Fourier representation for signals

Introduction:

Fourier series has long provided one of the principal methods of analysis for mathematical physics, engineering, and signal processing. It has spurred generalizations and applications that continue to develop right up to the present. While the original theory of Fourier series applies to periodic functions occurring in wave motion, such as with light and sound, its generalizations often relate to wider settings, such as the time-frequency analysis underlying the recent theories of wavelet analysis and local trigonometric analysis.

• In 1807, Jean Baptiste Joseph Fourier Submitted a paper of using trigonometric series to represent "any" periodic signal.

• But Lagrange rejected it!

• In 1822, Fourier published a book "The Analytical Theory of Heat" Fourier‟s main contributions: Studied vibration, heat diffusion, etc. and found that a series of harmonically related sinusoids is useful in representing the temperature distribution through a body.

• He also claimed that "any" periodic signal could be represented by Fourier series. These arguments were still imprecise and it remained for P. L. Dirichlet in 1829 to provide precise conditions under which a periodic signal could be represented by a FS.

• He however obtained a representation for aperiodic signals i.e., Fourier integral or transform

• Fourier did not actually contribute to the mathematical theory of Fourier series.

• Hence out of this long history what emerged is a powerful and cohesive framework for the analysis of continuous- time and discrete-time signals and systems and an extraordinarily broad array of existing and potential application.

The Response of LTI Systems to Complex Exponentials:

We have seen in previous chapters how advantageous it is in LTI systems to represent signals as a linear combinations of basic signals having the following properties.

Key Properties: for Input to LTI System

1. To represent signals as linear combinations of basic signals.
2. Set of basic signals used to construct a broad class of signals.
3. The response of an LTI system to each signal should be simple enough in structure.
4. It then provides us with a convenient representation for the response of the system.
5. Response is then a linear combination of basic signal.

Eigenfunctions and Values :

• One of the reasons the Fourier series is so important is that it represents a signal in terms of eigen functions of LTI systems.

• When I put a complex exponential function like x(t) = ejωt through a linear time-invariant system, the output is y(t) = H(s)x(t) = H(s) ejωt where H(s) is a complex constant (it does not depend on time).

• The LTI system scales the complex exponential ejωt .

Historical background

There are antecedents to the notion of Fourier series in the work of Euler and D. Bernoulli on vibrating strings, but the theory of Fourier series truly began with the profound work of Fourier on heat conduction at the beginning of the century. In [5], Fourier deals with the problem of describing the evolution of the temperature of a thin wire of length X. He proposed that the initial temperature could be expanded in a series of sine functions:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx. \quad (2)$$

The Fourier sine series, defined in Eq.s (1) and (2), is a special case of a more general concept: the Fourier series for a *periodic function.* Periodic functions arise in the study of wave motion, when a basic waveform repeats itself periodically. Such periodic waveforms occur in musical tones, in the plane waves of electromagnetic vibrations, and in the vibration of strings. These are just a few examples. Periodic effects also arise in the motion of the planets, in ac-electricity, and (to a degree) in animal heartbeats.

A function f is said to have period P if $f(x + P) = f(x)$ for all x. For notational simplicity, we shall restrict our discussion to functions of period 2π. There is no loss of generality in doing so, since we can always use a simple change of scale $x = (P/2\pi)t$ to convert a function of period P into one of period 2π.

If the function f has period 2π, then its *Fourier series* is

$$c_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} \quad (4)$$

with *Fourier coefficients* c_0, a_n, and b_n defined by the integrals

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \quad (5)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad (6)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \quad (7)$$

The following relationships can be readily established, and will be used in subsequent sections for derivation of useful formulas for the unknown Fourier coefficients, in both time and frequency domains.

$$\int_0^T \sin(kw_0 t)dt = \int_0^T \cos(kw_0 t)dt \tag{1}$$
$$= 0$$

$$\int_0^T \sin^2(kw_0 t)dt = \int_0^T \cos^2(kw_0 t)dt \tag{2}$$
$$= \frac{T}{2}$$

$$\int_0^T \cos(kw_0 t)\sin(gw_0 t)dt = 0 \tag{3}$$

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where

$$w_0 = 2\pi f \tag{6}$$

$$f = \frac{1}{T} \tag{7}$$

where f and T represents the frequency (in cycles/time) and period (in seconds) respectively. Also, k and g are integers.

A periodic function $f(t)$ with a period T should satisfy the following equation

$$f(t+T) = f(t) \tag{8}$$

Example 1

Prove that

$$\int_0^\pi \sin(kw_0 t) = 0$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k is an integer.

Solution

Let

$$A = \int_0^T \sin(kw_0 t)dt \tag{9}$$

$$= -\left(\frac{1}{kw_0}\right)\left[\cos(kw_0 t)\right]_0^T$$

$$A = \left(\frac{-1}{kw_0}\right)\left[\cos(kw_0 T) - \cos(0)\right] \tag{10}$$

$$= \left(\frac{-1}{kw_0}\right)\left[\cos(k2\pi) - 1\right]$$

$$= 0$$

Example 2

Prove that

$$\int_0^{\pi} \sin^2(kw_0 t) = \frac{T}{2}$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k is an integer.

Solution

Let

$$B = \int_0^T \sin^2(kw_0 t)dt \tag{11}$$

Recall

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{12}$$

Thus,

$$B = \int_0^T \left[\frac{1}{2} - \frac{1}{2}\cos(2kw_0 t)\right]dt \tag{13}$$

$$= \left[\left(\frac{1}{2}\right)t - \left(\frac{1}{2}\right)\left(\frac{1}{2kw_0}\right)\sin(2kw_0 t)\right]_0^T$$

$$B = \left[\frac{T}{2} - \frac{1}{4kw_0}\sin(2kw_0 T)\right] - [0] \tag{14}$$

$$= \frac{T}{2} - \left(\frac{1}{4kw_0}\right)\sin(2k * 2\pi)$$

$$= \frac{T}{2}$$

Example 3

Prove that

$$\int_0^{\pi} \sin(gw_0 t)\cos(kw_0 t) = 0$$

for

$$w_0 = 2\pi f$$

$$f = \frac{1}{T}$$

and k and g are integers.

Solution

Let

$$C = \int_0^T \sin(gw_0 t)\cos(kw_0 t)dt \tag{15}$$

Recall that

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha) \tag{16}$$

Hence,

$$C=\int_0^T\left[\sin[(g+k)w_0t]-\sin(kw_0t)\cos(gw_0t)\right]dt \tag{17}$$

$$=\int_0^T\sin[(g+k)w_0t]dt-\int_0^T\sin(kw_0t)\cos(gw_0t)dt \tag{18}$$

From Equation (1),

$$\int_0^T[\sin(g+k)w_0t]dt=0$$

then

$$C=0-\int_0^T\sin(kw_0t)\cos(gw_0t)dt \tag{19}$$

Adding Equations (15), (19), $2C=\int_0^T\sin(gw_0t)\cos(kw_0t)dt-\int_0^T\sin(kw_0t)\cos(gw_0t)dt$

$$=\int_0^T\sin[(gw_0t)-(kw_0t)]dt=\int_0^T\sin[(g-k)w_0t]dt \tag{20}$$

$2C=0$, since the right side of the above equation is zero (see Equation 1). Thus,

$$C=\int_o^T\sin(gw_0t)\cos(kw_0t)dt=0 \tag{21}$$

$$=0$$

Example 4

Prove that

$$\int_0^T\sin(kw_0t)\sin(gw_0t)dt=0$$

for

$w_0=2\pi f$

$f=\frac{1}{T}$

$k, g=$ integers

Solution

Let $D=\int_0^T\sin(kw_0t)\sin(gw_0t)dt$ (22)

Since

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

or

$$\sin(\alpha)\sin(\beta)=\cos(\alpha)\cos(\beta)-\cos(\alpha+\beta)$$

Thus,

$$D=\int_0^T\cos(kw_0t)\cos(gw_0t)dt-\int_0^T\cos[(k+g)w_0t]dt \tag{23}$$

From Equation (1)

$$\int_0^T \cos[(k+g)w_0 t]dt = 0$$

then

$$D = \int_0^T \cos(kw_0 t)\cos(gw_0 t)dt - 0$$

Adding Equations (23), (26)

$$2D = \int_0^T \sin(kw_0 t)\sin(gw_0 t) + \int_0^T \cos(kw_0 t)\cos(gw_0 t)dt$$

$$= \int_0^T \cos[kw_0 t - gw_0 t]dt$$

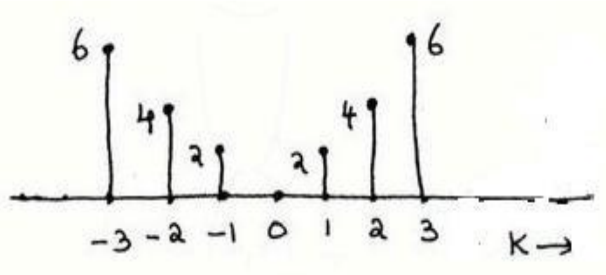
$$= \int_0^T \cos[(k-g)w_0 t]dt$$

2D = 0, since the right side of the above equation is zero (see Equation 1). Thus,

$$D \equiv \int \sin(kw_0 t)\sin(gw_0 t)dt = 0$$

<u>Recommended Questions</u>

1. Find x(t) if the Fourier series coefficients are shown in fig. The phase spectrum is a null spectrum.



2. Prove the following properties of Fourier series. i) Convolution property ii) Parsevals relationship.

3. Find the DTFS harmonic function of x(n) = A Cos (2πn/No). Plot the magnitude and phase spectra.

4. Determine the complex Fourier coefficients for the signal. X(t)= {t+1 for -1 < t< 0; 1-t for 0 < t < 1 which repeats periodically with T=2 units. Plot the amplitude and phase spectra of the signal.

5. State and prove the following of Fourier transform. i) Time shifting property ii) Time differentiation property iii) Parseval's theorem.