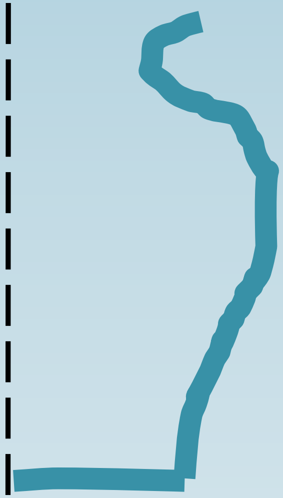
- To model this:

- We just need this:

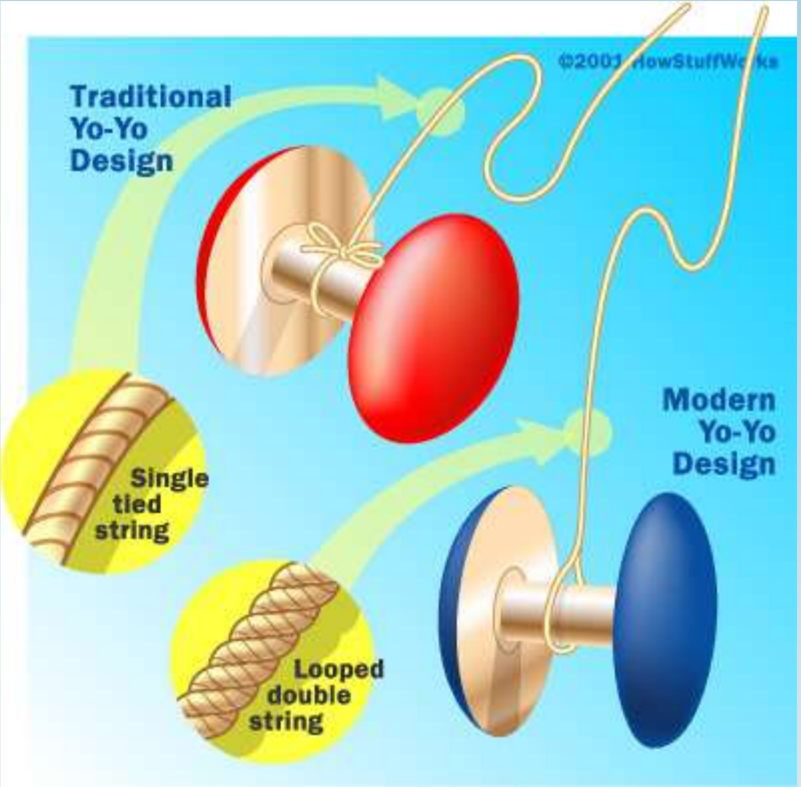


How would you model this?

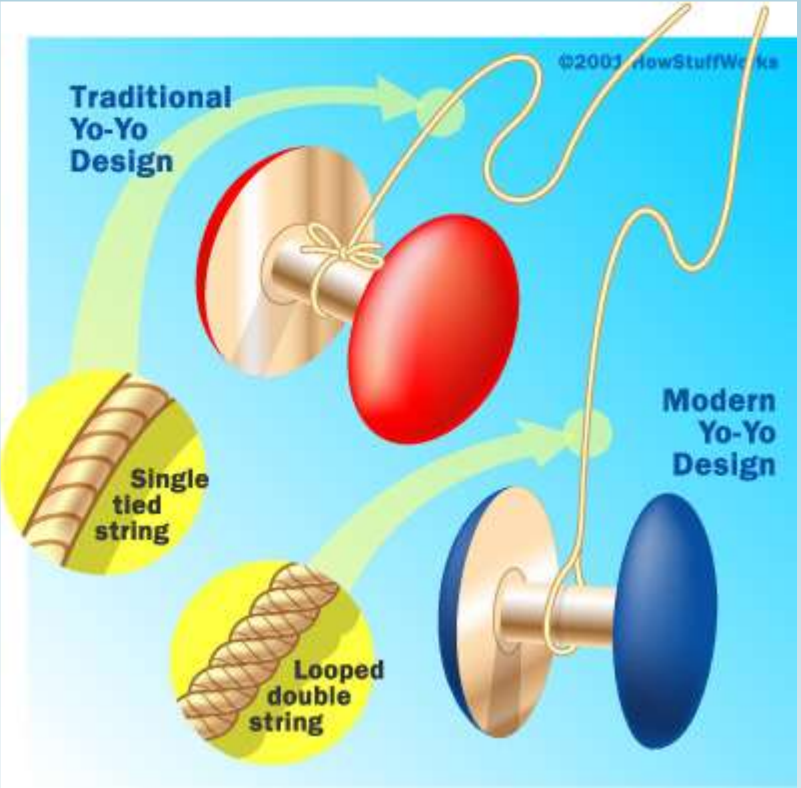
How would you model this?

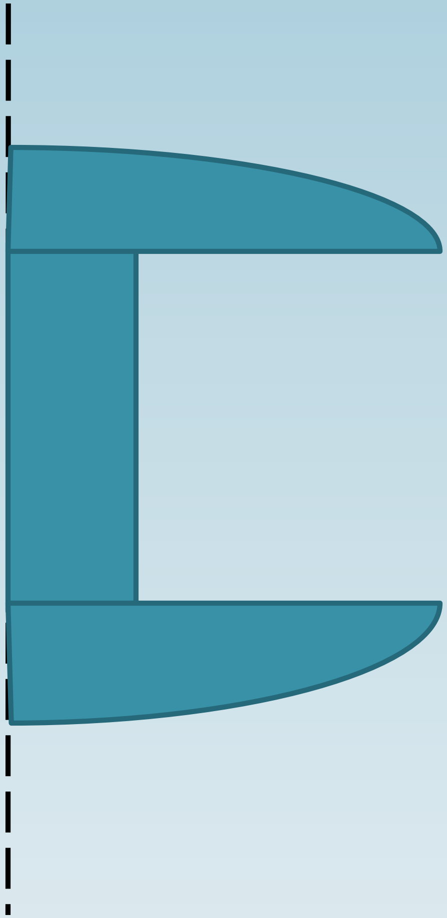


How would you model this?



How would you model this?





Note

- In ANSYS, axisymmetric models must be drawn in the x-y plane.
- The x-direction is the radial direction.
- The 2-D model will be rotated about the y-axis (and always about x=0)
- Nothing in your model should be in the region x<0
- In postprocessing, σ_x will be the radial stress, σ_y will be the axial stress, and σ_z will be the “hoop” stress

Subject: Finite Element Methods
Sub. Code: 15ME61
Topic: Fluid flow through porous media
Presented by: S A Goudadi
Class : VI A

GeneralStepstobefollowedwhilesolvingaproblemonFluidflowthroughporousmediabyusingFEM:

1.Discretize and select the element type

2.Choose a potential function

3.Define the gradient / potential and velocity / gradient relationship

4.Derive the element stiffness matrix and equations

5.Assemble the element equations to obtain the global equations and introduce boundary conditions

6.Solve for the nodal potential

7.Solve for the element velocities and volumetric flow rates

Points to be remembered:

- 1. This is similar to one dimensional heat conduction problem
- 2. The temperature function T is to be replaced by fluid velocity potential Φ
- 3.The nodal temperature vector should be replaced by vector of nodal potential denoted by
- 4. Fluid velocity v replaces heat flux q and permeability coefficient K for flow through porous medium replaces the conduction coefficient K
- 5. If fluid flow through a pipe or around a solid body is considered, then K is taken as unity

- Navier-Stokes equations:

$$\frac{\partial v}{\partial t} + v\nabla v = -\frac{1}{\rho}\nabla p + \nu\Delta v + f$$

- Darcy law:

$$q = -\frac{K}{\mu}\nabla p$$

- K is the matrix of permeability: porous media characteristic

- Poiseuille flow in a tube:
  single-phase, horizontal flow steady and laminar   no entrance and exit effects

$$v = -\frac{\pi R^2}{8\mu}\frac{\Delta p}{L}$$

v mean velocity
R radius
L length
Δp pressure gradient

- Defining the porosity as ϕ=ΔVf/ΔVϕ=ΔVf/ΔV, where Δ*V* is the volume of the representative elementary volume and ΔVfΔVf is the volume of fluid in the representative elementary volume, the flux of fluid per unit area (the "Darcy velocity") is given by the volume av

General Description of the finite element method

Application of the finite element method to a structural problem demands the subdivision of the structure into a number of discrete elements. Each of these elements must satisfy three conditions

1. Equilibrium of forces;
2. Compatibility of strains; and The force displacement relationship specified by the geometric and elastic properties of the discrete element

An equivalent set of conditions for a pipe network exist; hence, the ability to draw the analogy:

- The algebraic sum of the flows at any joint or node must be zero.
- The value of the piezometric head at a joint or node is the same for all pipes connected to that joint; and
- The flow-head relationship {such as Darcy-Weisbach or Hazen-Williams} must be satisfied for each element or pipe.

For Direct application of the finite element method involving a matrix solution, a linear relationship is required to define the element or pipe.

Hence there is a relationship of the form:

q = c h (1)

In which q = flow; h = head loss and c = the hydraulic properties of the pipe (to be assumed).

The solution technique can be subdivided into three steps:

1. An initial value of the pipe coefficient. c, is selected for each pipe and is then combined to yield the system matrix coefficient {C}. The system matrix is then solved for the value of piezometric head at each joint.
2. The individual pipe flows, q, are computed by means of Eq. (1) using the difference between the determined piezometric heads. These flows are then substituted in the Darcy-Weisbach equation to calculate the pipe head losses. If the pipe head losses obtained from Q the Darcy-Weisbach equation correspond to those obtained from the matrix solution, then the unique solution, satisfying both the Darcy-Weisbach equation and the linear equation (1) has been found.
3. If there is a difference between the values of head loss calculated by the two methods, the values of c are changed to cause the problem to converge to a solution.