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S&S
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Department of Electronics & Communication Engg.

Course : Signals and Systems Engg-15EC44. Sem.: 4^{th} (2017-18, Even)

Course Coordinator:

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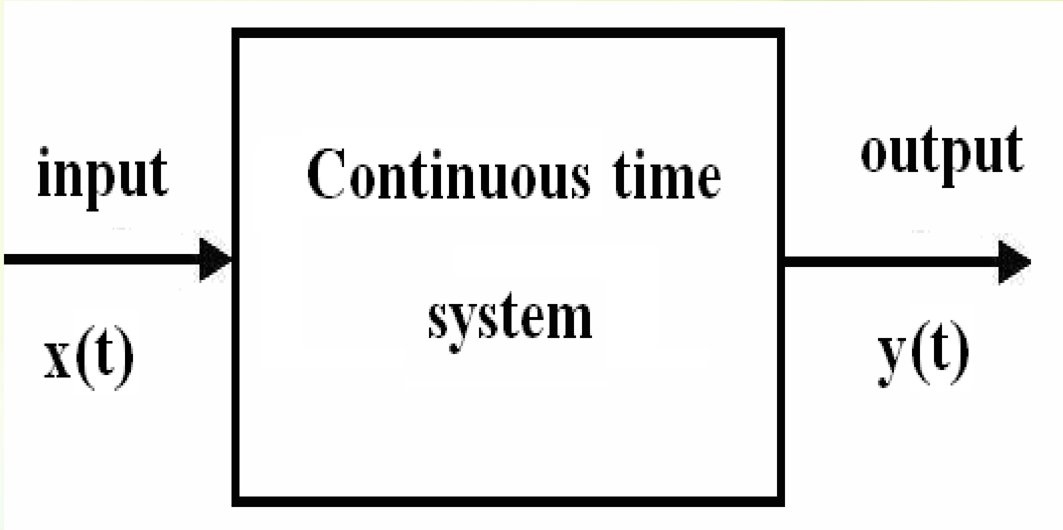
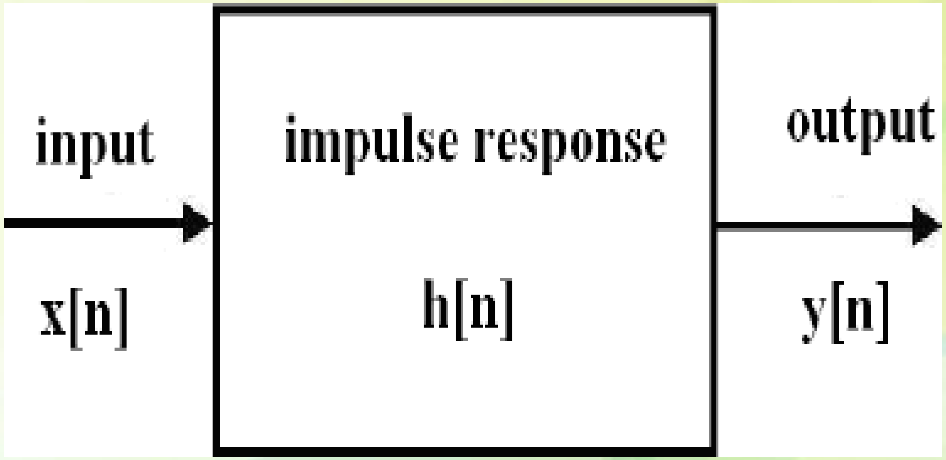
Convolution



Discrete time Convolution



$$x(n) \longrightarrow \boxed{\mathbf{L}} \longrightarrow y(n)$$

DT convolution is based on an earlier result where we showed that any signal $x(n)$ ca
expressed as a sum of impulses.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

So let us consider $x(n)$ written in this form to be our input to the LTI system.

$$y(n) = \mathrm{L}\left[x(n)\right] = \mathrm{L}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

This looks like our general linear form with a scalar $x(k)$ and a signal in n, $\delta(n-k)$. Re
that for an LTI system:

- Linearity (L): $ax_1(n) + bx_2(n) \longrightarrow \boxed{\mathbf{L}} \longrightarrow ay_1(n) + by_2(n)$
- Time Invariance (TI): $x(n-n_o) \longrightarrow \boxed{\mathbf{L}} \longrightarrow y(n-n_o)$

We can use the property of linearity to distribute the system **L** over our input.

$$y(n) = \mathrm{L}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)\mathrm{L}\left[\delta(n-k)\right]$$

then we can infer

$$x(n) \longrightarrow \boxed{\mathrm{L}} \longrightarrow y(n)$$

which gives us the following.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

This is the *convolution sum* for DT LTI systems.

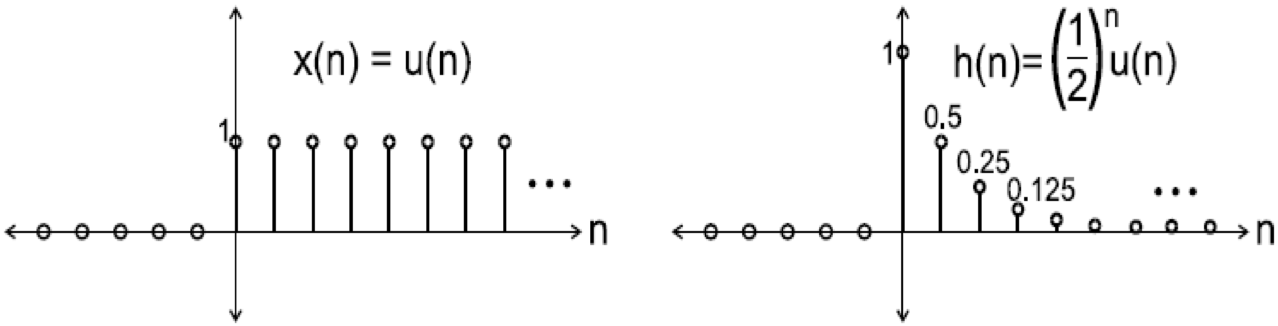
The convolution sum for $x(n)$ and $h(n)$ is usually written as shown here.

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Example 2.1: DT Convolution: Step Response

Say we are given the following signal $x(n)$ and system impulse response $h(n)$.

$$x(n) = u(n) \quad \text{and} \quad h(n) = \left(\frac{1}{2}\right)^n u(n)$$



We wish to find the step response $s(n)$ of the system (i.e. the response of the system to the unit step input $x(n) = u(n)$. This is shown below.

$$s(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

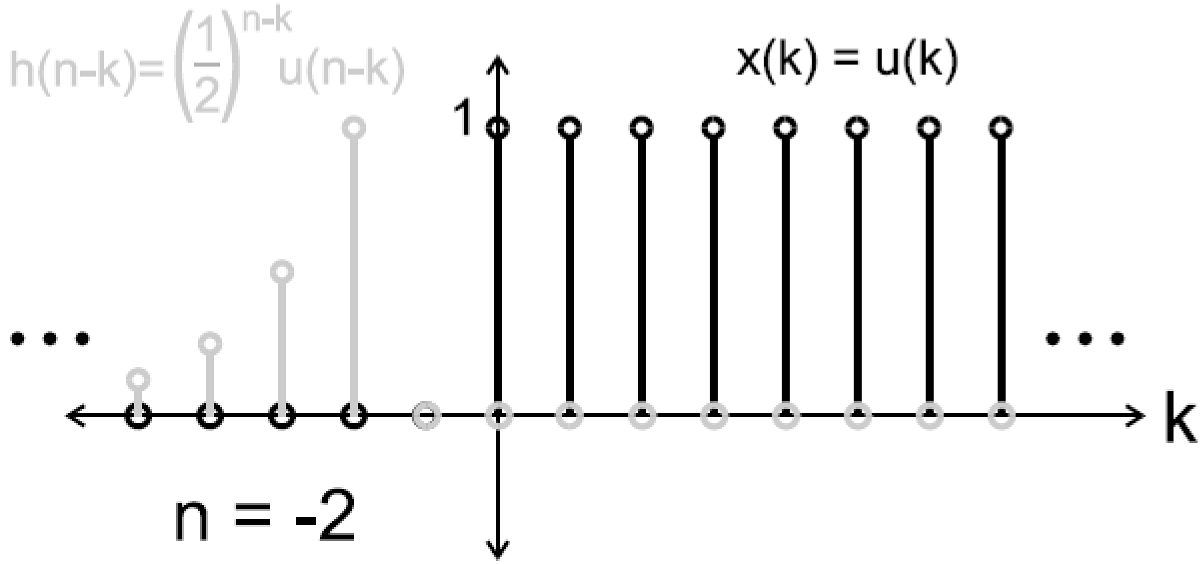
Thus the step response is as follows, found by substituting our actual signals into the general convolution sum.

$$s(n) = \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{2}\right)^{n-k} u(n-k)$$

Let's look at this step response in smaller ranges to see what happens.

- First, consider the case where $n < 0$.

Here, $s(n) = 0$. This is because $u(n-k)$ (and the associated exponential) will be starting at a point less than 0 in the k domain, and will extend to $-\infty$, whereas $u(k)$ starts at 0 and extends to $+\infty$. We can visualize this, say for a value of $n = -2$.



Notice that there is no non-zero overlap of $x(k)$ and $h(n-k)$. Since they are multiplied together, the zero part of one signal cancels out the non-zero part of the other, and vice versa. Thus, $s(n) = 0$ for $n < 0$.

- The more interesting case is when $n \geq 0$.

  Recall the convolution sum we are using to determin $s(n)$.

$$s(n) = \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{2}\right)^{n-k} u(n-k)$$

  Note that $u(k)$ means we know the summation will be 0 for all values of $k < 0$, so we can change the lower limit of the summation to 0. Similarly, the $u(n-k)$ term means that the summation for all values of $k > n$ will be 0, since that unit step is flipped and extends toward $-\infty$. So, we can change the upper limit of the summation to n. In the range $0 \leq k \leq n$, both of the unit steps will have a value of 1. This is shown below.

$$
\begin{aligned}
s(n) &= \sum_{k=-\infty}^{\infty} u(k)\left(\frac{1}{2}\right)^{n-k} u(n-k) \\
&= \sum_{k=0}^{n} 1 \cdot \left(\frac{1}{2}\right)^{n-k} \cdot 1
\end{aligned}
$$

We can pull out any terms only in n since that is not the summation variable.

$$
\begin{aligned}
&= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{-k} \\
&= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{-k} \\
&= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} 2^{k}
\end{aligned}
$$

Now we have a form consistent with a geometric series. We can use that to solve.

$$\text{Recall} \quad \sum_{k=0}^{n} 2^k = \frac{1-2^{n+1}}{1-2} = 2^{n+1} - 1$$

So we have $s(n)$ as follows.

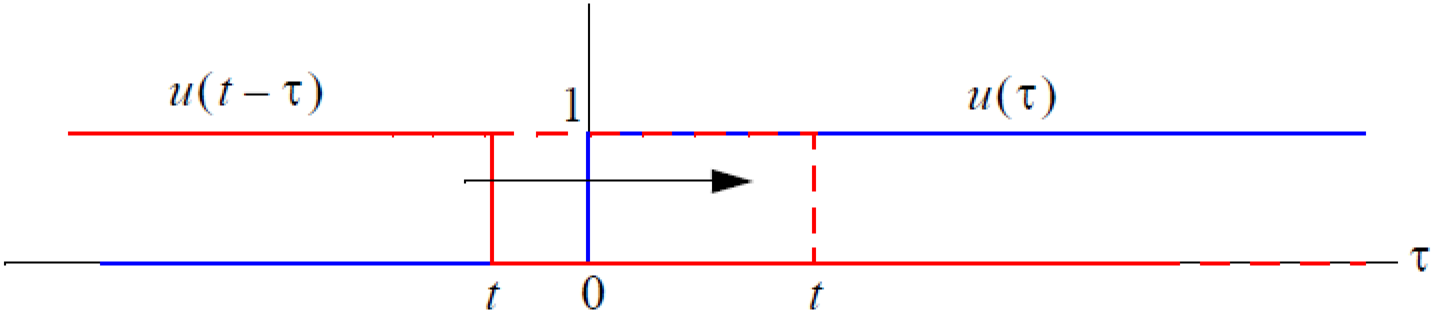
$$\begin{aligned}
s(n) &= \left(\frac{1}{2}\right)^n \left(2^{n+1} - 1\right) \\
&= \left(\frac{1}{2}\right)^n (2 \cdot 2^n - 1) \\
&= \left(\frac{1}{2}\right)^n \left(2 \cdot \left(\frac{1}{2}\right)^{-n} - 1\right) \\
&= 2 \cdot \left(\frac{1}{2}\right)^{-n} \left(\frac{1}{2}\right)^n - 1 \cdot \left(\frac{1}{2}\right)^n \\
s(n) &= 2 - \left(\frac{1}{2}\right)^n
\end{aligned}$$

We can visualize this, say for $n = 2$, as shown below. Note how the system

Example: $y(t) = x(t) * h(t) = u(t) * u(t)$

- Setting up the convolution integral we have

$$y(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau$$



$$y(t) = \begin{cases} 0, & t < 0 \\ \int_0^t d\tau, & t \geq 0 \end{cases}$$

$$= \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$

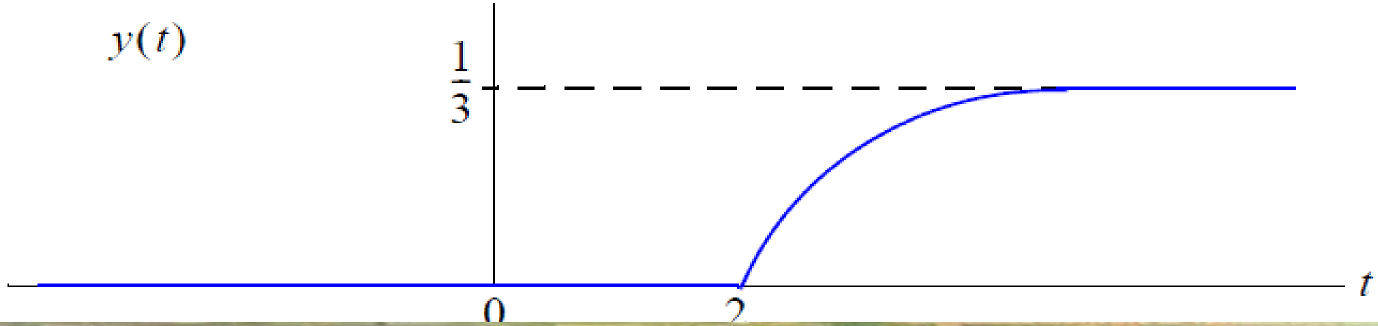
or simply

$$y(t) = tu(t) \equiv r(t),$$

which is known as the *unit ramp*

- For $t-2<0$ or $t<2$ there is no overlap in the product that comprises the integrand, so $y(t) = 0$
- For $t-2>0$ or $t>2$ there is overlap for $\tau \in [0, t-2)$, so here

$$\begin{aligned} y(t) &= \int_0^{t-2} e^{-3\tau} d\tau \\ &= \left.\frac{e^{-3\tau}}{-3}\right|_0^{t-2} \\ &= \frac{1}{3}[1-e^{-3(t-2)}]u(t-2) \end{aligned} \qquad (9.39)$$



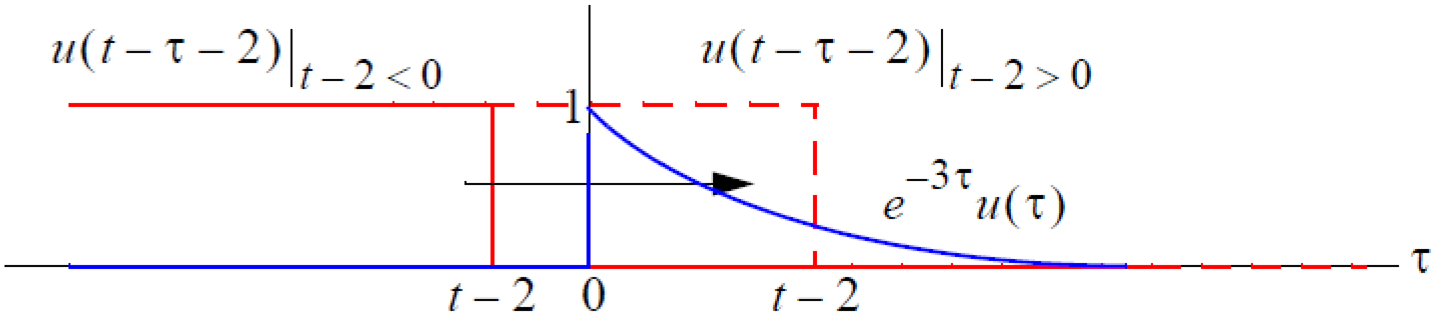
Evaluating Convolution Integrals

Step and Exponential

- Consider $x(t) = u(t-2)$ and $h(t) = e^{-3t}u(t)$
- We wish to find $y(t) = x(t)*h(t)$

$$y(t) = \int_{-\infty}^{\infty} e^{-3\tau}u(\tau)u(t-\tau-2)d\tau \quad (9.38)$$

- To evaluate this integral we first need to consider how the step functions in the integrand control the limits of integration



Properties of Convolution

- Commutativity:

$$x(t) * h(t) = h(t) * x(t)$$

- Associativity:

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

- Distributivity over Addition:

$$x(t) * [h_1(t) * h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

- Identity Element of Convolution:

$$x(t) * h(t) = h(t)$$

What is $x(t)$?

– It turns out that $x(t) = \delta(t) \Rightarrow \delta(t) * h(t) = h(t)$

<u>proof</u>

$$\int_{-\infty}^{\infty} \delta(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau) h(t-0) d\tau$$

$$= h(t) \int_{-\infty}^{\infty} \delta(\tau) d\tau = h(t)$$

Integrator

$$h(t) = \int_{-\infty}^{t} x(\tau)d\tau \Big|_{x(\tau) = \delta(\tau)} = u(t)$$

Ideal delay

$$h(t) = x(t - t_d)\Big|_{x(t) = \delta(t)} = \delta(t - t_d)$$

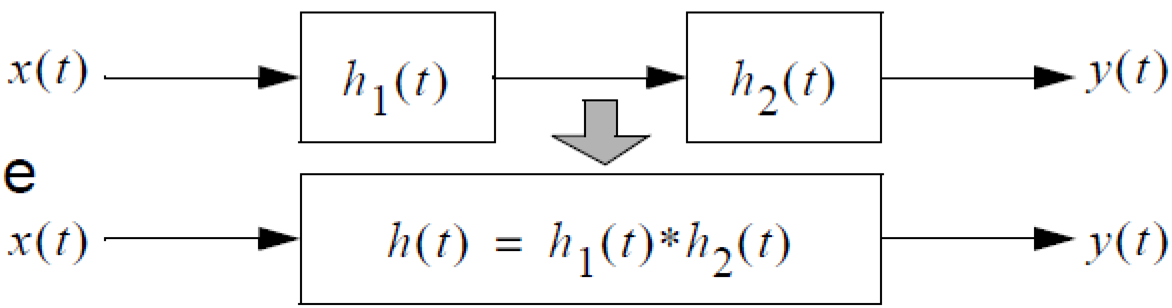
- Note that this means that

$$x(t) * \delta(t - t_d) = x(t - t_d)$$

- For a cascade of two LTI systems having impulse responses $h_1(t)$ and $h_2(t)$ respectively, the impulse response of the cascade is the convolution of the impulse responses

$$h_{\text{cascade}}(t) = h_1(t) * h_2(t) \tag{9.44}$$

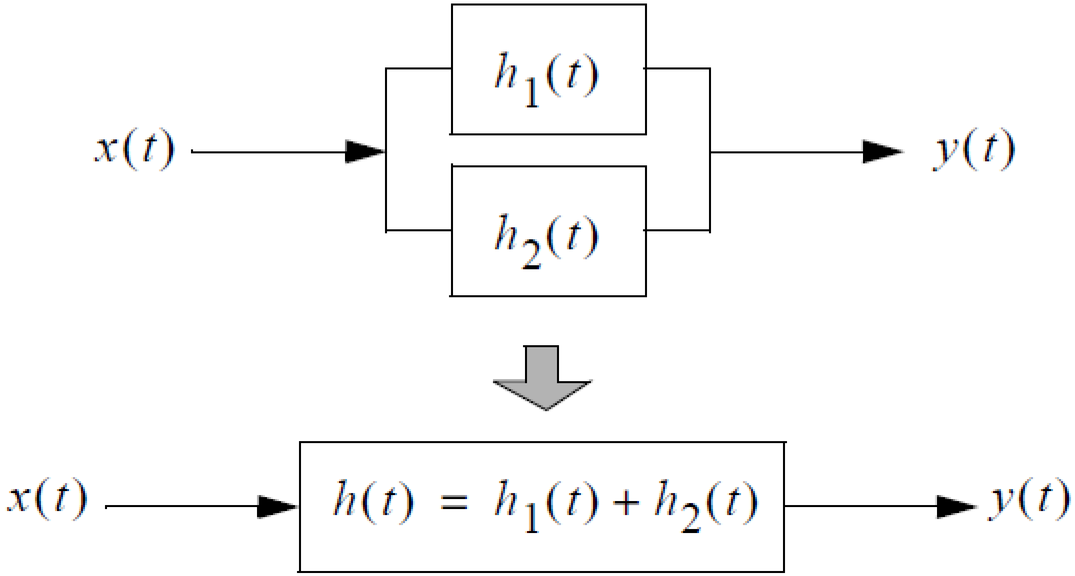
Cascade



- For two systems connected in parallel, the impulse response is the sum of the impulse responses

$$h_{\text{parallel}}(t) = h_1(t) + h_2(t) \quad (9.45)$$

Parallel



Queries ….?