Module 2: Time-domain representations for LTI systems – 1

Time-domain representations for LTI systems – 1: Convolution, impulse response representation, Convolution Sum and Convolution Integral.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

REFERENCE BOOKS :

1. **Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab,** "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002

2. **H. P Hsu, R. Ranjan**, "Signals and Systems", Scham‟s outlines, TMH, 2006

3. **B. P. Lathi**, "Linear Systems and Signals", Oxford University Press, 2005

4. **Ganesh Rao and Satish Tunga**, "Signals and Systems", Sanguine Technical Publishers, 2004

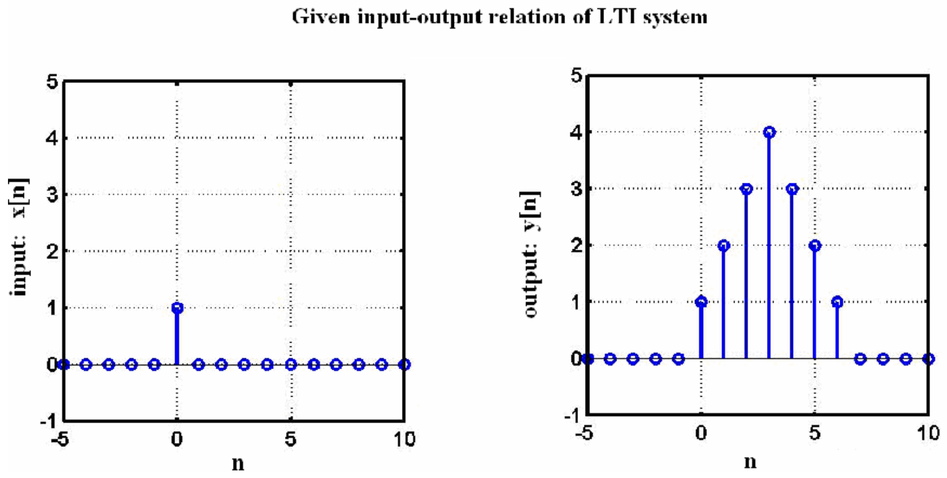
Module 2
Time-domain representations for LTI systems – 1

2.1 Introduction:

The Linear time invariant (LTI) system:

Systems which satisfy the condition of linearity as well as time invariance are known as linear time invariant systems. Throughout the rest of the course we shall be dealing with LTI systems. If the output of the system is known for a particular input, it is possible to obtain the output for a number of other inputs. We shall see through examples, the procedure to compute the output from a given input-output relation, for LTI systems.

Example – I:



2.1.1 <u>Convolution</u>:

A continuous time system as shown below, accepts a continuous time signal x(t) and gives out a transformed continuous time signal y(t).

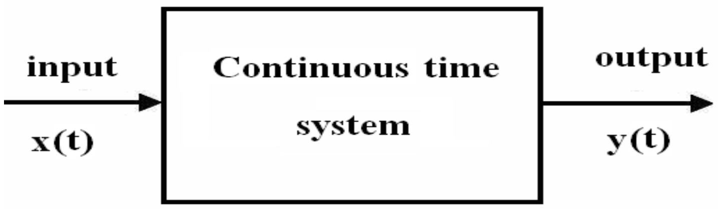


Figure 1: The continuous time system

Some of the different methods of representing the continuous time system are:

i) Differential equation
ii) Block diagram
iii) Impulse response
iv) Frequency response
v) Laplace-transform

vi) Pole-zero plot

It is possible to switch from one form of representation to another, and each of the representations is complete. Moreover, from each of the above representations, it is possible to obtain the system properties using parameters as: stability, causality, linearity, invertibility etc. We now attempt to develop the convolution integral.

2.2 Impulse Response

The impulse response of a continuous time system is defined as the output of the system when its input is an unit impulse, $\delta(t)$. Usually the impulse response is denoted by $h(t)$.

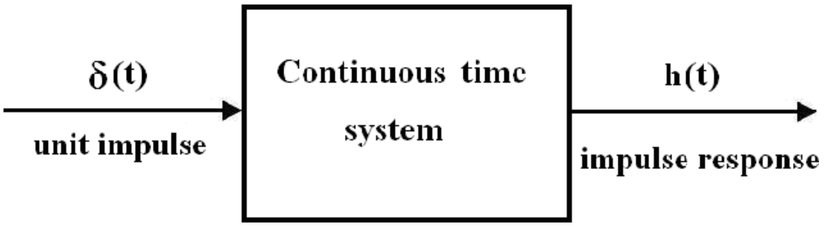
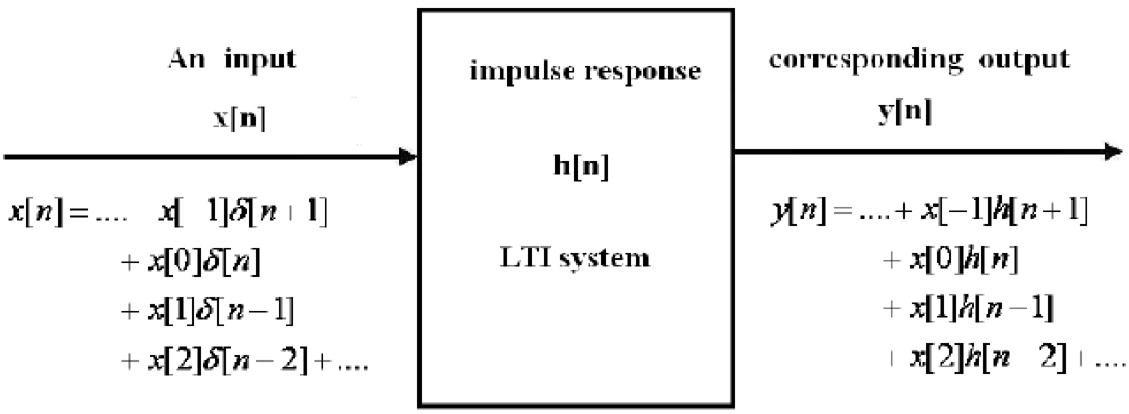
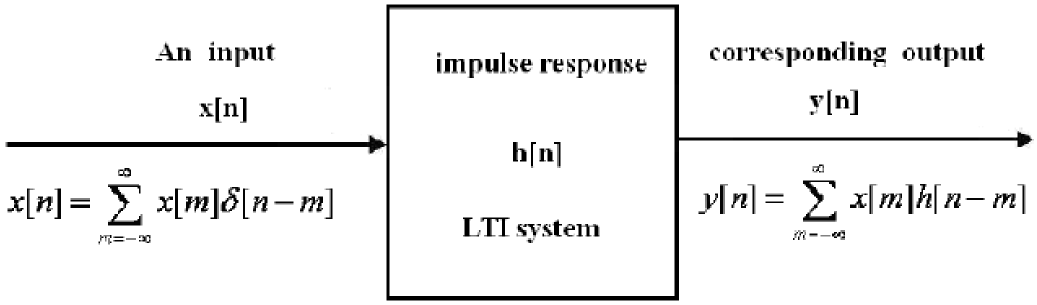


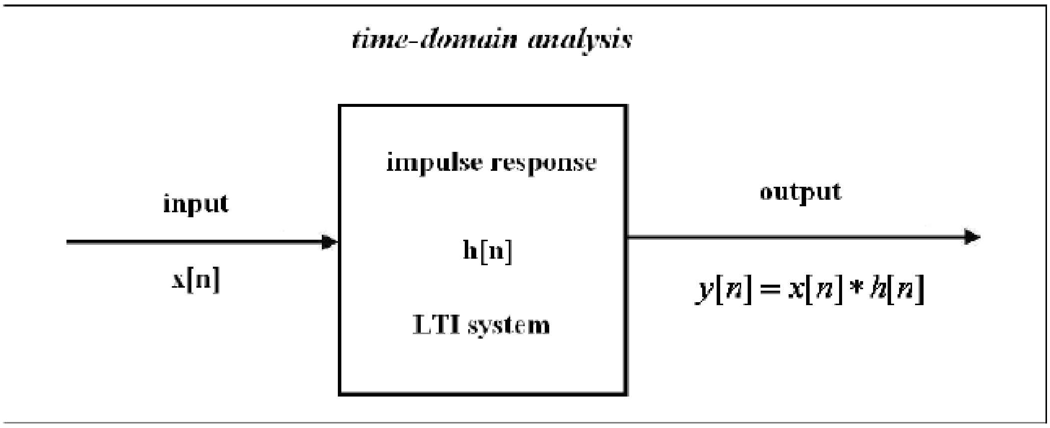
Figure 2: The impulse response of a continuous time system

2.3 Convolution Sum:

We now attempt to obtain the output of a digital system for an arbitrary input x[n], from the knowledge of the system impulse response h[n].







$$y[n]=x[n]*h[n]$$

Methods of evaluating the convolution sum:

Given the system impulse response h[n], and the input x[n], the system output y[n], is given by the convolution sum:

$$y[n]=\sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Problem:
To obtain the digital system output y[n], given the system impulse response h[n], and the system input x[n] as:

$$h[n]=[1,\ -1.5,\ 3]$$

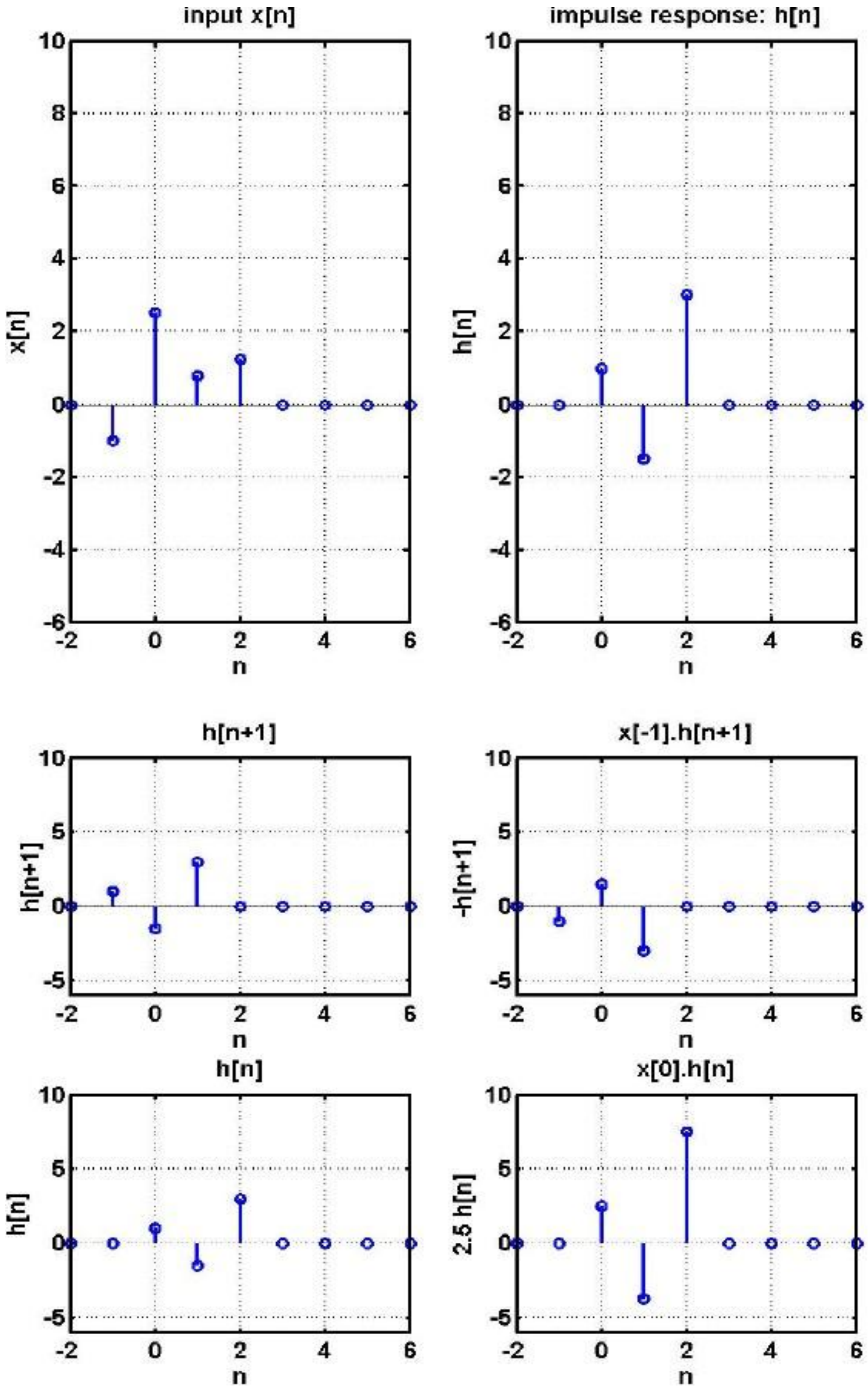
$$x[n]=[-1\ ,\ \underset{\uparrow}{2.5},\ 0.8,\ 1.25]$$

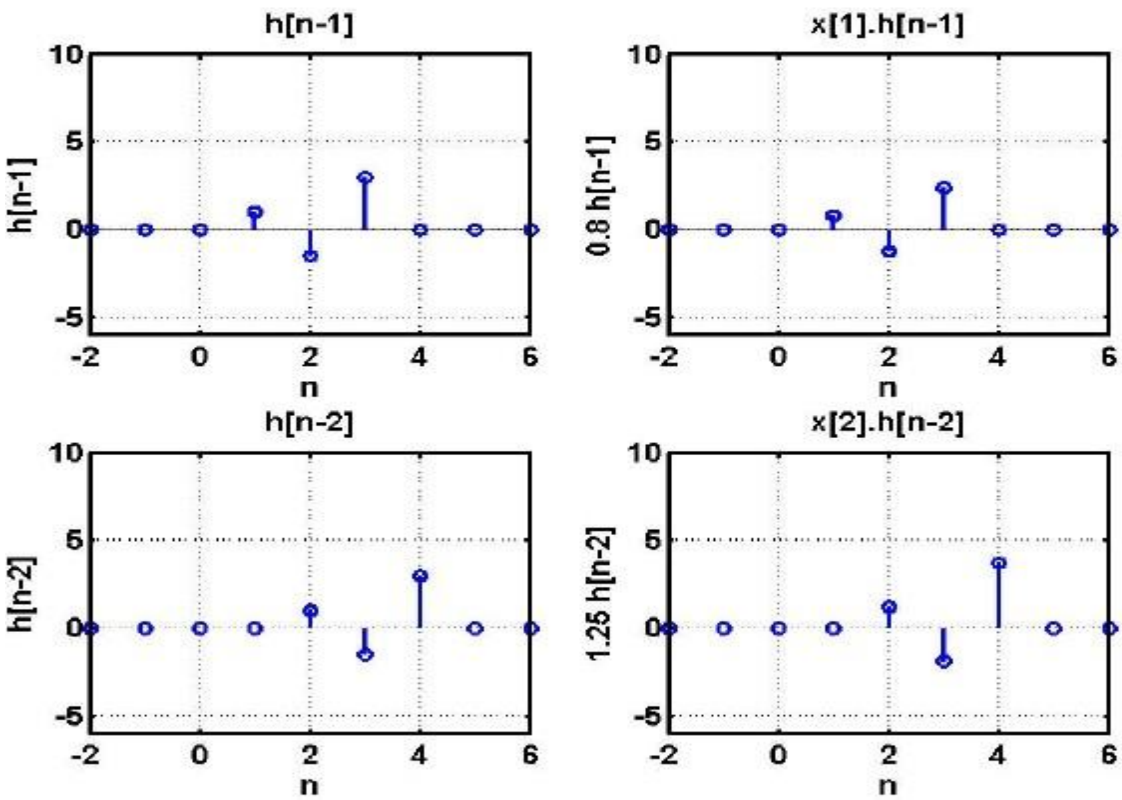
-1 4 -5.95 7.55 0.525 3.75

1. Evaluation as the weighted sum of individual responses

The convolution sum of equation (…), can be equivalently represented as:

$$y[n] = \ldots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \ldots$$





Convolution as matrix multiplication:

Given

$$x[n] = [x_1 \quad x_2 \quad \ldots \quad x_L] \qquad \text{starting from } N_X$$

and

$$h[n] = [h_1 \quad h_2 \quad \ldots \quad h_M] \qquad \text{starting from } N_H$$

Step 1: Length of convolved sequence is NUM = (L+M-1)

Step 2: The convolved sequence starts at $i = N_X + N_H$

Step 3: The convolution is given by the following matrix multiplication

$$\begin{bmatrix} y[i] \\ y[i+1] \\ y[i+2] \\ y[i+3] \\ y[i+4] \\ y[i+5] \\ . \\ . \end{bmatrix} = \begin{bmatrix} x_1 & 0 & . & . & 0 \\ x_2 & x_1 & . & . & 0 \\ x_3 & x_2 & . & . & 0 \\ . & x_3 & . & . & 0 \\ . & . & . & . & x_1 \\ x_L & . & . & . & x_2 \\ 0 & x_L & . & . & . \\ 0 & 0 & . & . & x_L \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ . \\ . \\ h_M \end{bmatrix} = \begin{bmatrix} h_1 & 0 & . & . & 0 \\ h_2 & h_1 & . & . & 0 \\ h_3 & h_2 & . & . & 0 \\ . & h_3 & . & . & 0 \\ . & . & . & . & h_1 \\ h_M & . & . & . & h_2 \\ 0 & h_M & . & . & . \\ 0 & 0 & . & . & h_M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_L \end{bmatrix}$$

The dimensions of the above matrices are:

$$[NUM \quad by \quad 1] = [NUM \quad by \quad M][M \quad by \quad 1] = [NUM \quad by \quad L][L \quad by \quad 1]$$

For the given example:

x[n] is of length L=4, and starts at $N_X = -1$

h[n] is of length M=3 and starts at $N_H = 0$

Step 1: Length of convolved sequence is NUM = (L+M-1)=6

Step 2: The convolved sequence starts at i=(-1+0)=(-1)

$$\begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2.5 & -1 & 0 \\ 0.8 & 2.5 & -1 \\ 1.25 & 0.8 & 2.5 \\ 0 & 1.25 & 0.8 \\ 0 & 0 & 1.25 \end{bmatrix} \begin{bmatrix} 1 \\ -1.5 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -5.95 \\ 7.55 \\ 0.525 \\ 3.75 \end{bmatrix}$$

or

$$\begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.5 & 1 & 0 & 0 \\ 3 & -1.5 & 1 & 0 \\ 0 & 3 & -1.5 & 1 \\ 0 & 0 & 3 & -1.5 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2.5 \\ 0.8 \\ 1.25 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -5.95 \\ 7.55 \\ 0.525 \\ 3.75 \end{bmatrix}$$

Evaluation using graphical representation:

Another method of computing the convolution is through the direct computation of each value of the output y[n]. This method is based on evaluation of the convolution sum for a single value of *n*, and varying *n* over all possible values.

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Step 1: Sketch x[m]

Step 2: Sketch h[-m]

Step 3: Compute y[0] using:

$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[-m]$$

which is the 'sum of the product of the two signals x[m] & h[-m]'

Step 4: Sketch h[1-m], which is right shift of h[-m] by 1.

Step 5: Compute y[1] using:

$$y[1] = \sum_{m=-\infty}^{\infty} x[m]h[1-m]$$

which is the 'sum of the product of the two signals x[m] & h[1-m]'

Step 6: Sketch h[2-m], which is right shift of h[-m] by 2.

Step 7: Compute y[2] using:

$$y[2] = \sum_{m=-\infty}^{\infty} x[m]h[2-m]$$

which is the 'sum of the product of the two signals x[m] & h[2-m]'

Step 8: Proceed this way until all possible values of y[n], for positive 'n' are computed

Step 9: Sketch h[-1-m], which is left shift of h[-m] by 1.

Step 10: Compute y[-1] using:

$$y[-1] = \sum_{m=-\infty}^{\infty} x[m]h[-1-m]$$

which is the 'sum of the product of the two signals x[m] & h[-1-m]'

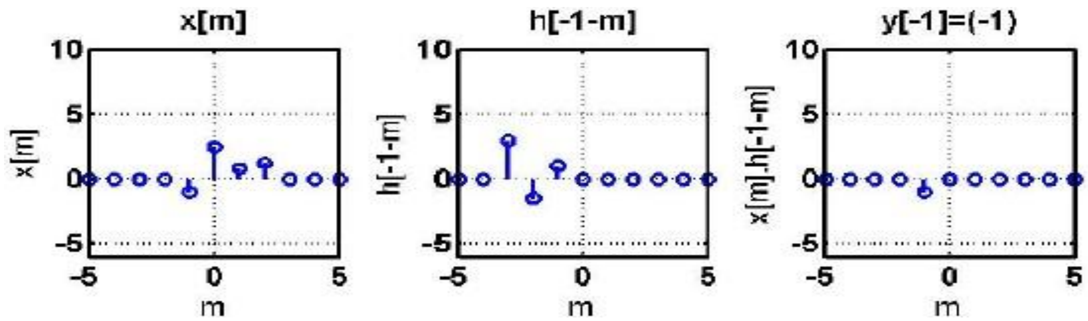
Step 11: Sketch h[-2-m], which is left shift of h[-m] by 2.

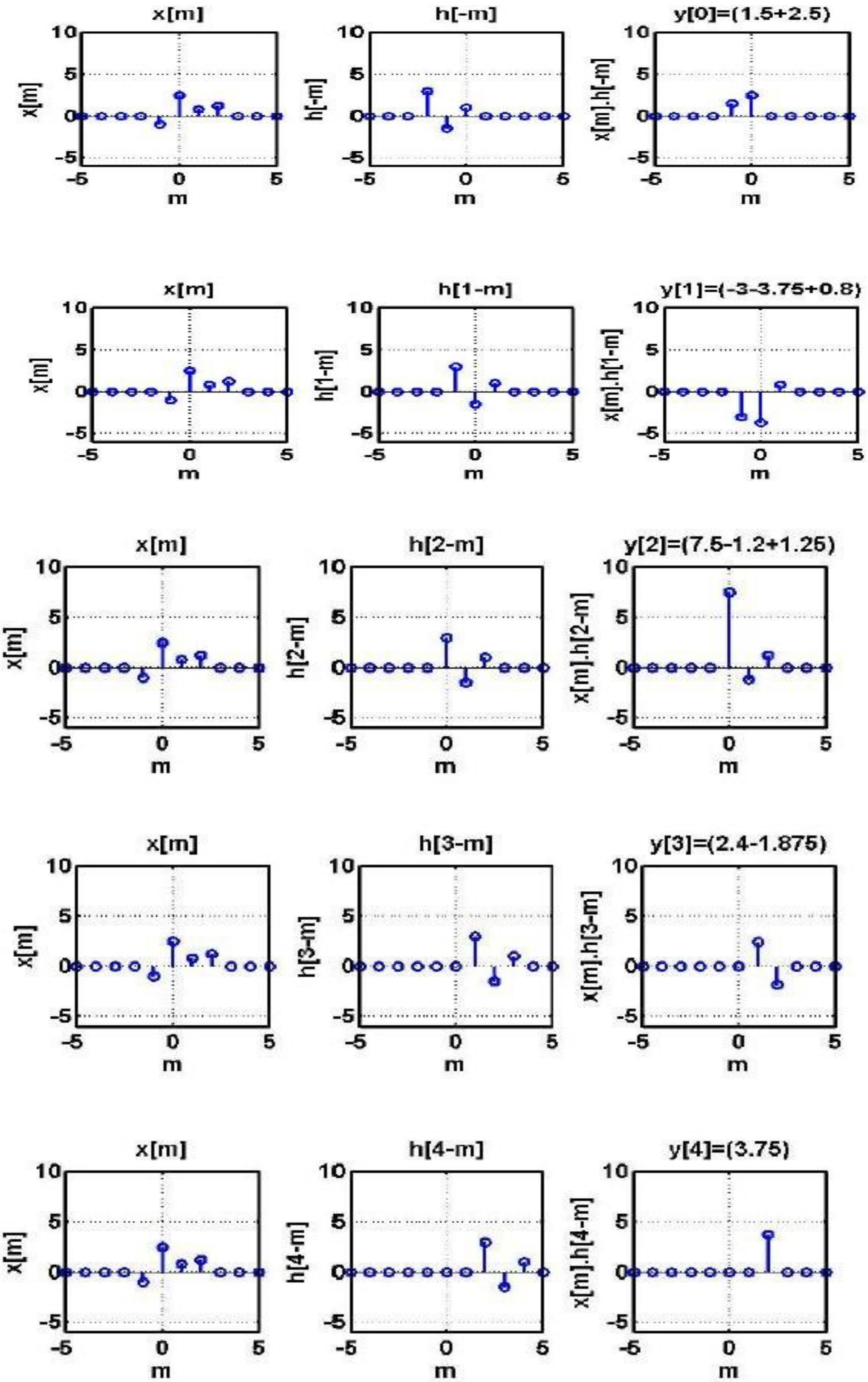
Step 12: Compute y[-2] using:

$$y[-2] = \sum_{m=-\infty}^{\infty} x[m]h[-2-m]$$

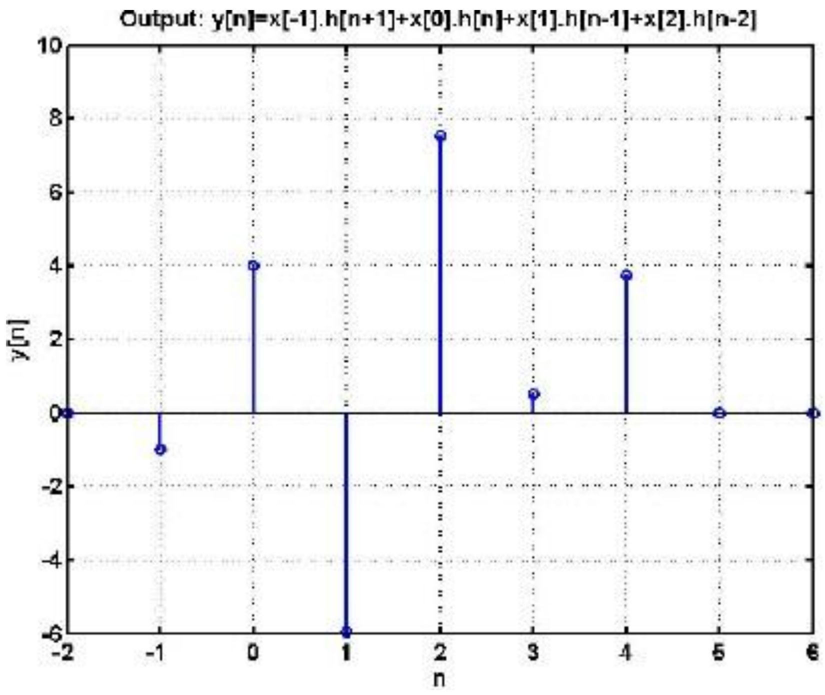
which is the 'sum of the product of the two signals x[m] & h[-2-m]'

Step 13: Proceed this way until all possible values of y[n], for negative 'n' are computed





x[m]
h[-m]
y[0]=(1.5+2.5)
x[m]
h[1-m]
y[1]=(-3-3.75+0.8)
x[m]
h[2-m]
y[2]=(7.5-1.2+1.25)
x[m]
h[3-m]
y[3]=(2.4-1.875)
x[m]
h[4-m]
y[4]=(3.75)
x[m].h[-m]
x[m].h[1-m]
x[m].h[2-m]
x[m].h[3-m]
x[m].h[4-m]
m



Evaluation from direct convolution sum:

While small length, finite duration sequences can be convolved by any of the above three methods, when the sequences to be convolved are of infinite length, the convolution is easier performed by direct use of the „convolution sum" of equation (…).

$$since: \quad u[m] = \begin{cases} 0 & for \quad m < 0 \\ 1 & for \quad m \geq 0 \end{cases}$$

$$\begin{aligned} u[n-m] &= \begin{cases} 0 & for \quad (n-m) < 0 \\ 1 & for \quad (n-m) \geq 0 \end{cases} \\ &= \begin{cases} 0 & for \quad (-m) < n \\ 1 & for \quad (-m) \geq n \end{cases} \\ &= \begin{cases} 0 & for \quad m > n \\ 1 & for \quad m \leq n \end{cases} \end{aligned}$$

Example: A system has impulse response $h[n] = \exp(-0.8n)u[n]$. Obtain the unit step response.
Solution:

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[m] \\ &= \sum_{m=-\infty}^{\infty} \{\exp(-0.8(m))u[m]\}\{u[n-m]\} \end{aligned}$$

$$= \sum_{m=0}^{\infty} \{\exp(-0.8(m))\}\{u[n-m]\}$$

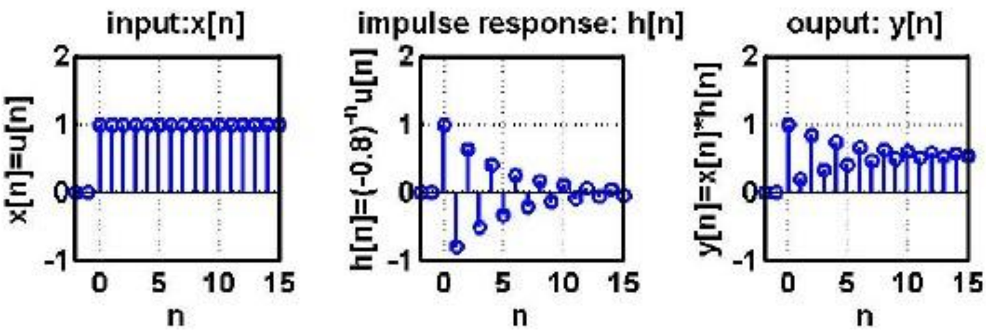
$$= \sum_{m=0}^{n} \{\exp(-0.8(m))\}$$

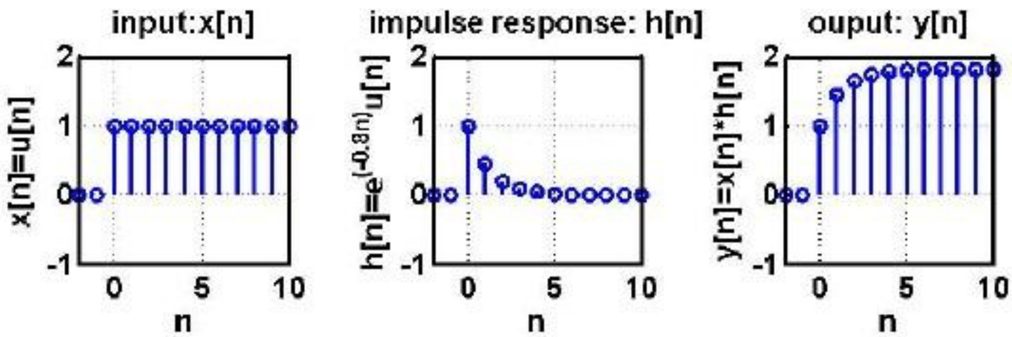
$$= \sum_{m=0}^{n} \{\exp(-0.8(m))\}$$

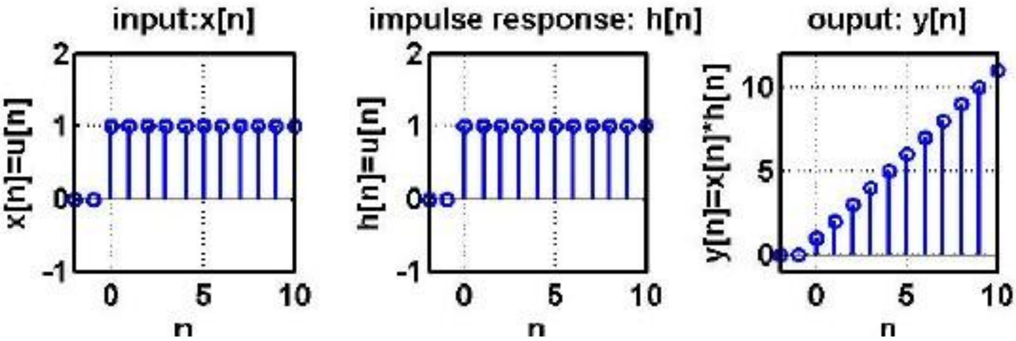
$$= \frac{\left(1-(-0.8)^{n+1}\right)}{\left(1-(-0.8)\right)}$$

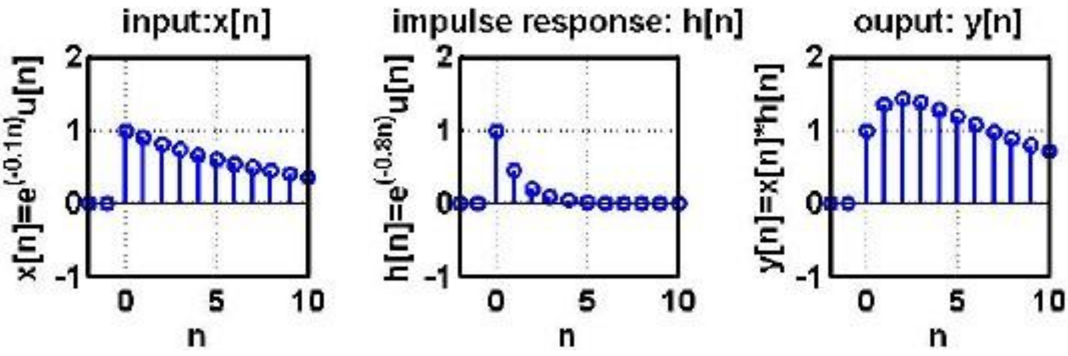
$$y[n] = \sum_{m=-\infty}^{\infty} \left\{(-0.8)^{(n-m)} u[n-m]\right\}$$

$$= \sum_{m=0}^{\infty} \{\exp(-0.8(n-m)) u[n-m]\}$$

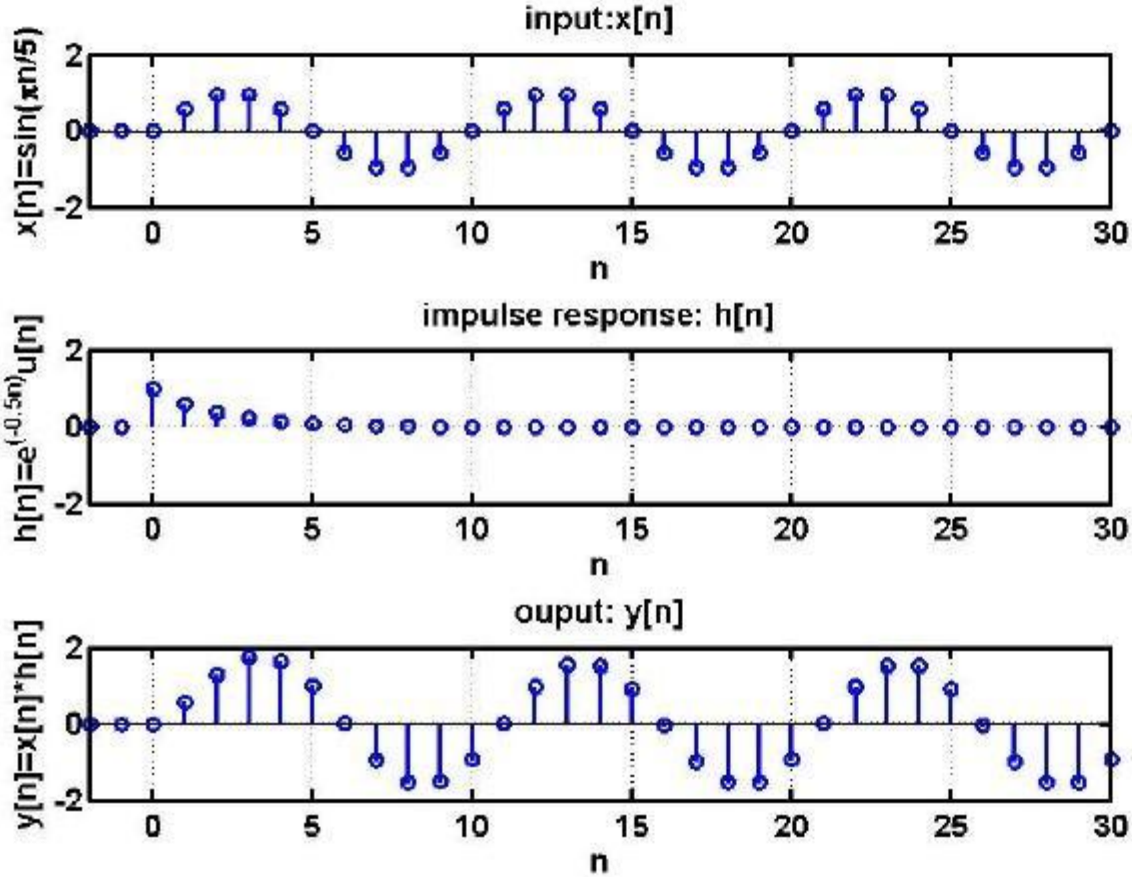




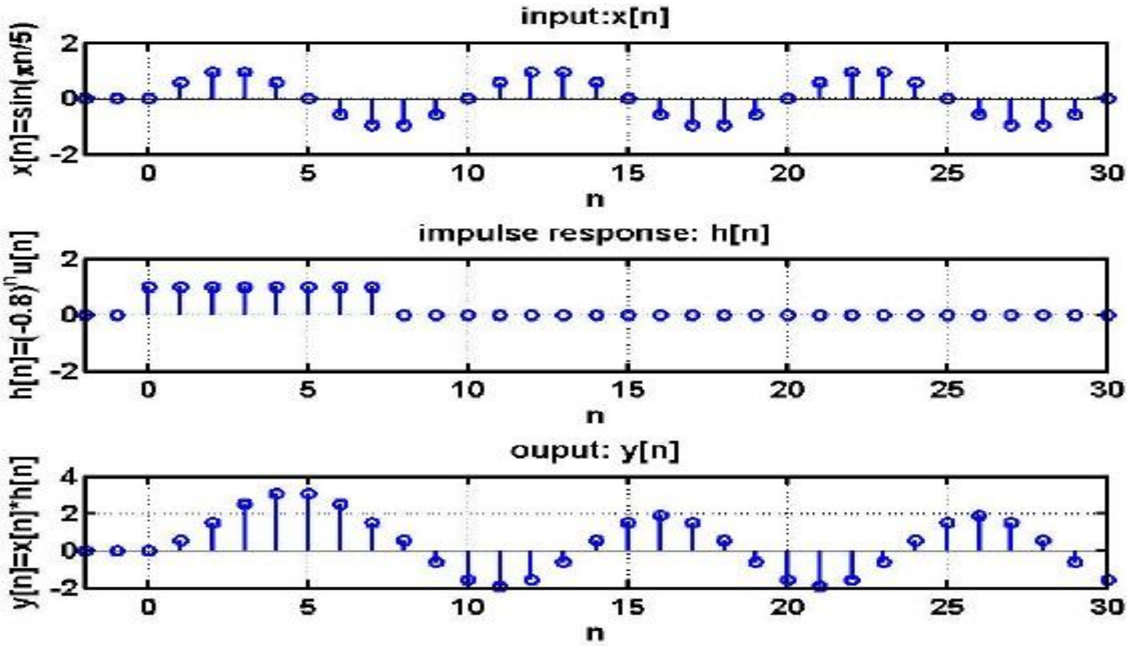




input:x[n]
x[n]=e^(-0.1n)u[n]
impulse response: h[n]
h[n]=e^(-0.8n)u[n]
ouput: y[n]
y[n]=x[n]*h[n]
n



input:x[n]
x[n]=sin(πn/5)
impulse response: h[n]
h[n]=e^(-0.5n)u[n]
ouput: y[n]
y[n]=x[n]*h[n]
n



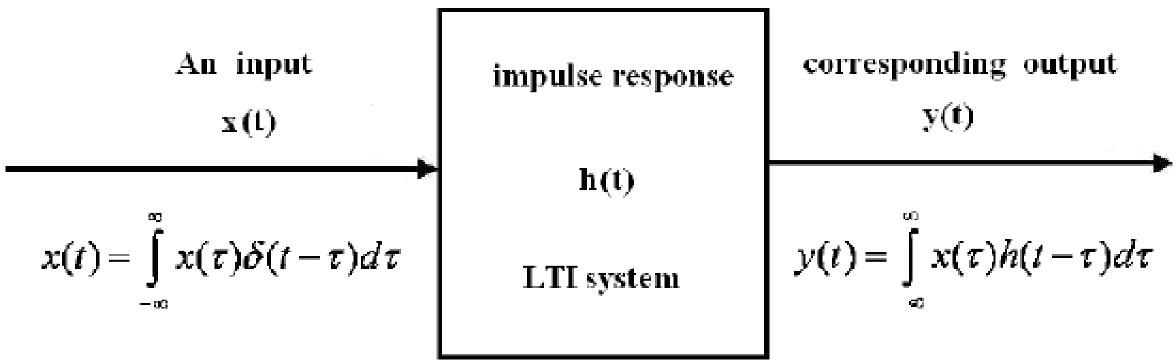
input:x[n]
x[n]=sin(πn/5)
impulse response: h[n]
h[n]=(-0.8)^n u[n]
ouput: y[n]
y[n]=x[n]*h[n]
n

2.4 Convolution Integral:

We now attempt to obtain the output of a continuous time/Analog digital system for an arbitrary input x(t), from the knowledge of the system impulse response h(t), and the properties of the impulse response of an LTI system.

The output y(t) is given by, using the notation, y(t)=R{x(t)}.

$$
\begin{aligned}
y(t) &= R\{x(t)\} \\
&= R\left\{ \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \right\} \\
&= \int_{-\infty}^{\infty} x(\tau)R\{\delta(t-\tau)\}\,d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= x(t) * h(t)
\end{aligned}
$$



Methods of evaluating the convolution integral: (Same as Convolution sum)

Given the system impulse response h(t), and the input x(t), the system output y(t), is given by the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Some of the different methods of evaluating the convolution integral are: Graphical representation, Mathematical equation, Laplace-transforms, Fourier Transform, Differential equation, Block diagram representation, and finally by going to the digital domain.

Recommended Questions

1. Show that if x(n) is input of a linear time invariant system having impulse response h(n), then the output of the system due to x(n) is

$$y(n)=\sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

2. Use the definition of convolution sum to prove the following properties
1. x(n) * [h(n)+g(n)]=x(n)*h(n)+x(n)*g(n) (Distributive Property)
2. x(n) * [h(n)*g(n)]=x(n)*h(n) *g(n) (Associative Property)
3. x(n) * h(n) =h(n) * x(n) (Commutative Property)
3. Prove that absolute summability of the impulse response is a necessary condition for stability of a discrete time system.
4. Compute the convolution y(t)= x(t)*h(t) of the following pair of signals:

$$(a)\quad x(t)=\begin{cases}1 & -a<t\le a\\ 0 & \text{otherwise}\end{cases},\quad h(t)=\begin{cases}1 & -a<t\le a\\ 0 & \text{otherwise}\end{cases}$$

$$(b)\quad x(t)=\begin{cases}t & 0<t\le T\\ 0 & \text{otherwise}\end{cases},\quad h(t)=\begin{cases}1 & 0<t\le 2T\\ 0 & \text{otherwise}\end{cases}$$

$$(c)\quad x(t)=u(t-1),\ h(t)=e^{-3t}u(t)$$

5. Compute the convolution sum y[n] =x[n]* h[n] of the following pairs of sequences:

$$(a)\quad x[n]=u[n],\ h[n]=2^n u[-n]$$

$$(b)\quad x[n]=u[n]-u[n-N],\ h[n]=\alpha^n u[n],\ 0<\alpha<1$$

$$(c)\quad x[n]=(\tfrac{1}{2})^n u[n],\ h[n]=\delta[n]-\tfrac{1}{2}\delta[n-1]$$

6. Show that if y (t) =x(t)* h(t), then

$$y'(t)=x'(t)*h(t)=x(t)*h'(t)$$

7. Let ***y[n]*** = ***x[n]* h[n].*** Then show that

$$x[n-n_1]*h[n-n_2]=y[n-n_1-n_2]$$

8. Show that

$$x_1[n]\otimes x_2[n]=\sum_{k=n_0}^{n_0+N-1} x_1[k]x_2[n-k]$$

for an arbitrary starting point ***no.***