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(1)

(Module 1)
GATE: Section 7

Information Theory and Channel Capacity

Part of GATE Syllabus (ECE) Section 7, Communications

Meaning of Information: Message ~~of~~ or Intelligence.

Source → Which produces message or intelligence

Information source → Encoder → Transmitter $\xrightarrow{\text{Channel}}$ Decoder → Receiver

↑ Noise

Information Source : 1) Analog information sources : microphone, TV camera o/p etc

2) Digital (Discrete) information sources : Fax o/p (Encoded), teletype etc.

→ Discrete information sources are characterized by: a) source alphabet, b) symbol rate c) source alphabet - probabilities and d) probabilistic dependance of symbols in a sequence.

Symbol rate : Rate at which teletype produces characters

e.g., 100 chars/sec. ⇒ symbol rate : 10 symbols/sec.

Source alphabets & their probabilities:

$$S = \{s_1, s_2, s_3, \dots s_q\} \;\&\; P = \{p_1, p_2, p_3 \cdots p_q\}$$

i.e., $s_1 \to p_1$; $s_2 \to p_2$; $s_3 \to p_3$ so on..

Then $p_1 + p_2 + p_3 + \cdots p_q = 1$

or $\sum_{i=1}^{q} p_i = 1$.

C.E. Shannon → Father of Information Theory → presented a paper on Mathematical Theory of Communication → Theoretical bounds for the performances of Communication Systems.

Measure of Information:

Let $S = \{s_1, s_2, s_3, \dots s_q\}$ with probabilities $P = \{p_1, p_2, \dots, p_q\}$

Then Self information or Amount of information

$$I_K = \log \frac{1}{P_K}$$

Note: log base is '2'. i.e., (Bits) (common)

Also, when base is '10' units → Hartleys or Decits

and, when base is 'e' units → NATs

or In general for the base of 'r' units are called r-ary units.

Example: On a particular day during winter season in

1. Sun will rise in east on the day of trip $P=1$ (certain) ; $I=0$ no surprise!

2. It will be a very cold day $P \neq 0$; $I \neq 0$

3. There will be snowfall on that particular day in $P=0$; $I \to$ Max Full of surprise!
uncertain

$\Rightarrow$ The only equation which satisfies #1 to #3 is:

$$\Rightarrow I = \log \frac{1}{P}$$

example 1: The binary symbols 0 & 1 are transmitted with probabilities 1/4 & 3/4 respectively. Find 'I' in each case:

$\rightarrow$ Symbol '0' with $P_0 = 1/4$

$$I_0 = \log_2 \frac{1}{P_0} = \log_2 \frac{1}{1/4} = 2 \qquad \left\{ \log_2 2 = 1 \ \& \ \log m^n = n \log m \right.$$

III^{ly}

$$I_1 = \log \frac{1}{P_1} = \log_2 \left(\frac{1}{3/4}\right) = 0.415 \text{ bits} \Rightarrow \text{less inf}^n \text{ for high 'P'}.$$

Why 'log': Inf^n can't be -ve.
$I=0$ for sure event (lower bound) $(P=1)$
$I>0$ for $P \neq 0$
$I \to$ max for least probable events $(P=0)$.

Property: When independent symbols are transmitted, the total self information must be equal to the sum of individual self informations.

proof: Let us consider two independent symbols S_K and S_L are transmitted with probabilities P_K and P_L.

$$\rightarrow I_{KL} = \log \frac{1}{P(S_K \text{ and } S_L)} = \log \frac{1}{(S_K \cap S_L)}$$

$$= \log \frac{1}{P(S_K) \cdot P(S_L)} = \log \frac{1}{P_K P_L} = \log \frac{1}{P_K} + \log \frac{1}{P_L} = I_K + I_L$$

$$\Rightarrow I_{KL} = I_K + I_L.$$

$\rightarrow$ Total self information is equal to the sum of individual self informations.

Zero-Memory Source: Statistically independent symbols i.e., no connection between the symbols i.e., source has no memory. ⇒ Zero memory or Memoryless.

Average Information (H) OR Entropy: (GATE Syllabus)

$\triangleq$ Average self information = Itotal/L

Where $I_{tot} = L \sum_{i=1}^{q} P_i \log \frac{1}{P_i}$ bits

⇒ Average self information is also called "ENTROPY".

∴ Entropy of Source 'S' is $H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i}$

Other names of H(S): Average uncertainty or Average amount of surprise".

(1) Example: Find H(S) for $S = \{s_1, s_2\}$ with $P = \{\frac{1}{256}, \frac{255}{256}\}$

$\rightarrow \quad H(S) = \frac{1}{256} \log_2 \frac{1}{1/256} + \frac{255}{256} \log_2 \frac{1}{255/256}$

$= 0.037$ bits/symbol

Observation: Avg. uncertainty is very small as occurance of s_1 & s_2 can be predicted easily (prob. difference)

(2) Example: Repeat (1) for $S = \{s_3, s_4\}$ with $P = \{1/2, 1/2\}$

$\rightarrow \quad H(S) = \frac{1}{2} \log_2 \frac{1}{1/2} + \frac{1}{2} \log_2 \frac{1}{1/2} \Rightarrow \frac{2}{2} \log_2 2 = \frac{2}{2} = 1$ bits/symbol

Observation: Uncertainty (avg.) is maximum : Impossible to predict.

Information Rate: Let r_s be the rate at which source emits the symbols Then Average source information rate 'R_s' in bit/sec is the product of H(S) & r_s: i.e., Average information content per symbol & message symbol rate

∴ $R_s = r_s \cdot H(S)$ bits/sec

Example: $S = \{s_1, s_2, s_3\}$ with $P = \{1/2, 1/4, 1/4\}$

Find a) Self Information

b) Entropy of Source

$$P_1 I_1 + P_2 I_2 + P_3 I_3$$

Solution: $H(S) = P_1 \log\frac{1}{P_1} + P_2 \log\frac{1}{P_2} + P_3 \log\frac{1}{P_3} = \sum_{i=1}^{3} P_i \log\frac{1}{P_i}$

$$= \frac{1}{2}\log_2\frac{1}{(1/2)} + \frac{1}{4}\log_2\frac{1}{(1/4)} + \frac{1}{4}\log_2\frac{1}{(1/4)}$$

$$= \frac{1}{2} + 1/2 + 1/2 = 1.5 \text{ bits/symbol}$$

Relationship among the parameters: Bits, Nats, and Hartleys

→ Difference in 'log' base

i.e., $I = \log_{10}\frac{1}{P}$ Hartleys base: 10

$= \log_e \frac{1}{P}$ Nats base: e

$= \log_2 \frac{1}{P}$ bits base: 2

a) Relationship between Hartley & Nats

$I = \log_{10}\frac{1}{P}$ Hartleys

$\therefore$ 1 Hartley $= \dfrac{I}{\log_{10}\frac{1}{P}} = \dfrac{\log_e \frac{1}{P}}{\log_{10}\frac{1}{P}} = \dfrac{-\log_e P}{-\log_{10} P}$

$\rightarrow \dfrac{\frac{1}{\log_P e}}{\frac{1}{\log_P 10}} = \dfrac{\log_P 10}{\log_P e}$ Nats $\left\{ \because \log_a b = \dfrac{1}{\log_b a} \right.$

$\log_n m = \dfrac{\log_e m}{\log_e n}$

$= \dfrac{\frac{\log_e 10}{\cancel{\log_e P}}}{\frac{\log_e e}{\cancel{\log_e P}}} = \log_e 10 = 2.303$ Nats $\log_e e = \log_2 2 = 1.$

$\therefore$ 1 Hartley = 2.303 Nats

Similarly: 1 Hartley = $\log_2 10$ bits ~~Nats~~ = 2.303 Nats

⇒ 1 Hartley = 3.32 bits = 2.303 Nats

And, 1 nat = $\log_2 e$ bits

$\therefore$ 1 nat $= \dfrac{1}{\log_e 2} = 1.443$ bits.

Example: A fair coin is tossed repeatedly. Let

A = event of getting 3 heads out of 5 trials

B = event of getting 5 heads out of 8 trials

Which event conveys more information? Support answer by numerical computation of respective amounts of information.

Soln: Let X be the R.V. = no of heads

= Binomial R.V. (since tossing a coin results in only 2 outcomes)

$n \to$ no. of trials $\qquad p = q = 0.5$ equiprobable & mutually exclusive

And, $p \to$ prob. of getting H,

$1-p \to$ " " " T

$${}^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

Then $\triangleq$ $P(X=x) = {}^{n}C_x\, p^{x} \cdot q^{n-x}$

e.g. $${}^{4}C_3 = \frac{4!}{3!\,(4-3)!}$$

Also $0! = 1! = 1$.

$\Rightarrow P(A) = P(X=3) = {}^{5}C_3\, p^{3} \cdot q^{5-3} = 0.3125$

$|||^{ly}$ $P(B) = P(X=5) = {}^{8}C_5\, p^{5} \cdot q^{8-3} = 0.21875$

$\to I_A = \log_2 \frac{1}{P(A)} = \log_2 \frac{1}{0.3125} = 1.678$ bits

$|||^{ly}$ $I_B = \log_2 \frac{1}{P(B)} = \log_2 \frac{1}{0.21875} = 2.193$ bits

$\therefore I_B > I_A$ event B conveys more information!

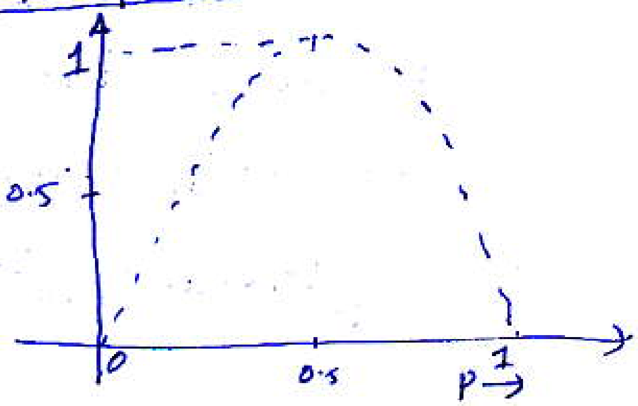
Example: A binary source is emitting an independent sequence of 0s and 1s with probabilities p and $1-p$ respectively. Plot the entropy of the source versus p.

$\to$ Entropy $H(S) = \sum_{i=1}^{2} p_i \log \frac{1}{p_i} = p \log_2 \frac{1}{p} + (1-p) \cdot \log_2 \frac{1}{1-p}$

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
H(S)	0	0.469	0.722	0.881	0.971	1	0.971	0.881	0.722	0.469	0

$\to$ $H(S)$ is max when $p = 0.5$
i.e., max uncertainty

$\to$ $H(S)$ is symmetric



Properties of Entropy:

For a source alphabet $S = \{s_1, s_2 \cdots s_q\}$ with $P = \{P_1, P_2, \cdots P_q\}$
where q = number of source symbols, then

$$H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} = \sum_{i=1}^{q} P_i I_i \quad \text{bits/symbol}$$ Entropy or Avg. Infⁿ (self)

Following properties can be noted:

a) The entropy function is continuous for every independent variable P_K in the interval $(0,1)$ i.e., $P_K \updownarrow ; H(S) \updownarrow$

b) The entropy function is a symmetrical function of its arguments.
i.e., $H[P_K, (1-P_K)] = H[(1-P_K), P_K] \ \forall K ; K = 1, 2, 3, \cdots q$
the value of $H(S)$ remains same irrespective of location of probabilities
e.g., $P_A = \{P_1\ P_2, P_3\}$, $P_B = \{P_2, P_3, P_1\}$, and $P_C = \{P_3, P_1, P_2\}$
Then $H(S_A) = H(S_B) = H(S_C)$; S_A, S_B, S_C are sources.

c) Extermal property: Let $S = \{s_1, s_2, s_3, \cdots s_q\}$ with $P = \{P_1, P_2, \cdots P_q\}$
The entropy of S is given by

$$H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} \quad \text{and} \quad \sum_{i=1}^{q} P_i = 1$$

Then upper & Lower bound of $H(S)$ are

$\Rightarrow$ It is obvious that lower bound for $H(S) = 0$.

Upper bound: $\log q - H(S) = \left(\sum_{i=1}^{q} P_i\right) \log q - \sum_{i=1}^{q} P_i \log \frac{1}{P_i}$ —(1) $\because \sum_{i=1}^{q} P_i = 1$.

$$\sum_{i=1}^{q} P_i \left[\log q - \log \frac{1}{P_i}\right]$$

$\log\left(\frac{m}{n}\right) = \log m - \log n$

$$= \sum_{i=1}^{q} P_i \log q P_i$$

$$= \sum_{i=1}^{q} P_i \frac{\log_e q P_i}{\log_e 2}$$

$\log_2^m = \frac{\log_e m}{\log_e 2}$

$$\therefore \log q - H(S) = \log_2 e \sum_{i=1}^{q} P_i \ln q P_i \quad -(2)$$

Using log inequality $y = x - 1$ & $y = \ln x \Rightarrow \ln x \le x - 1$
equality only at $x = 1$

Let $x = \frac{1}{qP_i} \rightarrow \ln qP_i \geq 1 - \frac{1}{qP_i}$ or $\ln \frac{1}{x} \geq 1 - x$

$$\Rightarrow \sum_{i=1}^{q} P_i \ln q P_i \geq \sum_{i=1}^{q} P_i \left(1 - \frac{1}{qP_i}\right) \quad -(3)$$

$\because$ Multiplying both sides by $\sum_{i=1}^{q} P_i$

Multiplying (3) by $\log_2 e$ bothsides we get-

$$\log_2 e \sum_{i=1}^{q} p_i \ln q p_i \geq \log_2 e \left[\sum_{i=1}^{q} p_i - \sum_{i=1}^{q} \frac{1}{q} \right] \quad -(4)$$

$\because x = \frac{1}{q p_i}$

RHS is always zero & LHS is $\log q - H(S)$ equality only when $x=1$

i.e., $\log q - H(S) \geq 0$ $\therefore p_i = \frac{1}{q}$

or $H(S) \leq \log_2 q$

The equality sign holds good when $p_i - \frac{1}{q} = 0 \ \forall i$

i.e., $p_i = \frac{1}{q} \ \forall q$; $i = 1, 2, 3, \cdots q$.

$\therefore H(S)_{max} = \log_2 q$ bits/symbol. $q \rightarrow$ number of symbols.

$\therefore$ Entropy attains max when all Symbols are equiprobable.

Example: $S = \{s_1, s_2, s_3, s_4\}$ with $P = \{1/4, 1/4, 1/4, 1/4\}$ $p_i = \frac{1}{q}$ for $\forall i$

$$\Rightarrow H(S) = H(S)_{max} = \log_2 4 = 2 \text{ bits/symbol}$$

Alternatively, $4 \times \frac{1}{4} \log_2 \frac{1}{(1/4)} = 2$ bits/symbol.

(d) Property of Additivity: Suppose splitting the symbols S_q into $S_{q1}, S_{q2} \cdots S_{qn}$ with probabilities $P_{q1}, P_{q2}, \cdots P_{qn}$ such that-

$$P_q = P_{q1} + P_{q2} + P_{q3} + \cdots P_{qn} = \sum_{j=1}^{n} P_{qj} \quad -(1)$$

Then, the splitted symbol entropy is

$$H'(S) = H(p_1, p_2, p_3, \cdots p_{q-1}, P_{q1}, P_{q2} \cdots P_{qn}) \text{ ~~such that~~}$$

$$= \sum_{i=1}^{q-1} p_i \log \frac{1}{p_i} + \sum_{j=1}^{n} P_{qj} \log \frac{1}{P_{qj}}$$

$$= \sum_{i=1}^{q} p_i \log \frac{1}{p_i} - P_q \log \frac{1}{P_q} + \sum_{j=1}^{n} P_{qj} \log \frac{1}{P_{qj}} \quad -(2)$$

$$= \sum_{i=1}^{q} p_i \log \frac{1}{p_i} - \sum_{j=1}^{n} P_{qj} \log \frac{1}{P_q} + \sum_{j=1}^{n} P_{qj} \log \frac{1}{P_{qj}}$$

$$= H(S) - \sum_{j=1}^{n} P_{qj} \left(\log \frac{1}{P_{qj}} - \log \frac{1}{P_q} \right)$$

After arranging & simplifying

$\therefore H'(S) = H(S) +$ a positive quantity $\because P_{qj} \leq P_q \ \forall j$

$$\therefore H'(S) \geq H(S).$$

" partitioning the symbols into subsymbols will not reduce entropy

(e) Source Efficiency: Ratio of entropy to max entropy and is given by

$$\eta_s = \frac{H(S)}{H(S)_{max}}$$ and the source reduandancy denoted by

'$R_{\eta s}$' is given by $R_{\eta s} = 1 - \eta_s$

Note: η_s and $R_{\eta s}$ can also be expressed in %.

Example: Verify the rule of additivity.

$S = \{ S_1\ S_2\ S_3\ S_4 \}$ with $P = \{ 1/2, 1/3, 1/12, 1/12 \} = (P_1, P_2, P_3, P_4)$ say

→ $H(S) = 1.625$ bits/symbol

Using property of additivity

$$H'(S) = P_1 \log \frac{1}{P_1} + (1 - P_1) \log \frac{1}{1-P_1} + 1 - P_1 \left\{ \frac{P_2}{1-P_1} \log \frac{1-P_1}{P_2} + \frac{P_3}{1-P_1} \log \frac{1-P_1}{P_3} + \frac{P_4}{1-P_1} \log \frac{1-P_1}{P_4} \right\}$$

$$\therefore H'(S) = \frac{1}{2} \log_2 2 + \frac{1}{2} \log_2 2 + \frac{1}{2} \left\{ \frac{\frac{1}{3}}{\frac{1}{2}} \log \left(\frac{\frac{1}{2}}{1/3} \right) + \frac{\frac{1}{12}}{\frac{1}{2}} \log \frac{\frac{1}{2}}{\frac{1}{12}} + \frac{\frac{1}{12}}{1/2} \log \frac{\frac{1}{2}}{1/12} \right\}$$

$$H'(S) = 1.625 = H(S).$$

Example: A black and white TV picture consists of 525 lines of picture information. Assume that each line consists of 525 picture elements & that each element can have 256 brightness levels. Picture are repeated at the rate of 30 frames/sec. Calculate the average rate of information conveyed by a TV set to a viewer

Solution: No of lines: 525; Picture elements/line = 525

$\therefore$ Total no. of elements = 525×525

Also, each element can have 256 brightness levels.

Then each frame can have brightness levels of $256^{525 \times 525}$

$\therefore$ Max ~~Avg~~ Infn $= H(S)_{max} = \log_2 256^{525 \times 525} = 22.05 \times 10^5$ bits/frame.

$\therefore$ Avg. Infn rate $R_s = r_s \cdot H(S)_{max} = r_s\, I_{max}(A_s)$

$$= 30 \times 22.05 \times 10^5$$

$$R_s = 66.15 \times 10^6 \text{ bits/sec}$$

5 Suppose that S_1 and S_2 are two zero-memory sources with probabilities $P_1, P_2, \cdots P_n$ for source S_1 and $q_1, q_2, \cdots q_n$ for source S_2. Show that the entropy of source S_1

$$H(S_1) \leq \sum_{K=1}^{n} P_K \log \frac{1}{q_K}$$

Solution: Given S_1 & S_2 are Zero Memory Sources.

$$\therefore \quad H(S_1) = \sum_{K=1}^{n} P_K \log \frac{1}{P_K} \quad \text{—(1)}$$

$$\& \quad \sum_{K=1}^{n} P_K = 1 \quad (\text{By defn of probability})$$

IIIly $\quad H(S_2) = \sum_{K=1}^{n} q_K \log \frac{1}{q_K} \quad \& \quad \sum_{K=1}^{n} q_K = 1. \quad \text{—(2)}$

Imp step

Consider: $H(S_1) - \sum_{K=1}^{n} P_K \log \frac{1}{q_K} = \sum_{K=1}^{n} P_K \log \frac{1}{P_K} - \sum_{K=1}^{n} P_K \log \frac{1}{q_K}$

$$= \sum_{K=1}^{n} P_K \left(\log \frac{1}{P_K} - \log \frac{1}{q_K} \right) = \sum_{K=1}^{n} P_K \cdot \log_2 \left(\frac{q_K}{P_K} \right)$$ {log rule

$$= \sum_{K=1}^{n} P_K \left[\frac{\log_e (q_K/P_K)}{\log_e^2} \right] = \log_2^e \sum_{K=1}^{n} P_K \ln \left(\frac{q_K}{P_K} \right)$$ $\left\{ \log_2^e = \frac{\log_e^e}{\log_e^2} = \frac{1}{\log_e^2} \right.$

W.K.T $\ln 1/x \geq 1 - x$ {log inequality

$-\ln x \geq 1 - x$

Removing -ve sign i.e., $\ln x \leq x - 1$

let $x = \frac{q_K}{P_K} \therefore \ln \left(\frac{q_K}{P_K} \right) \leq \frac{q_K}{P_K} - 1$

Multiply by P_K taking summation for $\forall K$ $(K = 1, 2 \ldots n)$ & then multiply-ing by $\log_2^e$ on both sides we get

$$\log_2^e \sum_{K=1}^{n} P_K \ln \left(\frac{q_K}{P_K} \right) \leq \log_2^e \sum_{K=1}^{n} P_K \left(\frac{q_K}{P_K} - 1 \right)$$

$$= H(S_1) - \sum_{K=1}^{n} P_K \ln \left(\frac{q_K}{P_K} \right) \leq \underbrace{\log_2^e \sum_{K=1}^{n} [q_K - P_K]}_{} = 0$$

$$\therefore H(S_1) \leq \sum_{K=1}^{n} P_K \log \frac{1}{q_K}.$$

Example: A discrete message source "S" emits two independent symbols X (S1) and Y (S2) with probabilities 0.55 and 0.45 respectively. Calculate the efficiency of the source and its redundancy.

$$P_x = P(X) = 0.55, \quad P_y = P(Y) = 0.45 \qquad (x, y) = (s_1, s_2)$$

$$\therefore \text{Entropy } H(S) = \sum_{i=x}^{y} P_i \log \frac{1}{P_i}$$

$$\therefore H(S) = P_x \log_2 \frac{1}{P_x} + P_y \log_2 \frac{1}{P_y}$$

$$= 0.55 \log_2 \frac{1}{0.55} + 0.45 \log_2 \frac{1}{0.45}$$

$$= 0.9928 \text{ bits/symbol}$$

$$\eta_s = \frac{H(S)}{H(S)_{max}} = \frac{0.9928}{1} = 99.28\%$$

And, the source redundancy is given by

$$R_{\eta s} = 1 - \eta_s = 1 - 0.9928 = 0.0072$$

$$\therefore R_{\eta s} = 0.72\%$$

Example: In a facsimile transmission of picture, there are about 2.25×10^6 pixels/frame. For a good reproduction 12 brightness levels are necessary. Assume all these levels are equally likely to occur. Find the rate of information if one picture is to be transmitted every 3 minutes. What is the source efficiency of this fax transmitter.

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Solution: Total number of pixels in one frame $= 2.25 \times 10^6$

Number of brightness levels $= 12$

$\therefore$ Total no. of different frames possible $= 12^{2.25 \times 10^6}$

Since all levels are equally likely to occur, the net max info content per frame is

$$I = H(S)_{max} = \log_2 q = \log_2 (12)^{2.25 \times 10^6}$$

$$= 2.25 \times 10^6 \log_2 12 \qquad \therefore I = 8.066 \times 10^6 \text{ bits/frame} = H(S)_{max}.$$

Given that one picture is transmitted in 3 minutes

Therefore, the rate of transmission is given by

$$r_s = \frac{1}{3 \text{ minutes}} = \frac{1}{3 \times 60} \text{ pictures/sec}$$

⑥ ∴ The average rate of information is given by

$$R_s = r_s I = \frac{1}{3\times60} \times 8.066\times10^6$$

$$R_s = 44812 \text{ bits/sec}$$

Since the information transmitted is maximum (as all levels are equiprobable) the source efficiency

$$\eta_s = \frac{H(S)}{H(S)_{max}} = \frac{H(S)_{max}}{H(S)_{max}} = 1 = 100\%$$

<u>EXTENSION OF ZERO-MEMORY SOURCE</u>: Let us consider a binary source S emitting symbols s_1 & s_2 with probabilities P_1 & P_2 such that $P_1 + P_2 = 1$

Then the 2nd extension of this binary source ~~are~~ is

s_1s_1, s_1s_2, s_2s_1, and s_2s_2 & their probabilities are

P_1P_1, P_1P_2, P_2P_1, and P_2P_2 $\Rightarrow$ $P_1 \cdot P_1 = P_1^2$ & $P_2P_2 = P_2^2$

$\Rightarrow$ Sum of all probabilities $P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = P_1^2 + 2P_1P_2 + P_2^2 = 1$

$$(P_1+P_2)^2 = P_1^2 + P_1P_2 + P_2^2 = (1)^2 = 1^2 = 1$$

$\Rightarrow$ Entropy of 2nd extension of source $S = \{s_1, s_2\}$ with $P = (P_1, P_2)$

∴ $H(S)$ = Entropy of the basic binary source

$$H(S) = \sum_{i=1}^{2} P_i \log\frac{1}{P_i} = P_1\log\frac{1}{P_1} + P_2\log\frac{1}{P_2}$$

The entropy of 2nd extension is given by

$$H(S^2) = \sum_{i=1}^{4} P_j \log\frac{1}{P_j}$$

$P_1P_2 = P_2P_1$

$$= P_1^2 \cdot \log\frac{1}{P_1^2} + P_1P_2\log\frac{1}{P_1P_2} + P_2P_1\log\frac{1}{P_2P_1} + P_2^2\log\frac{1}{P_2^2}$$

$$= 2P_1^2\log\frac{1}{P_1} + 2P_1P_2\log\frac{1}{P_1P_2} + 2P_2^2\log\frac{1}{P_2}$$

$$= 2P_1^2\log\frac{1}{P_1} + 2P_1P_2\log\frac{1}{P_1} + 2P_1P_2\log\frac{1}{P_2} + 2P_2^2\log\frac{1}{P_2}$$

$$= 2P_1(P_1+P_2)\log\frac{1}{P_1} + 2P_2(P_1+P_2)\log\frac{1}{P_2}$$

$\underline{\underline{P_1 + P_2 = 1}}$

$$2\left(P_1\log\frac{1}{P_1} + P_2\log\frac{1}{P_2}\right) = 2H(S)$$

111ly $\boxed{H(S)^n = n \cdot H(S)}$

Example: A ZMS has a source alphabet $S=\{s_1, s_2, s_3\}$ with $P=\{1/2$ ...
Find the entropy of this source. Also determine the entropy of its 2^{nd} extension and verify that $H(S^2)=2H(S)$

Soln: For the basic source $H(S)=\frac{1}{2}\log_2\frac{1}{1/2}+2\times\frac{1}{4}\log\frac{1}{1/4} = 1.5$ bits.

2^{nd} Extension of a source: $s_1s_1,\ s_1s_2,\ s_1s_3,\ s_2s_1,\ s_2s_2,\ s_2s_3,\ s_3s_1,\ s_3s_2$ and s_3s_3.

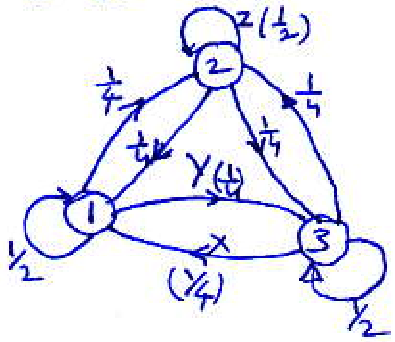
$\frac{1}{2}\times\frac{1}{2}\quad \frac{1}{2}\times\frac{1}{4},\ \frac{1}{2}\times\frac{1}{4},\ \frac{1}{4}\times\frac{1}{2},\ \frac{1}{4}\times\frac{1}{4},\ \frac{1}{4}\times\frac{1}{4},\ \frac{1}{4}\times\frac{1}{2},\ \frac{1}{4}\times\frac{1}{4}$, $1/4\times1/4$

$$\therefore H(S^2) = \frac{1}{4}\log_2 4 + 4\times\frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 = 2\times1.5 \text{ bits/message} = 2\times H(S)$$

ENTROPY OF SOURCE WITH MEMORY (MARKOV Sources):

In real life sources, there ~~are~~ is intersymbol ~~interference~~ influence present such that the occurance of x_i in zeroth position s_0 of the message depends on the previous q symbols $\{s_1, s_2, \dots s_q\}$. Such a source is known as q^{th} order Markoff source or Markov source. These sources are specified by a set of conditional probabilities $P(x_i/s_1, s_2, \dots s_q)$ where x_i is the symbol in the s_0 position and each 's' has an m-symbol alphabet $\{x_1, x_2, \dots x_m\}$. Since $P(x_i)$ now depends on the earlier 'q' symbols. The transitional probabilities may be shown in a state diagram with m^q possible states symbols.

Example: For the first order Markov source shown in figure, draw the tree diagram representing the states at the end of second symbol interval & find the corresponding probabilities. Assume $P(1)=P(2)=P(3)=1/3$.



Soln: From the tree diagram, the symbol XX can be generated by either one of the following transitions $1\to1\to1$ or $2\to1\to1$ or $3\to1\to1$. Thus the probability of the source emitting 2 symbol seq. XX is given by

$$P(XX) = P[(1\to1\to1) \text{ or } (2\to1\to1) \text{ or } (3\to1\to1)]$$

Since all transition paths are disjoint

$$P(XX) = P(1\to1\to1) + P(2\to1\to1) + P(3\to1\to1)$$
$$= \frac{1}{3}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{6}$$

iii)

$$P(XZ) = \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{12}$$

$$P(XY) = \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{12}$$

$$P(ZX) = \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{12}$$

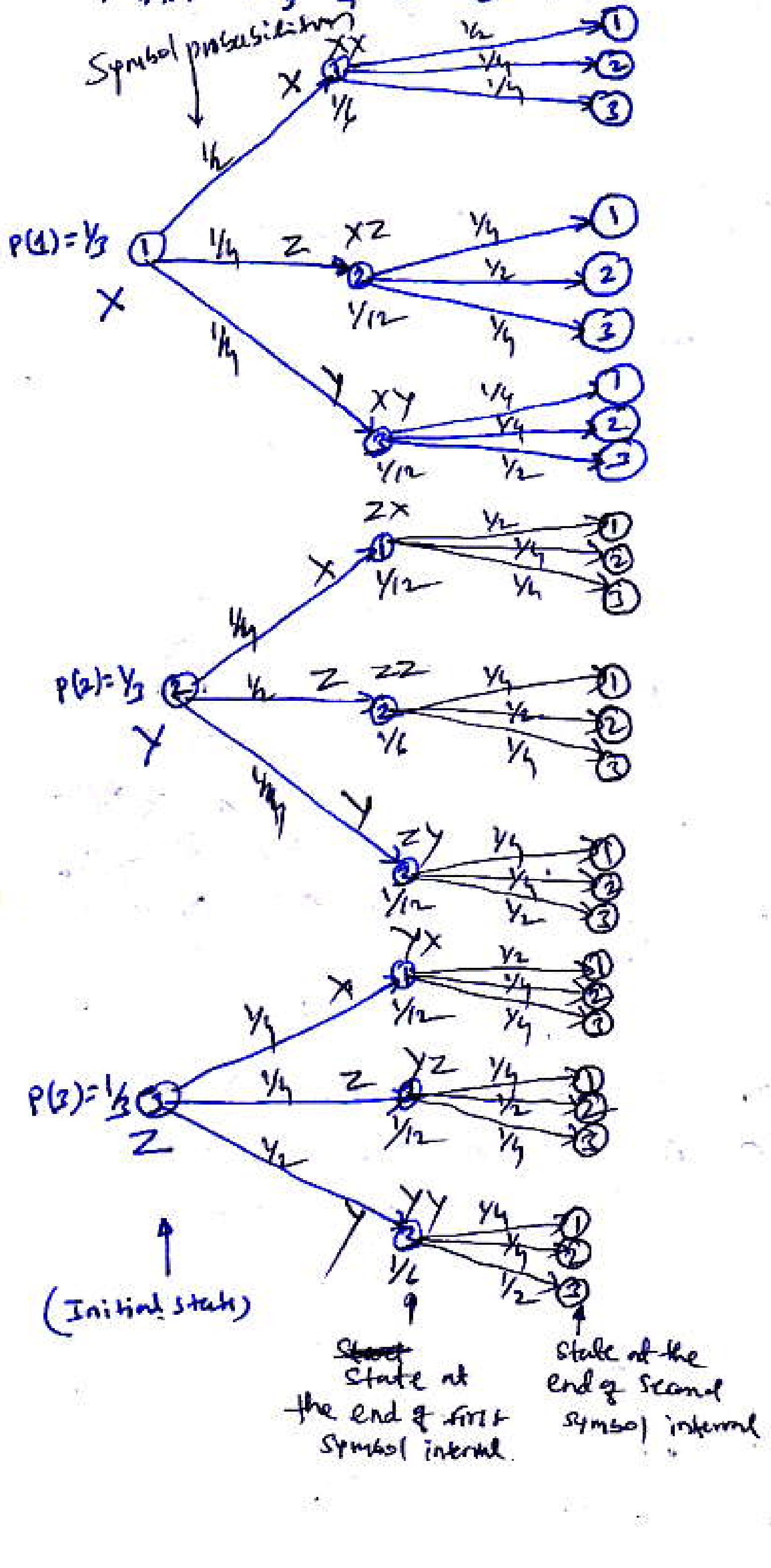
$$P(ZZ) = \frac{1}{3}\times\frac{1}{4}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{2} = \frac{1}{6}$$

$$P(ZY) = \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{12}$$

$$P(YX) = \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4} = \frac{1}{12}$$

$$P(YZ) = \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4} = \frac{1}{12}$$

$$P(YY) = \frac{1}{3}\times\frac{1}{4}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{4}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{6}$$



Entropy and Information rate of Markoff Sources:

The entropy of the source is defined as the weighted average of the entropy of the symbols emitted from each state. Where the entropy of state i, denoted by H_i is defined as the average infn content of the symbols emitted from the i^{th} state.

$$\therefore \boxed{H_i = \sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}}} \text{ bits/message symbol} \quad -(1)$$

The entropy of the source is then the average of the entropy of each state i.e.,

$$H = \sum_{i=1}^{n} P_i H_i$$

$$= \sum_{i=1}^{n} P_i \left[P_{ij} \log \frac{1}{P_{ij}} \right] \text{ bits/message symbol.}$$

Where P_i is the probability that the source is in state i.

The average information rate R_s for the source is defined as

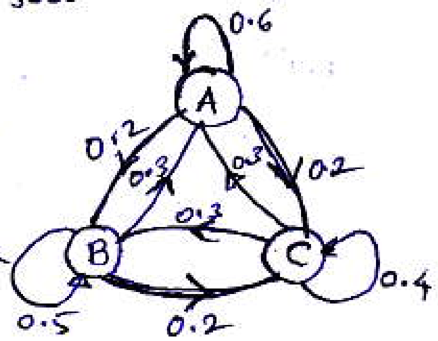
$$R_s = r_s \cdot H \text{ bits/sec}$$

Example: For the first order Markov source shown:

i) Find the stationary Distribution

ii) Find the entropy of each state and hence the entropy of the source

iii) Find the entropy of the adjoint source and verify that $H(S) < H(\bar{S})$



Solution: From the state diagram, the state equations are given by

State A: $P(A) = 0.6P(A) + 0.3P(B) + 0.3P(C) \rightarrow (1)$

$P(B) = 0.2P(A) + 0.5P(B) + 0.3P(C) \rightarrow (2)$

$P(C) = 0.2P(A) + 0.2P(B) + 0.4P(C) \rightarrow (3)$

$(2)-(3) \Rightarrow \quad P(B) - P(C) = 0.3P(B) - 0.1P(C).$

$\Rightarrow \quad 0.7P(B) = 0.9P(C)$ or $P(B) = \frac{9}{7}P(C). \quad -(4)$

$(1)-(2) = P(A) - P(B) = 0.4P(A) - 0.2P(B)$

$0.6P(A) = 0.8P(B)$

$\Rightarrow P(A) = \frac{4}{3}P(B) \quad -(5)$ or $P(B) = \frac{3}{4}P(A). \; -(5)$

Equating (4) & (5) $\frac{9}{7} P(C) = \frac{3}{4} P(A)$

$$\text{or } P(C) = \frac{7 \times \cancel{3}}{4 \times \cancel{9}_3} P(A)$$

$$= \frac{7}{12} P(A). \quad -(6) \quad \text{or} \quad \frac{12}{7} P(C) = P(A)$$

Also $P(A) + P(B) + P(C) = 1. \quad -(7)$

substituting (4), (5) & 6 in 7,

$$\cancel{P(A)} + \frac{12}{7} P(C) + \frac{9}{7} P(C) + P(C) = 1$$

$$\Rightarrow \frac{12 P(C) + 9 (P(C)) + 7 P(C)}{7} = 1 \text{ or } 28 P(C) = 7$$

$$\boxed{P(C) = \frac{7}{28} = 1/4}$$

∴ $P(A) = \frac{12}{7} \times P(C) = \frac{12}{7} \times \frac{1}{4} = 3/7 \Rightarrow \boxed{P(A) = \frac{3}{7}}$

& $P(B) = \frac{3}{4} P(A) = \frac{3}{4} \times \frac{3}{7} = \frac{9}{28} \Rightarrow \boxed{P(B) = \frac{9}{28}}$

ii) The entropy of each state is given by eq.

$$H_i = \sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}}$$

For state A $\quad H_A = \sum_{j=A}^{C} P_{Aj} \log \frac{1}{P_{Aj}}$

$$= P_{AA} \log \frac{1}{P_{AA}} + P_{AB} \log \frac{1}{P_{AB}} + P_{AC} \log \frac{1}{P_{AC}}$$

$$= 0.6 \times \log \frac{1}{0.6} + 0.2 \log \frac{1}{0.2} + 0.2 \log \frac{1}{0.2}$$

$$= \underline{1.371} \text{ bits/symbol}$$

State B $\quad H_B = \sum_{j=A}^{C} P_{Bj} \log \frac{1}{P_{Bj}}$

$$= P_{BA} \log \frac{1}{P_{BA}} + P_{BB} \log \frac{1}{P_{BB}} + P_{BC} \log \frac{1}{P_{BC}}$$

$$= 0.3 \log \frac{1}{0.3} + 0.5 \log \frac{1}{0.5} + 0.2 \log \frac{1}{0.2}$$

$$= \underline{1.485}$$

State C: $\quad \sum_{j=A}^{C} P_{Cj} \log \frac{1}{P_{Cj}} = P_{CA} \log \frac{1}{P_{CA}} + P_{CB} \log \frac{1}{P_{CB}} + P_{CC} \log \frac{1}{P_{CC}}$

$$\Rightarrow 0.3 \log \frac{1}{0.3} + 0.3 \log \frac{1}{0.3} + 0.4 \log \frac{1}{0.4} = \underline{1.571} \text{ bits/symbol}$$

$$\therefore H(S) = H = \sum_{i=A}^{C} P(i) \cdot H(i) = P(A) H(A) + P(B) H(B) + P(C) \cdot H(C)$$

$$= 3/7 \times 1.371 + \frac{9}{28} \times 1.485 + \frac{1}{4} \times 1.571 = 1.458 \text{ bits/symbol}$$

(iii) $G_1 = H(\bar{S}) = \sum_{i=1}^{3} P(m_i) \cdot \log \frac{1}{P(m_i)}$

$= P(A) \cdot \log \frac{1}{P(A)} + P(B) \cdot \log \frac{1}{P(B)} + P(C) \cdot \log \frac{1}{P(C)}$

$= 1.55$ bits / symbol.

$\therefore H(S) < H(\bar{S})$

4½

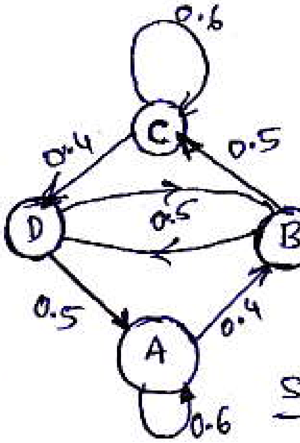
9/2

<u>Example</u>: Consider the state diagram of the Markov source :

i) Compute the state probabilities

ii) Find entropy of each state

iii) Find the entropy of the source.



<u>State Diagram</u>

<u>Solution</u>: <u>State equations</u>:

<u>State A</u>: $P(A) = 0.6\,P(A) + 0.5\,P(D)$ —(1)

B: $P(B) = 0.4\,P(A) + 0.5\,P(D)$ —(2)

C: $P(C) = 0.5\,P(B) + 0.6\,P(C)$ —(3)

D: $P(D) = 0.5\,P(B) + 0.4\,P(C)$ —(4)

From (1) & (2)

$P(A) - P(B) = 0.2\,P(A)$.

or $P(B) = 0.8\,P(A)$ —(5)

Also, $0.4\,P(A) = 0.5\,P(D)$

or $P(A) = 1.25\,P(D)$ —(6)

$0.4\,P(C) = 0.5\,P(B)$

or $P(C) = 1.25\,P(B)$ —(7)

$= 1.25\,P(D)$ —(8) $\quad \because P(B) = P(D)$

~~using (4)~~ $\cancel{P(D) = 0.5\,P(D) + 0.4 \times 1.25\,P(D)}$

WKT: $P(A) + P(B) + P(C) + P(D) = 1$.

$1.25\,P(D) + P(D) + 1.25\,P(D) + P(D) = 1$ or $4.5\,P(D) = 1$ or $P(D) = 1/4.5 = 2/9$

$P(D) = 2/9 \therefore P(B) = 2/9$, $P(A) = 1.25 \times P(D) = \frac{5}{\cancel{4}} \times \frac{\cancel{2}}{9} = \frac{5}{18}$

& $P(C) = 5/18$.

iii) $H(A) = 0.971$ bits/state, $H(B) = 1$ bit/state, $H(C) = 0.971$ bits/state

& $H(D) = 1$ bit/state

$\therefore H(S) = H = \sum_{i=1}^{n} P_i H_i = P(A) \cdot H_A + P(B) \cdot H_B + P(C) \cdot H_C + P(D) \cdot H_D$.

$\therefore H = 0.9839$ bits / binary digit