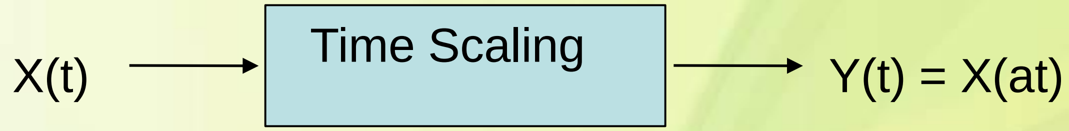
Department of Electronics & Communication Engg.

Course : Signals and Systems **Engg-15EC44.** **Sem.: 4^{th} (2017-18, Even)**

Course Coordinator:

Prof. S. S. Kamate

Time scaling



The signal $y(t) = x(at)$ is a time-scaled version of $x(t)$.

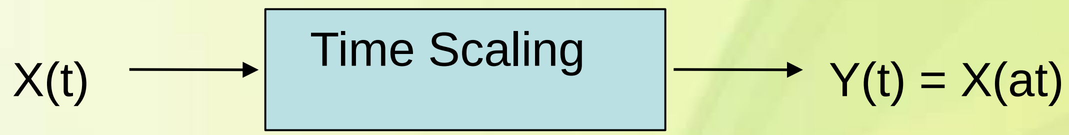
If $|a| > 1$, we are SPEEDING UP $x(t)$ by a factor of a.

If $|a| < 1$, we are SLOWING DOWN $x(t)$ by a factor of a.

The signal $y(t)$ has period $= \frac{T}{|a|}$, where T is the period of $x(t)$.

Example: Given $x(t)$, find $y(t) = x(2t)$. This SPEEDS UP $x(t)$

Time scaling



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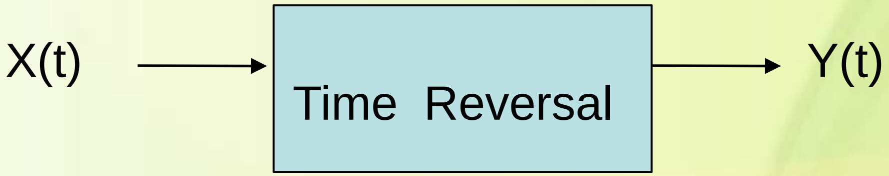
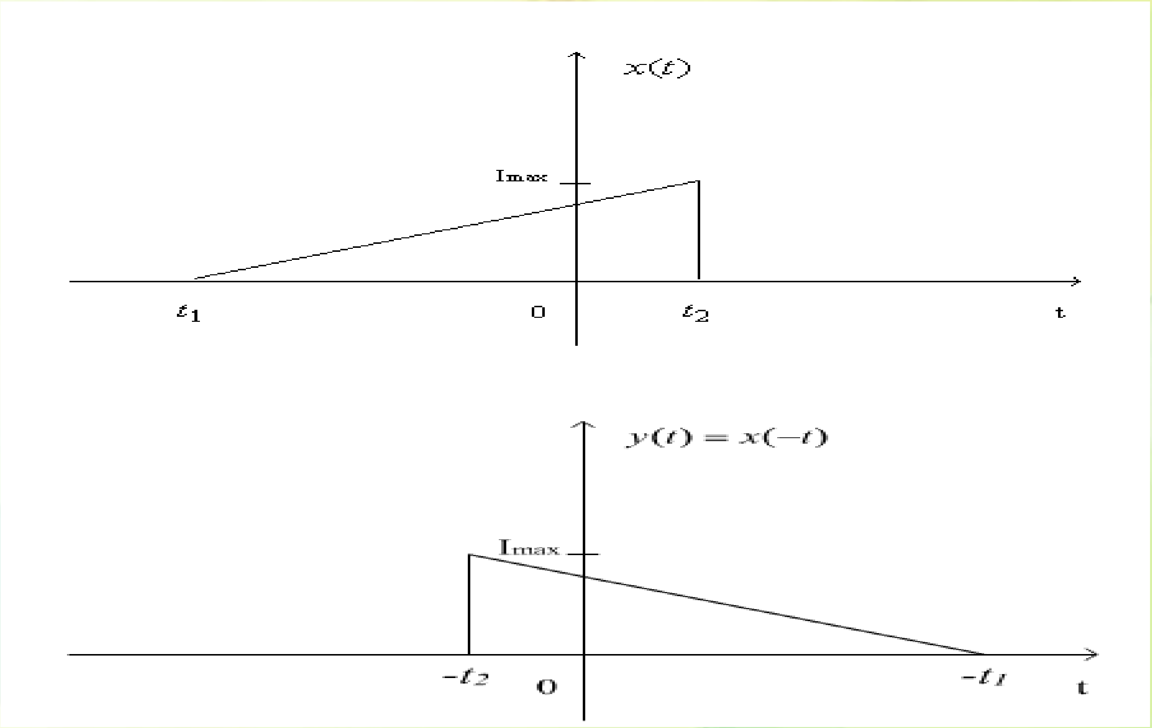
Time reversal

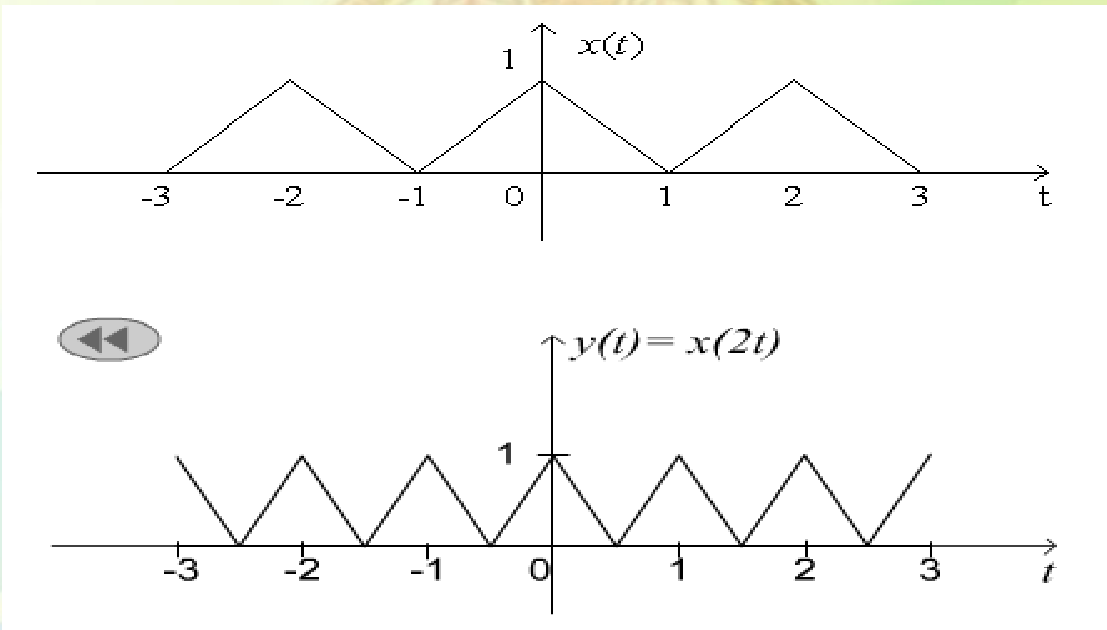




Contd…

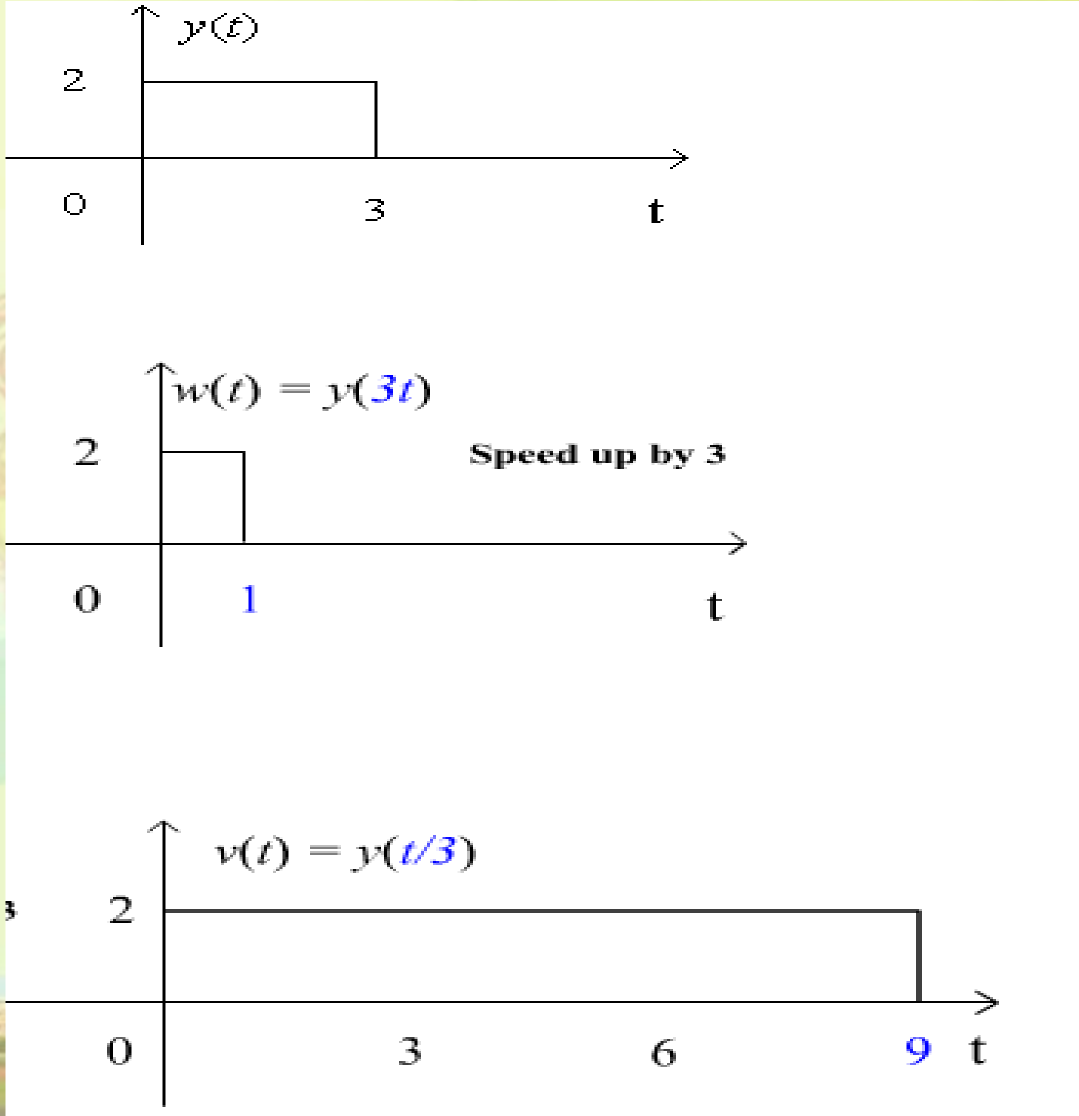
a>1 → Speeds up → Smaller period → Graph shrinks!
a<1 → slows down → Larger period → Graph expands

Example: Given $x(t)$, find $y(t) = x(2t)$. This SPEEDS UP $x(t)$ (the graph is shrinking



Contd…

- Given $y(t)$,
  - find $w(t) = y(3t)$
  - $v(t) = y(t/3)$.



Contd…

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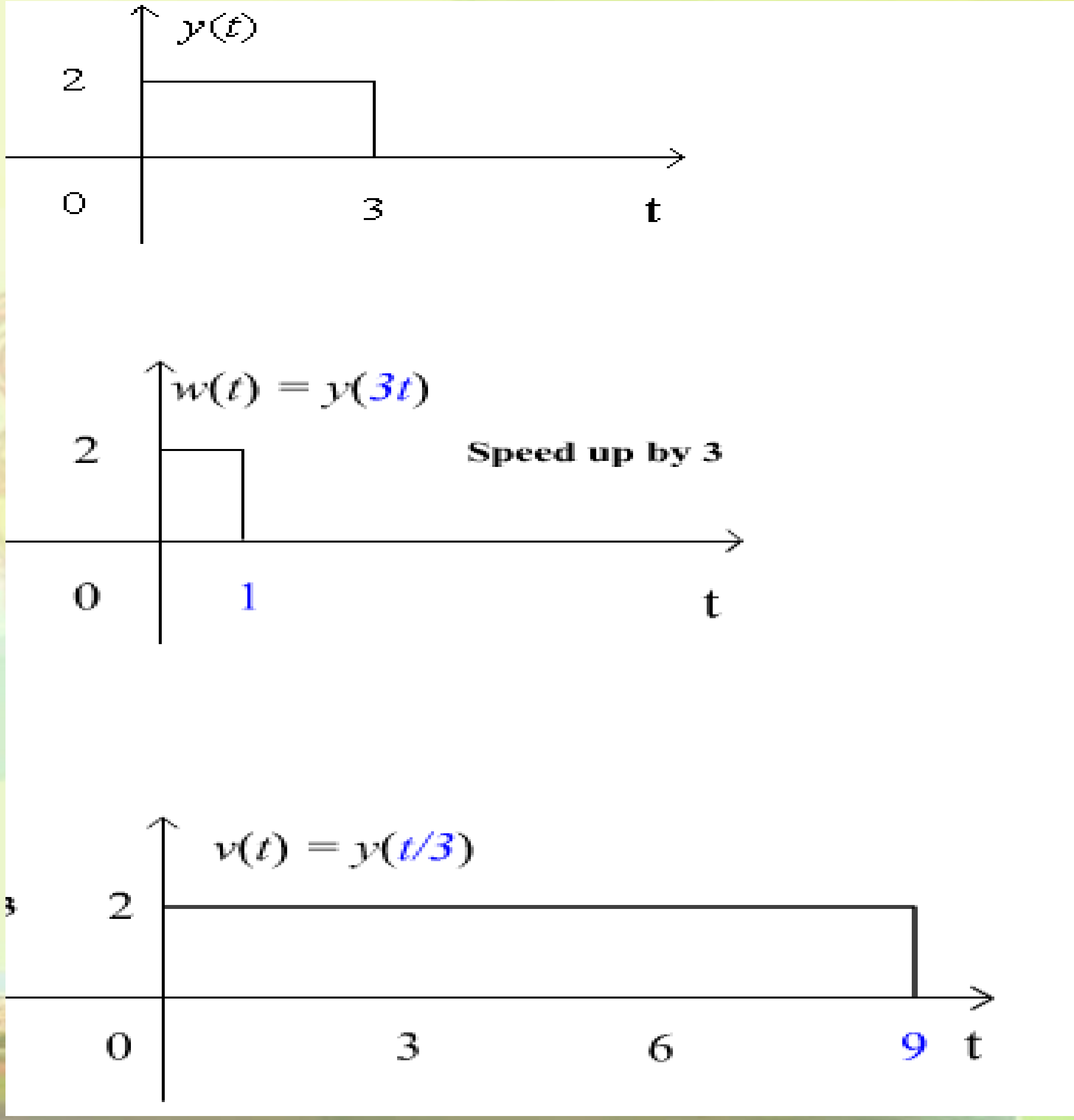
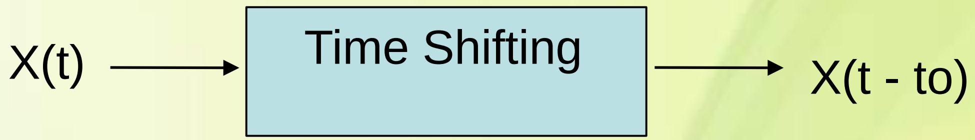


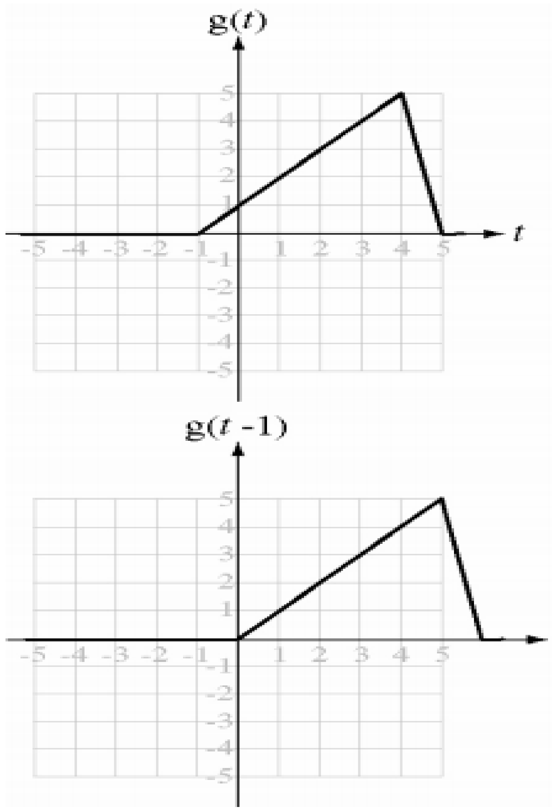
Time Shifting



- The original signal *g*(*t*) is shifted by an amount *t*0 .

Time Shift: y(t)=g(t-to)

- g(t)→g(t-to) ; to>0 →Signal Delayed→ Shift to the right

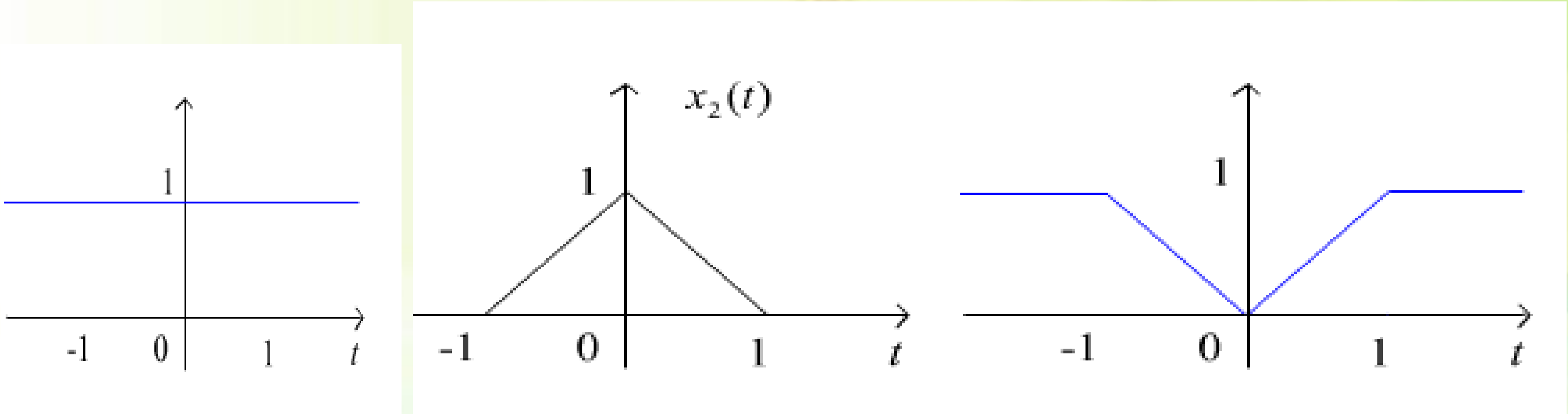


t	$g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	1
1	2
2	3
3	4
4	5
5	0

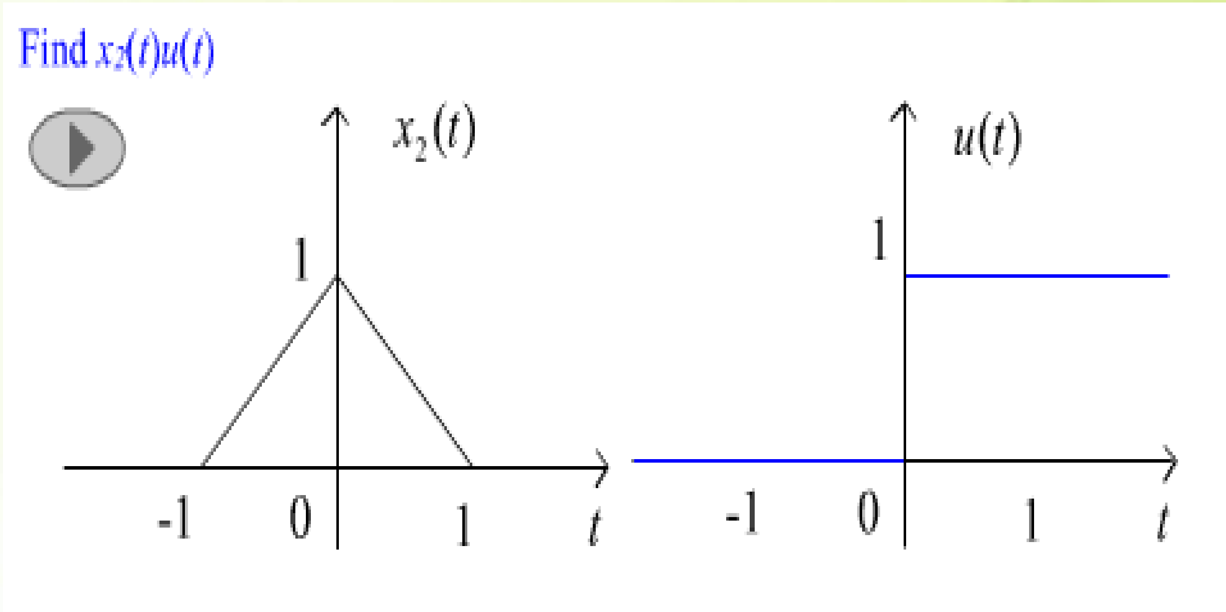
t	$t-1$	$g(t-1)$
−5	−6	0
−4	−5	0
−3	−4	0
−2	−3	0
−1	−2	0
0	−1	0
1	0	1
2	1	2
3	2	3
4	3	4
5	4	5

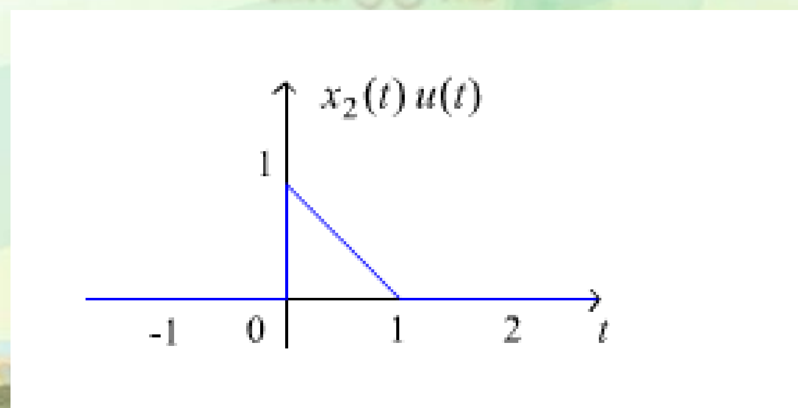
Amplitude Operations

Given $x_2(t)$, find $1 - x_2(t)$.



Multiplication of two signals: $x_2(t)u(t)$





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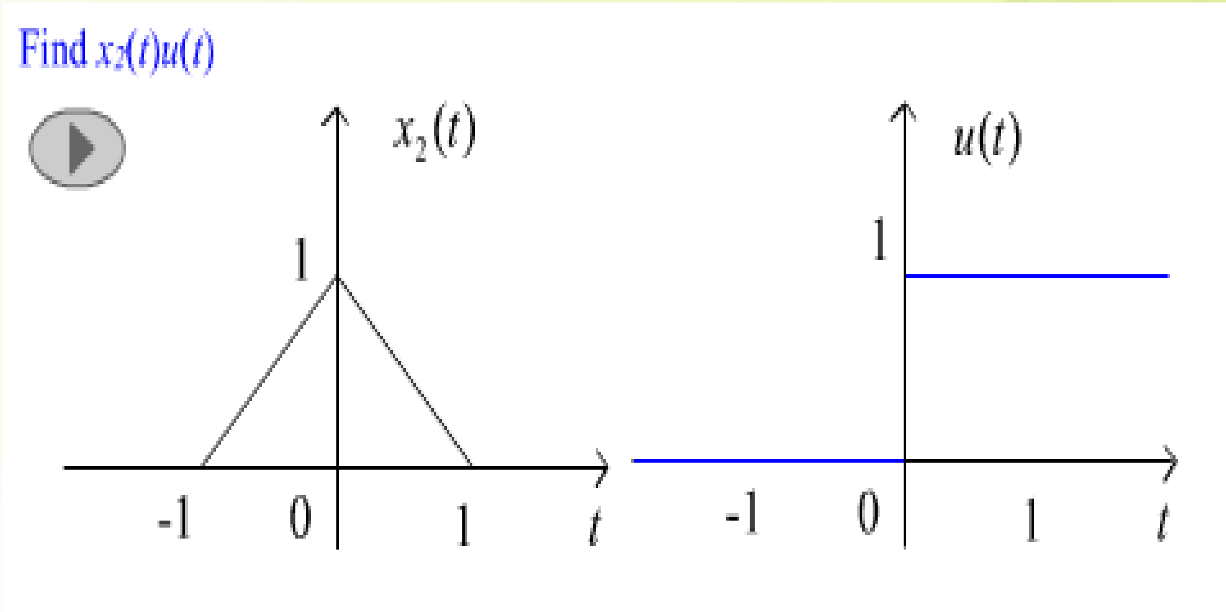
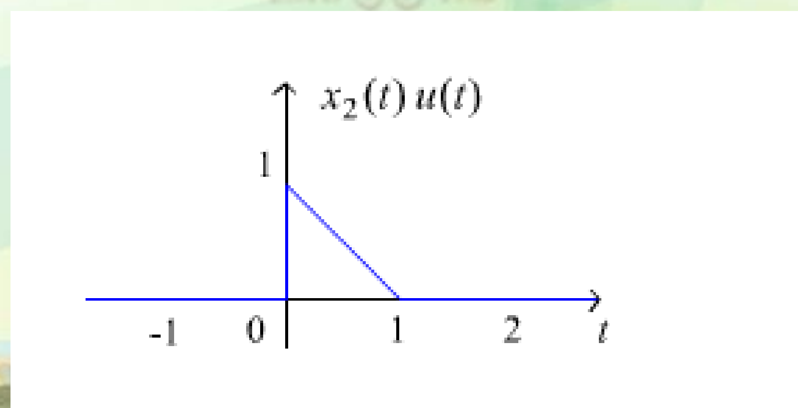


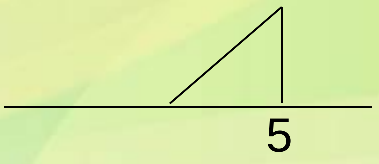


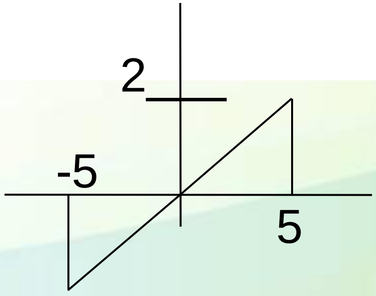
Example

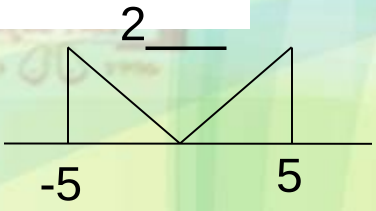
- Given x(t) find xe(t) and xo(t)

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$







Signal Characteristics

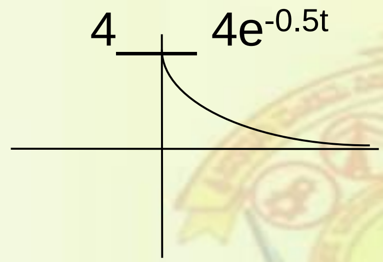
- Even and odd signals
  1. $X(t) = Xe(t) + Xo(t)$
  2. $X(-t) = X(t)$ ← Even
  3. $X(-t) = -X(t)$ ← Odd

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

Example

- **Given x(t) find xe(t) and xo(t)**



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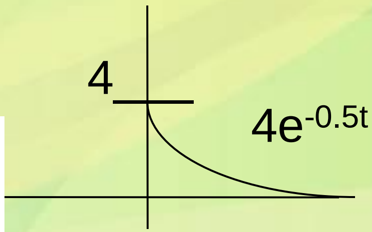

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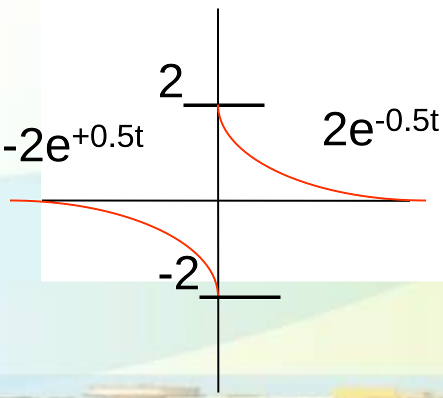
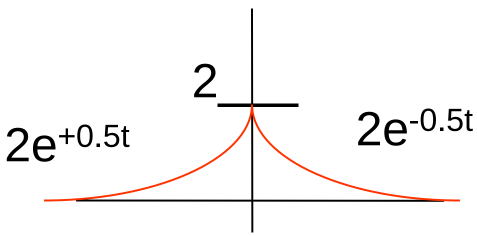




Periodic and Aperiodic Signals

- Given x(t) is a continuous-time signal
- X (t) is periodic iff X(t) = x(t+nT) for any T and any integer n
- Example
  - Is X(t) = A cos(ωt) periodic?
    - X(t+nT) = A cos(ω(t+Tn)) =
    - A cos(ωt+ω2nπ)= A cos(ωt)
  - Note: f0=1/T0; $\omega_o = 2\pi/T_o$ <= Angular freq.
  - T0 is fundamental period; T0 is the minimum value of T that satisfies X(t) = x(t+T)

Sum of periodic Signals

- X(t) = x1(t) + X2(t)
- X(t+nT) = x1(t+m_1T_1) + X2(t+m_2T_2)
- $m_1T_1 = m_2T_2$ = To = Fundamental period
- Example:
  - cos(tπ/3)+sin(tπ/4)
  - T1=(2π)/(π/3)=6; T2 =(2π)/(π/4)=8;
  - T1/T2=6/8 = ¾ = (rational number) = m2/m1
  - $m_1T_1 = m_2T_2$ → Find m1 and m2→
  - 6.4 = 3.8 = 24 = To (n=1)

Product of periodic Signals

$X(t) = x_a(t) * X_b(t) =$

$= 2\sin[t(7\pi/24)] * \cos[t(\pi/24)]$;

find the period of x(t)

- We know: $= 2\sin[t(7\pi/12)/2] * \cos[t(\pi/12)/2]$;
  - Using Trig. Itentity:

- **Sum-to-Product Formulas**

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

- $x(t) = \sin(t\pi/3)+\sin(t\pi/4)$
- Thus, $T_o=24$, as before!

http://www.sosmath.com/trig/Trig5/trig5/pdf/pdf.html

Sum of periodic Signals (cont.)

– X1(t) = cos(3.5t) → f1 = 3.5/2π → T1 = 2π /3.5

– X2(t) = sin(2t) → f2 = 2/2π → T2 = 2π /2

– X3(t) = 2cos(t7/6) → f3 = (7/6)/2π → T3 = 2π /(7/6)

- → T1/T2 = 4/7 Ratio or two integers
- → T1/T3 = 1/3 Ratio or two integers
- → Summation is periodic

T1	T2	T3
2π/3.5	2π/2	12π/7
m1	m2	m3
7.3=21	2.6=12	7.1=7
m1xT1	m2xT2	m3xT3
12π	12π	12π

– m1T1 = m2T2 = m3T3 = To ; Hence we find To

– The question is how to choose m1, m2, m3 such that the above relationship holds

– We know: 7(T1) = 4(T2) & 3(T1) = 1(T3) ; → m1(T1)=m2(T2)

– Hence:

21(T1) = 12(T2)= 7(T3);

Thus, fundamental period: To = 21(T1) = 21(2π /3.5)=12(T2)=12π

Find even and odd parts of v(t).

Important Engineering Signals

- Euler's Formula

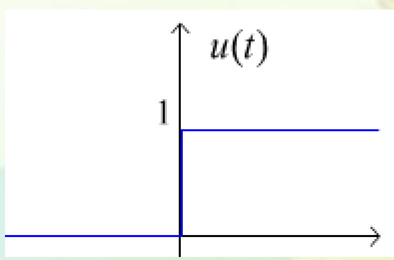
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

notes

- Unit Step Function (Singularity Function)
  - Can you draw $x(t) = \cos(t)[u(t) - u(t-2\pi)]$?



$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \\ \text{undefined}, & t = 0 \end{cases}$$

notes

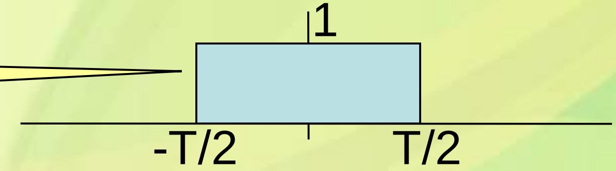
- Use Unit Step Function to express a block function (window)

Next→

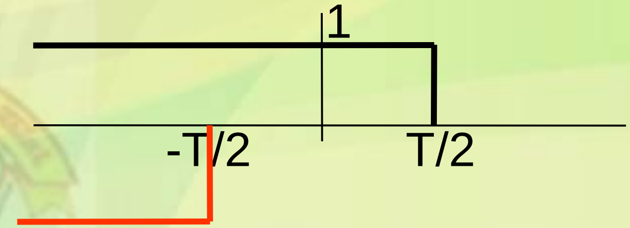
Unit Step Function Applications: Creating Block Function

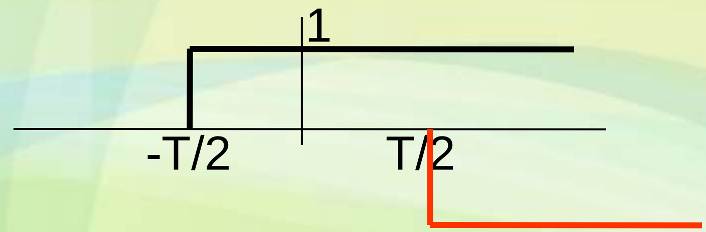
- rect(t/T)

Note:
Period is T; & symmetric



- Can be expressed as u(T/2-t)-u(-T/2-t)
  - Draw u(t+T/2) first; then reverse it!





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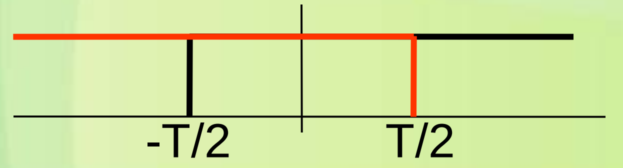


- Can be expressed as u(t+T/2).u(T/2-t)

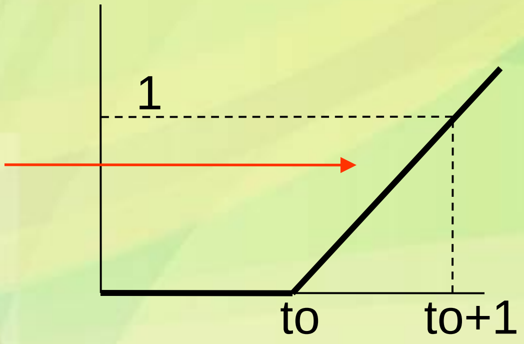
Unit Step Function Applications: Creating Unit Ramp Function

Unit ramp function can be achieved by:

notes

$$f(t)=\int_0^t u(\tau-t_0)\,d\tau=\int_{t_0}^t d\tau=(t-t_0)\,u(t-t_0)$$

Non-zero only for t>t0



The ramp function ($R(x):\mathbb{R}\to\mathbb{R}$) may be defined analytically in several ways. Possible definitions are:

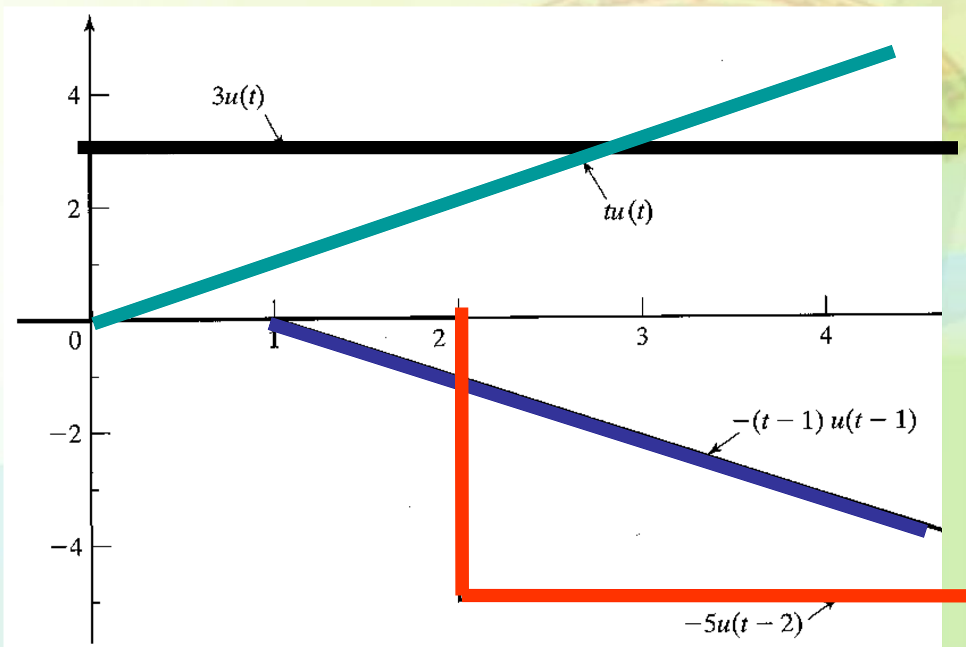
$$R(x):=\begin{cases}x, & x\geq 0;\\ 0, & x<0\end{cases}$$

Example: Using Mathematica: (t-2)*unitstep[t-2] – click here:

http://www.wolframalpha.com/input/?i=%28t-2%29*unitstep%5Bt-2%5D

Unit Step Function Applications: Example

$$f(t) = 3u(t) + tu(t) - [t-1]u(t-1) - 5u(t-2)$$



$$\begin{cases} -1 & t \geq 2 \\ 4 & 1 \leq t < 2 \\ t+3 & \text{(otherwise)} \end{cases}$$

3 * UnitStep (t) +
t * UnitStep (t) −
(t − 1) * UnitStep (t − 1) −
5 * UnitStep (t − 2)

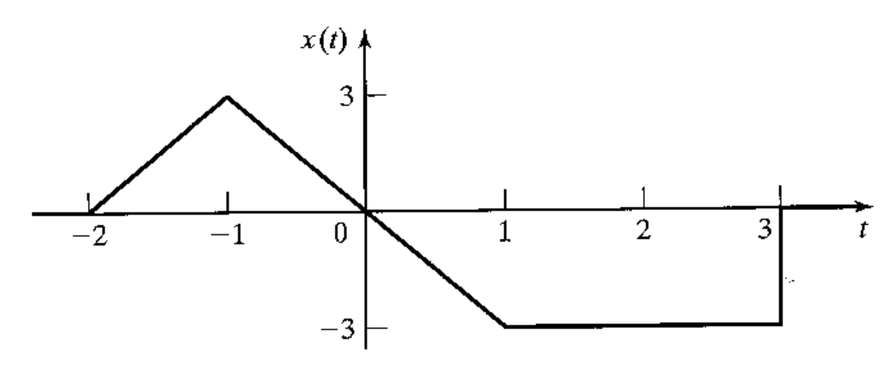
Computed by Wolfram|Alpha

Unit Step Function Applications: Example

- Plot

$$f(t)=3[t+2]u(t+2)-6[t+1]u(t+1)+3[t-1]u(t-1)+3\text{u}(t-3)$$

- t<-2 → f(t)=0
- -2<t<-1 → f(t)=3[t+2]
- -1<t<1 → f(t)=-3t
- 1<t<3 → f(t)=-3
- 3<t< → f(t)=0



Using Mathematica - click here:

http://www.wolframalpha.com/input/?i=3*%28t%2B2%29*UnitStep%28t%2B2%29-6*%28t%2B1%29*UnitStep%28t%2B1%29%2B3*%28t-1%29*UnitStep%28t-1%29%2B3*UnitStep%28t-3%29

Unit Impulse Function δ(t)

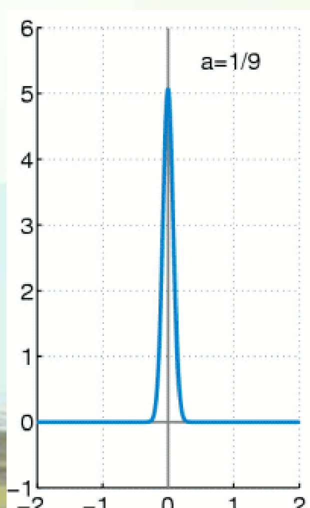
- Not real (does not exist in nature – similar to *i=sqrt(-1)*
- Also known as *Dirac delta* function
  - Generalized function or testing function
- The Dirac delta can be loosely thought of as a function of the real line which is zero everywhere except at the origin, where it is infinite
- Note that impulse function is not a true function – it is not defined for all values
  - It is a *generalized function*

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

$$\int_{-\infty}^{\infty} \phi(x)\delta(x)\, dx = \phi(0)$$

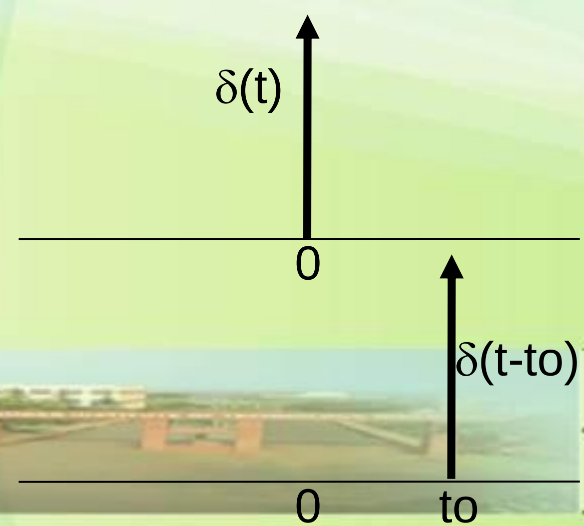
Mathematical definition



Mathematical definition

The Dirac delta function as the limit (in the sense of distributions) of the sequence of Gaussians

$$\delta_a(x) = \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$$ as $a \to 0.$



Unit Impulse Properties

- Integration of a test function

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = ?$$
$$= \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt$$
$$= x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = x(t_0)$$

  - Example

$$\int_{-\infty}^{\infty} \delta(t-a)\sin^2(\frac{t}{b})dt?$$

- Other properties:

- $\int_{-\infty}^{\infty} f(t-t_0)\delta(t-t_1)dt = f(t_1-t_0)$, if $f(t)$ is continuous at t_1-t_0
- $\int_{-\infty}^{t} \delta(\tau-t_0)d\tau = u(t-t_0)$
- $\delta(t) = \delta(-t)$
- $\delta(at) = \frac{1}{|a|}\delta(t)$

Make sure you can understand why!

Unit Impulse Properties

- Example: Verify $\delta(at) = \frac{1}{|a|}\delta(t)$

Schaum's p38

- Evaluate the following $\int_{-1}^{1} (3t^2+1)\delta(t) = ?$

Schaum's p40

Remember:

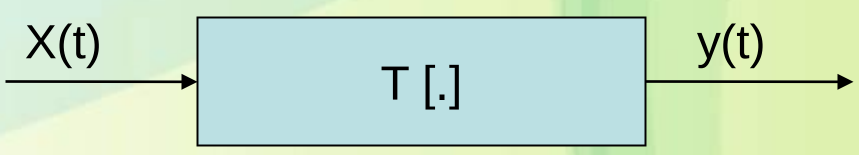
$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = ?$$

$$= \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt$$

$$= x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = x(t_0)$$

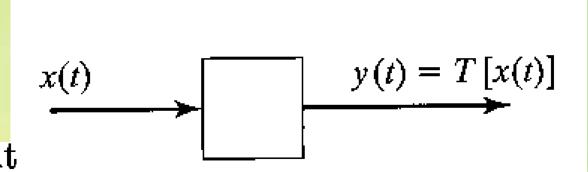
Continuous-Time Systems

- A system is an operation for which cause-and-effect relationship exists
  - Can be described by block diagrams
  - Denoted using transformation T[.]
- System behavior described by mathematical model



Continuous-Time Systems - Properties

- Systems with Memory:



A system $y(t_0)$ has memory if its output at time t_0 depends on the input $x(t)$ for $t > t_0$ or $t < t_0$, i.e. it depends on values of the input other than $x(t_0)$.

$y(t) = T[x(t)]$

$y(t) = S[x(t)]$

$x(t) \rightarrow y(t)$

- Examples – Memoryless/Memory?

$v(t_0) = Ri(t_0)$

$v(t_0) = \frac{1}{C}\int_{-\infty}^{t_0} i(t)dt$

$y(t) = (t+5)x(t)$

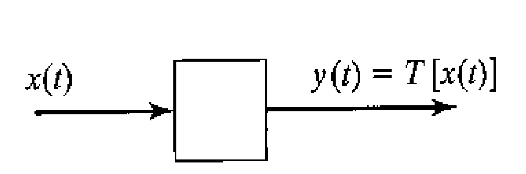
$z(t) = [x(t+5)]^2$

$a(t) = x(5)$

Has memory if output depends on inputs other than the one defined at current time

Continuous-Time Systems - Properties

Inverse of a System



A system is invertible if you can determine the input uniquely from the output, i.e. there is a one-to-one relationship between the input and output.

$$y(t) = T[x(t)]$$

$$y(t) = S[x(t)]$$

$$x(t) \to y(t)$$

Examples

$$x(t) = y(t)/R.$$

$$y(t) = x^5(t)$$

Noninvertible Systems

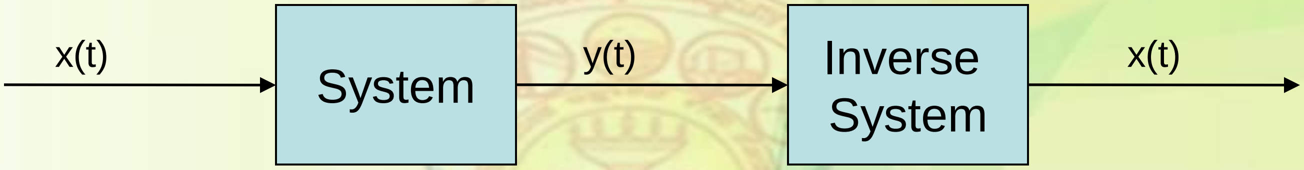
$$
\begin{aligned}
y(t) &= x(t)u(t) \Rightarrow \text{zeros out much of the input} \\
y(t) &= x^2(t) \Rightarrow \text{don't know sign} \\
y(t) &= \cos[x(t)] \Rightarrow \text{add } 2\pi \text{ to } x(t)
\end{aligned}
$$

Thermostat Example! (notes)

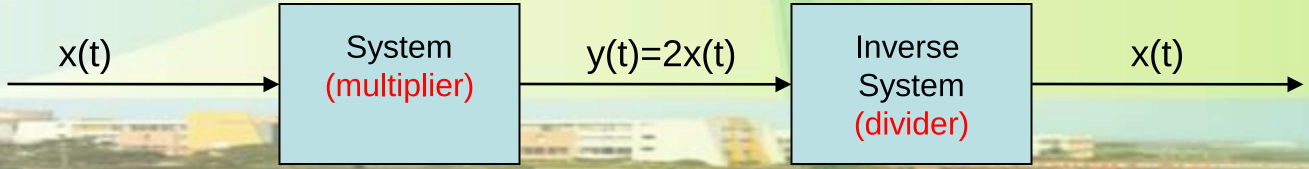
Each distinct input → distinct output

Continuous-Time Systems - Invertible

- If a system is invertible it has an Inverse System



- Example: y(t)=2x(t)
  - System is invertible: For any x(t) we get a distinct output y(t)
  - Thus, the system must have an Inverse
    - x(t)=1/2 y(t)=z(t)



If the system is not invertible it does not have an INVERSE!

Continuous-Time Systems - Causality

Causality (non-anticipatory system)

- A System can be causal with non-causal components!

Output $y(t)$ depends only on past and present inputs and **not on the future.**

All physical real-time systems are causal because we can not anticipate the future.

Image processing–Non-causal filters like blurring masks.

Music processing – record and process later – noncausal but not real-time

Examples

$v(t_0) = i(t_0)R \rightarrow$ memoryless $\Rightarrow$ Causal

Remember: Reverse is not TRUE!

$v(t_0) = \frac{1}{C}\int_{-\infty}^{t_0} i(t)dt \rightarrow$ Causal since only depends on past and present

$y(t_0) = \int_{-\infty}^{t_0+a} x(t)dt.$??

$y(t) = x(-t)$?? What it t<0?

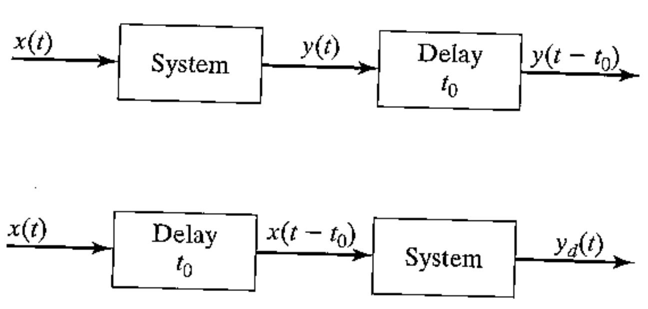
Depends on cause-and-effect → no future dependency

Continuous-Time Systems – Time Invariance

If you shift your input signal in time for a time-invariant system, all that will happen is you will get a similar shift in your output signal. Alternatively, the system behaves the same each day and does not change over time.

If $y(t-t_0) = S[x(t-t_0)]$, then system is Time-Invariant. Else it is Time-Varying.

Testing for Time Invariance



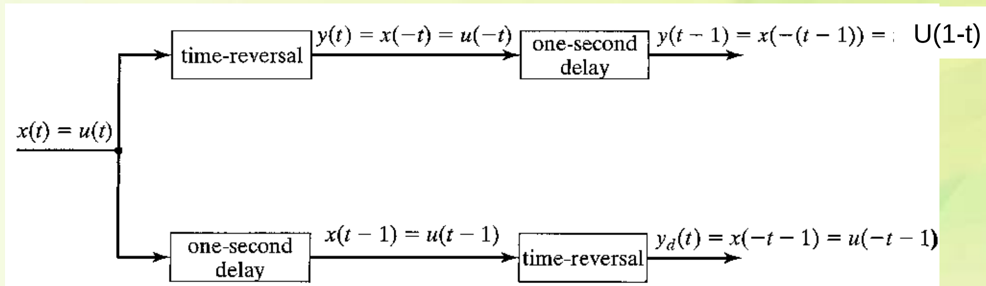
What if the system is time reversal? (next slide)

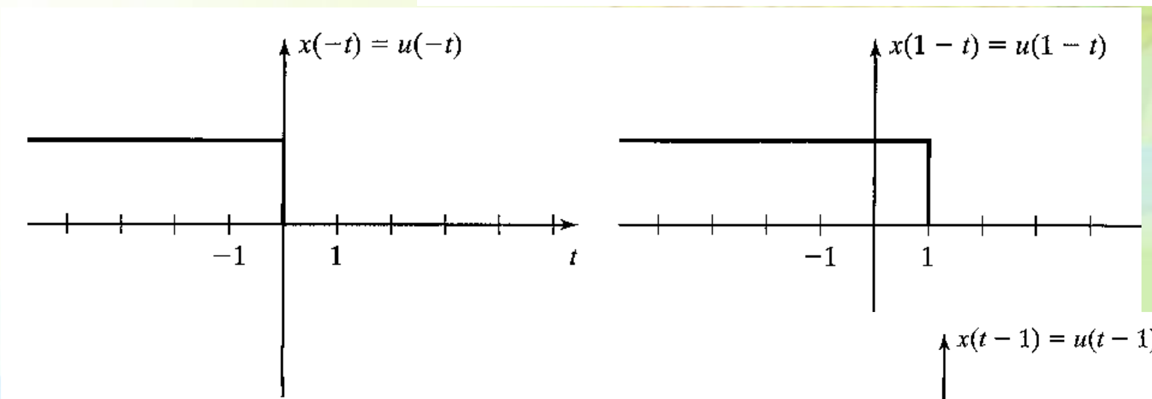
To test: y(t)|t-to → y(t)|x(t-to)
Example:
y(t) = e^x(t):
y(t)|t=t-to → e^x(t-to)
y(t)|x(t-to) → e^x(t-to)
→ Time Invariance

Time-shift in input results in time-shift in output → system always acts the same way (Fix System)

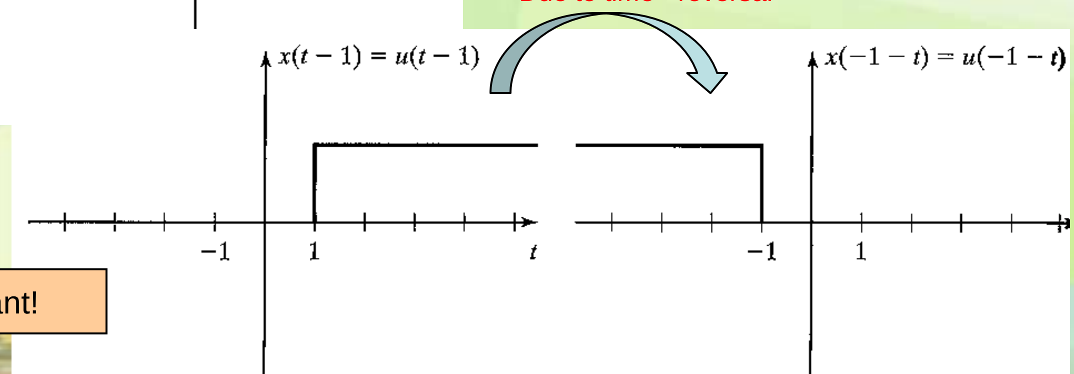
Continuous-Time Systems – Time Invariance

Example of a system:





Pay attention!
Due to time - reversal



Time reversal operation is NOT time invariant!

Models of Continuous Time Signals

Signals
Sinusoidal signals
Exponential signals
Complex exponential signals
Unit step and unit ramp
Impulse functions

Sinusoidal Signals

A sinusoidal signal is of the form

$x(t) = \cos(Wt + \theta)$:

where the radian frequency is !, which has the units of radians/s.

Also very commonly written as

$x(t) = A\cos(2\pi ft + \theta)$:

where f is the frequency in Hertz.

We will often refer to ! as the frequency, but it must be kept in mind

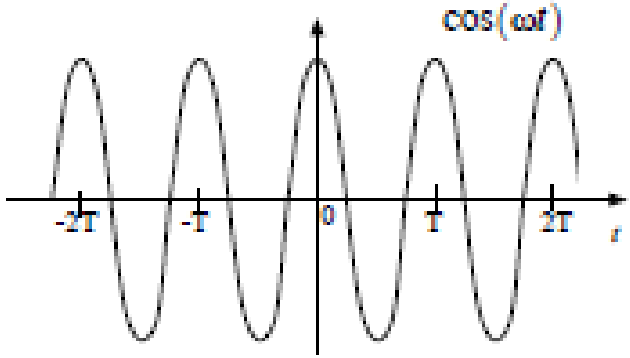
that it is really the radian frequency, and the frequency is actually f .

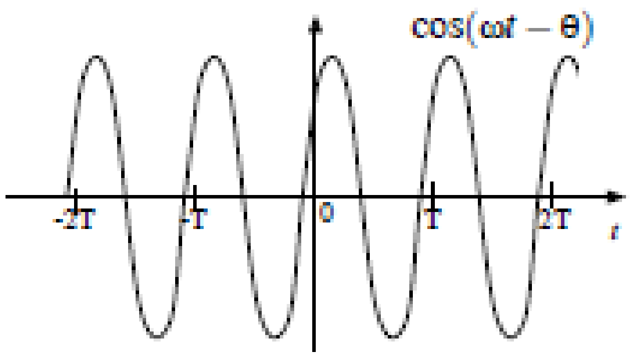
Contd…

- The period of the sinuoid is

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

  with the units of seconds.
- The *phase* or *phase angle* of the signal is θ, given in radians.

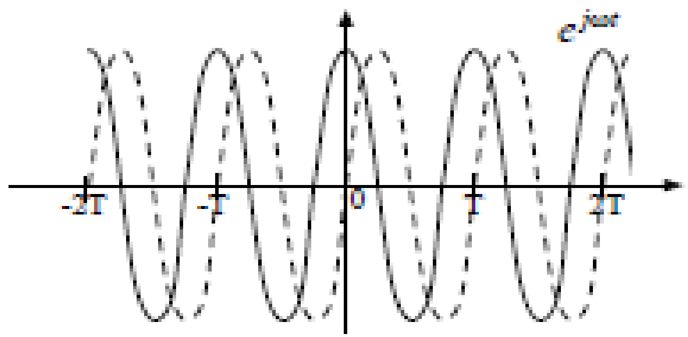




Contd…

- The Euler relation defines $e^{j\phi} = \cos\phi + j\sin\phi$.
- A complex sinusoid is

$$Ae^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + jA\sin(\omega t+\theta).$$



- Real sinusoid can be represented as the real part of a complex sinusoid

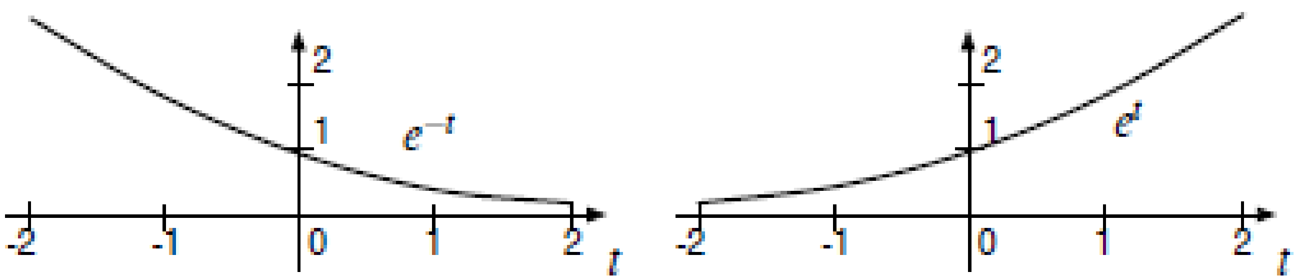
$$\Re\{Ae^{j(\omega t+\theta)}\} = A\cos(\omega t+\theta)$$

Exponential Signals

- An exponential signal is given by

$$x(t) = e^{\sigma t}$$

- If $\sigma < 0$ this is *exponential decay*.
- If $\sigma > 0$ this is *exponential growth*.

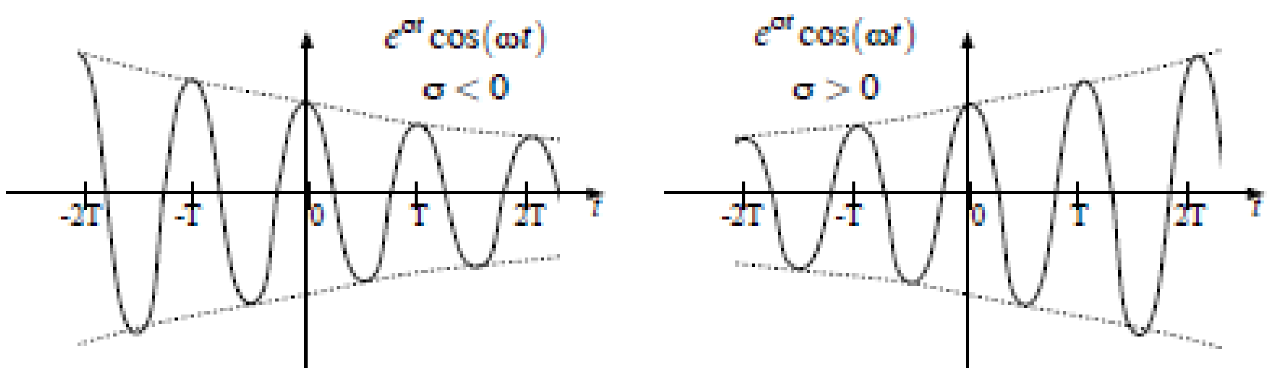


Exponentially damped signals

- A damped or growing sinusoid is given by

$$x(t) = e^{\sigma t}\cos(\omega t + \theta)$$

- Exponential growth ($\sigma > 0$) or decay ($\sigma < 0$), modulated by a sinusoid.

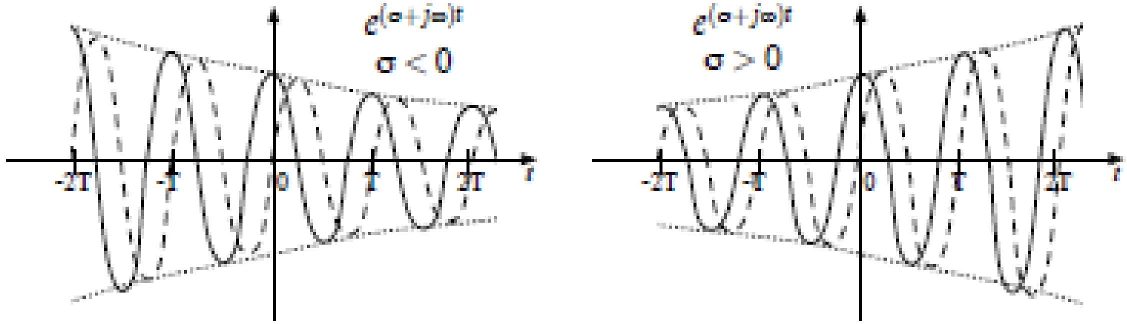


Contd…

- A complex exponential signal is given by

$$e^{(\sigma+j\omega)t+j\theta} = e^{\sigma t}(\cos(\omega t+\theta)+i\sin(\omega t+\theta))$$

- A exponential growth or decay, modulated by a complex sinusoid.
- Includes all of the previous signals as special cases.

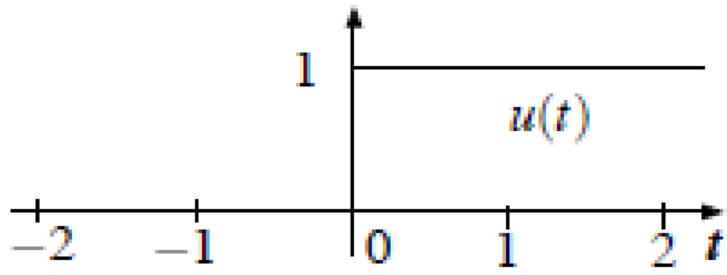


Unit Step Signal

- The *unit step function* $u(t)$ is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- Also known as the *Heaviside step function*.
- Alternate definitions of value exactly at zero, such as $1/2$.



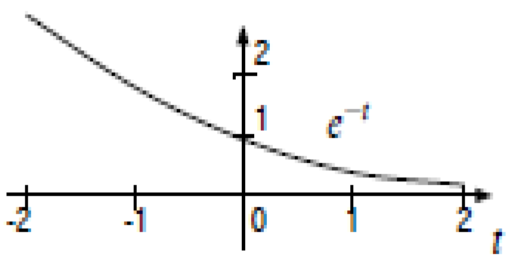
Example

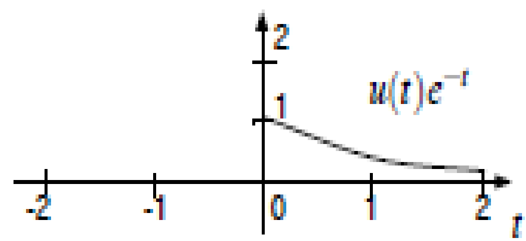
- Extracting part of another signal. For example, the piecewise-defined signal

$$x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

can be written as

$$x(t) = u(t)e^{-t}$$





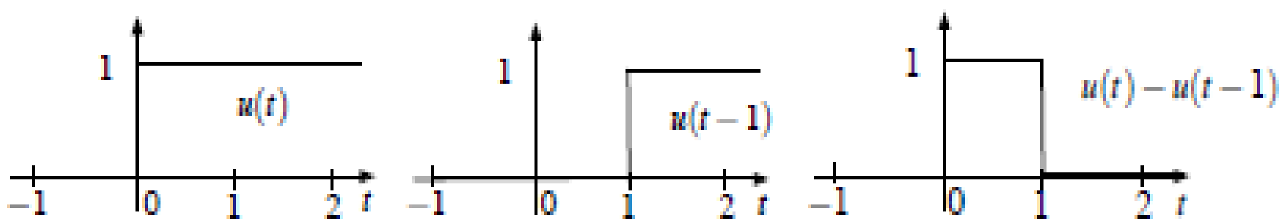
Example

- Combinations of unit steps to create other signals. The offset rectangular signal

$$x(t) = \begin{cases} 0, & t \geq 1 \\ 1, & 0 \leq t < 1 \\ 0, & t < 0 \end{cases}$$

can be written as

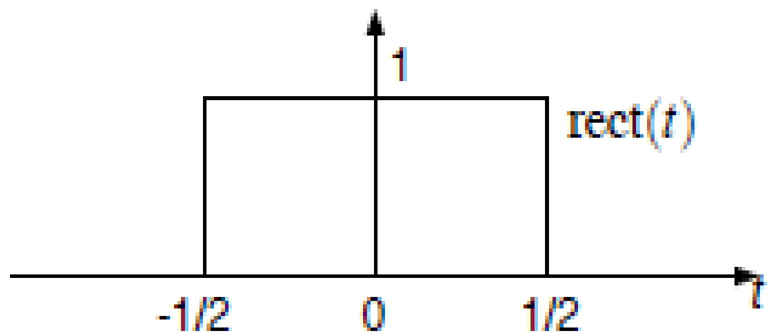
$$x(t) = u(t) - u(t-1).$$



Example

Unit rectangle signal:

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$



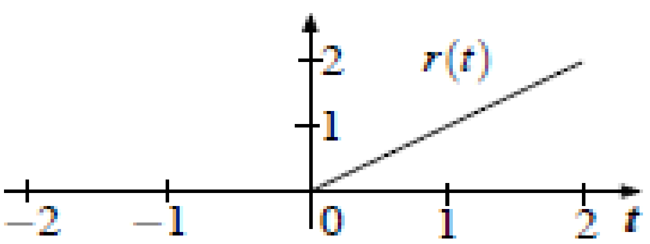
Unit Ramp Signal

- The *unit ramp* is defined as

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- The unit ramp is the integral of the unit step,

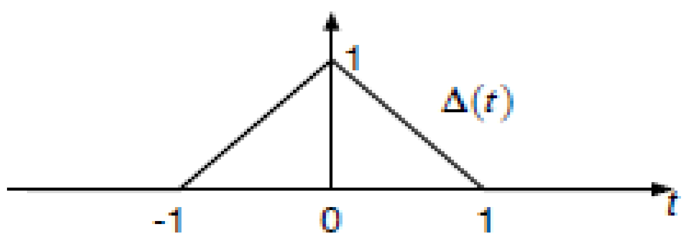
$$r(t) = \int_{-\infty}^{t} u(\tau) d\tau$$



Unit Triangular Signal

Unit Triangle Signal

$$\Delta(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



Linearity

- A linear system obeys the following

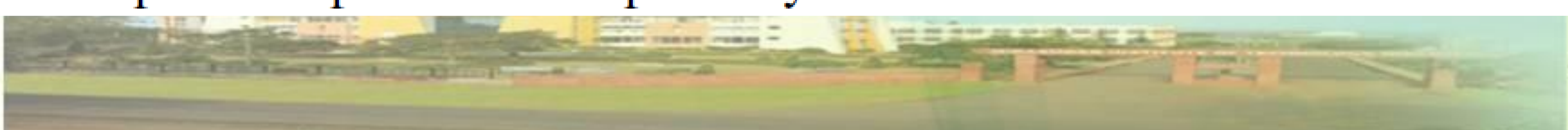
$$\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t) \tag{9.28}$$

  where the inputs are applied together or applied individually and combined via α and β later

- The <u>squarer is nonlinear</u> by virtue of the fact that

$$\begin{aligned} y(t) &= [\alpha x_1(t) + \beta x_2(t)]^2 \\ &= \alpha^2 x_1^2(t) + 2\alpha\beta x_1(t) x_2(t) + \beta^2 x_2(t) \end{aligned}$$

  produces a cross term which does not exist when the two inputs are processed separately and then combined



Time-Invariance

- A time invariant system obeys the following

$$x(t - t_0) \rightarrow y(t - t_0) \tag{9.26}$$

for any t_0

- Both the squarer and integrator are time invariant
- The system

$$y(t) = \cos(\omega_c t)x(t) \tag{9.27}$$

is not time invariant as the gain changes as a function of time

- The <u>integrator is linear</u> since

$$y(t) = \int_{-\infty}^{t} [\alpha x_1(\tau) + \beta x_2(\tau)]\,d\tau$$

$$= \alpha \int_{-\infty}^{t} x_1(\tau)\,d\tau + \beta \int_{-\infty}^{t} x_2(\tau)\,d\tau$$

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Queries ...?