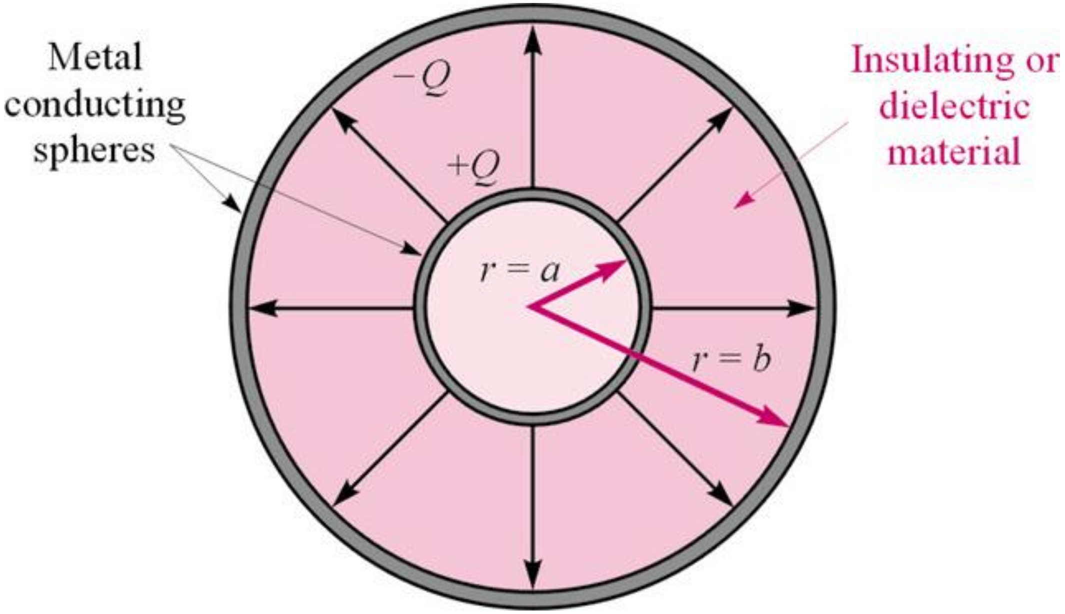
Electric Flux Density, Gauss's Law, and Divergence

3.1 Electric Flux Density

- Faraday's Experiment



Flux $= \Psi$, same units as Q

Ψ is responsible for creating $-Q$ on outer sphere

Electric Flux Density, **D**

- Units: C/m^2
- Magnitude: Number of flux lines (coulombs) crossing a surface normal to the lines divided by the surface area.
- Direction: Direction of flux lines (same direction as **E**).
- For a point charge: $$\mathbf{D} = \frac{Q}{4\pi r^2}\,\mathbf{a}_r$$
- For a general charge distribution,

$$\mathbf{D} = \epsilon_0\,\mathbf{E} = \int_{\text{vol}} \frac{\rho_v\,dv}{4\pi R^2}\,\mathbf{a}_r$$

D3.1

Given a 60-uC point charge located at the origin, find the total electric flux passing through:

(a) That portion of the sphere r = 26 cm bounded by 0 < theta < Pi/2 and 0 < phi < Pi/2

D3.2

Calculate D in rectangular coordinates at point P(2,-3,6) produced by : (a) a point charge QA = 55mC at Q(-2,3,-6)

$$P := \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \qquad Q := \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix} \qquad QA := 55 \cdot 10^{-3}$$

$$\varepsilon_0 := 8.854 \cdot 10^{-12} \qquad R := P - Q \qquad r := \frac{P - Q}{|P - Q|}$$

$$D := \frac{QA}{4 \cdot \pi \cdot (|R|)^2} \cdot r \qquad D = \begin{pmatrix} 6.38 \times 10^{-6} \\ -9.57 \times 10^{-6} \\ 1.914 \times 10^{-5} \end{pmatrix}$$

(b) a uniform line charge p_{LB} = 20 mC/m on the *x* axis

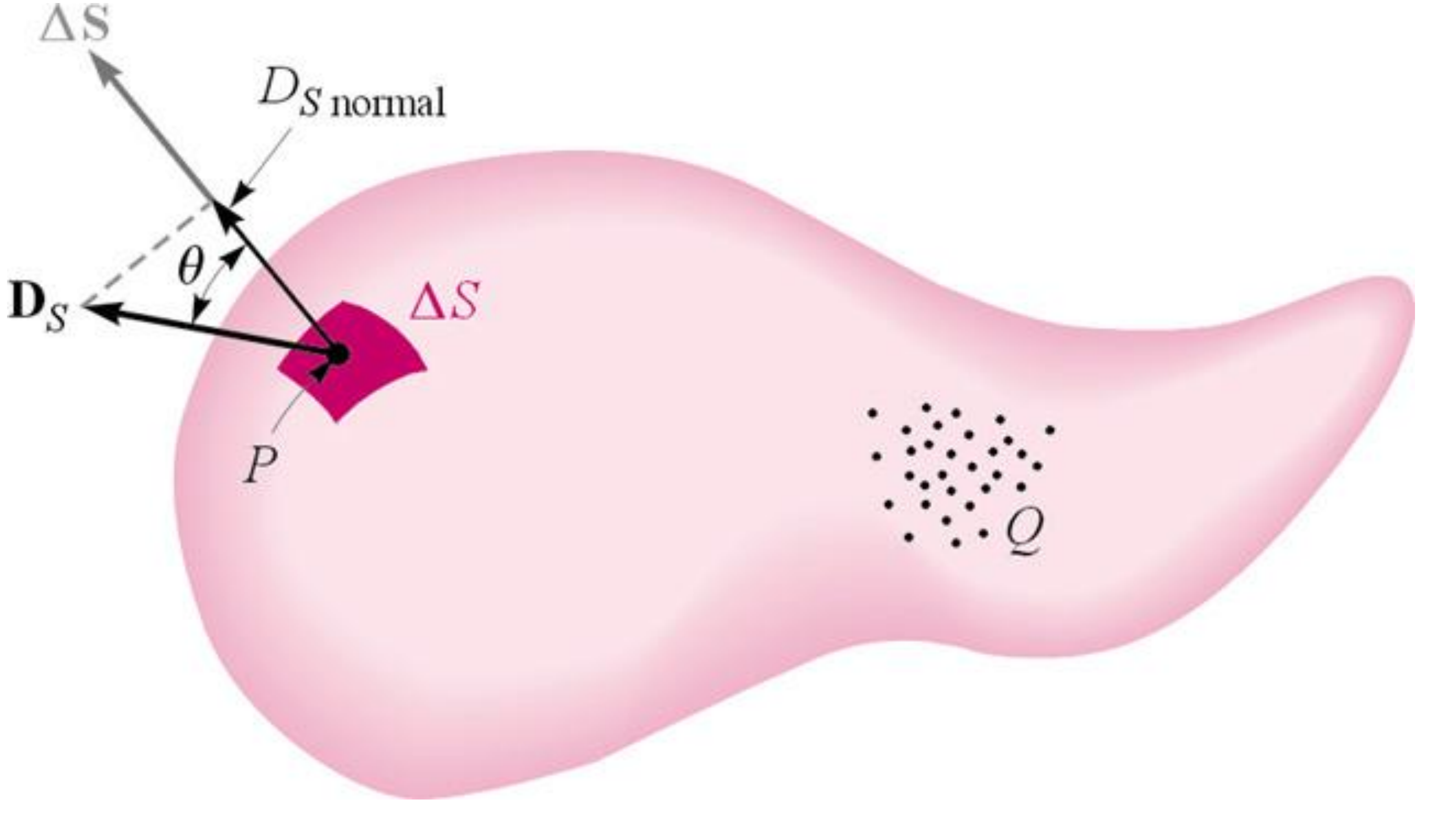
(c) a uniform surface charge density p_{SC} = 120 uC/m^2 on the plane z = -5 m.

Gauss's Law

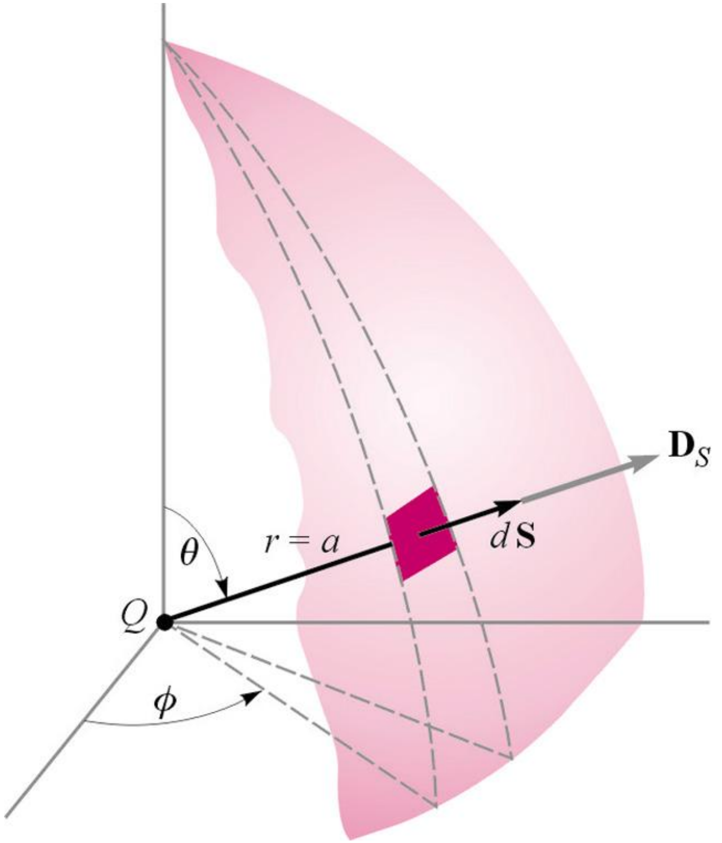
- "The electric flux passing through any closed surface is equal to the total charge enclosed by that surface."

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q$$

- The integration is performed over a *closed* surface, i.e. *gaussian surface*.



- We can check Gauss's law with a point charge example.



$$\int_0^{2\pi}\int_0^{\pi} \frac{q}{4\pi a^2}\, a^2 \,\text{Sin}[\theta]\, d\theta\, d\phi$$

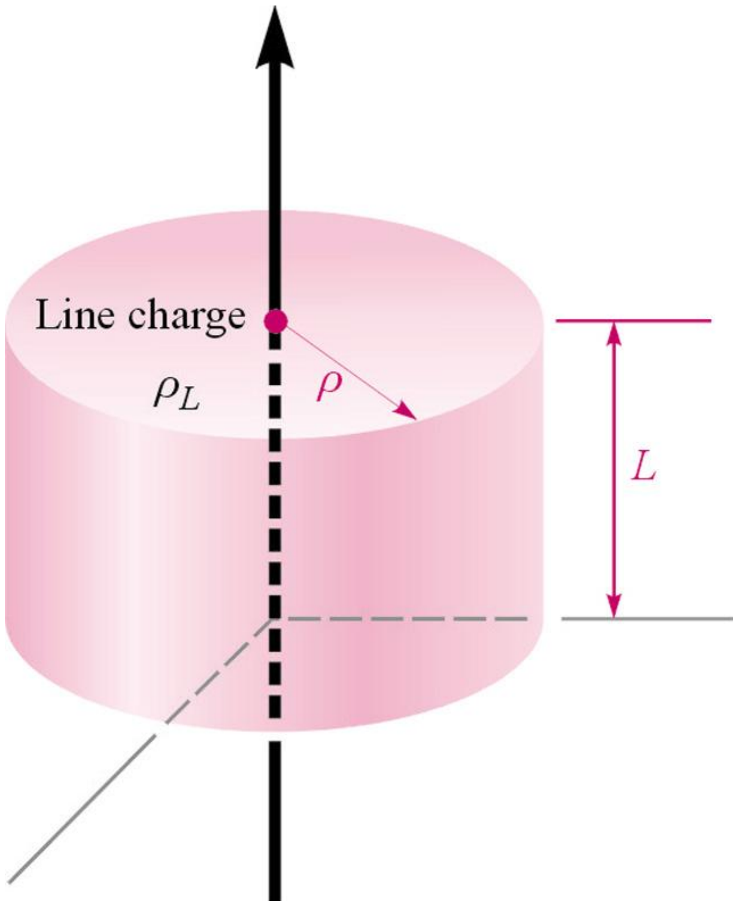
$$q$$

Symmetrical Charge Distributions

- Gauss's law is useful under two conditions.

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S\, dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot dS$ is not zero, D_S = constant.

Gauss's law simplifies the task of finding **D** near an infinite line charge.

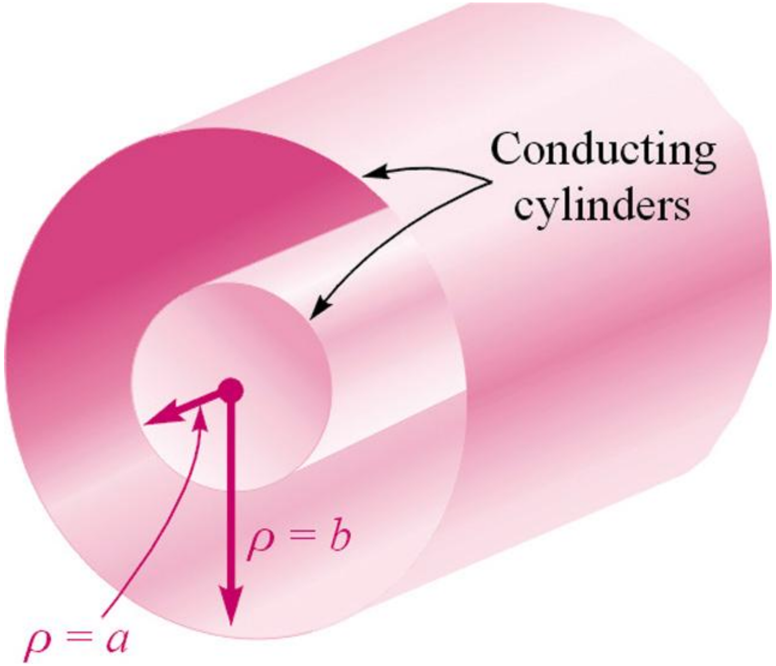


$$Q = \oint_{cyl} \mathbf{D}_S \cdot d\mathbf{S} = D_S \int_{sides} dS + 0 \int_{top} dS + 0 \int_{bottom} dS$$

$$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S \, 2\pi\rho L$$

$$D_S = D_\rho = \frac{Q}{2\pi\rho L} = \frac{\rho_L}{2\pi\rho}$$

Infinite coaxial cable:

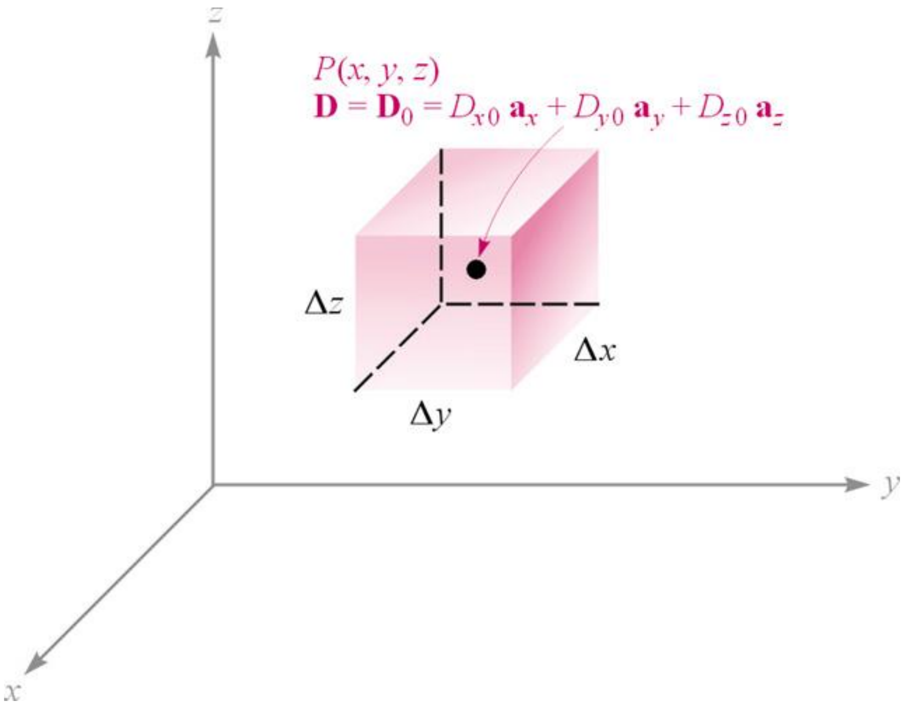


$$D = \frac{\rho_L}{2\pi\rho}\,\mathbf{a}_\rho \quad (a < \rho < b)$$

$$D = 0 \quad (\rho > b)$$

Differential Volume Element

- If we take a small enough closed surface, then **D** is almost constant over the surface.



$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

$$\int_{\text{front}} \doteq \left(D_{\text{x0}} + \frac{\Delta x}{2}\,\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

$$\int_{\text{back}} \doteq \left(-D_{\text{x0}} + \frac{\Delta x}{2}\,\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x}\,\Delta x \Delta y \Delta z$$

$$\vdots$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

$$\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times \text{volume } \Delta v$$

D3.6a

$$D(x,y,z) := \begin{pmatrix} 8\cdot x\cdot y\cdot z^4 \\ 4\cdot x^2\cdot z^4 \\ 16\cdot x^2\cdot y\cdot z^3 \end{pmatrix}$$

$$\int_1^3 \int_0^2 D(x,y,2)_2\cdot 10^{-12}\,dx\,dy = 1.365\times 10^{-9}$$

D3.6b

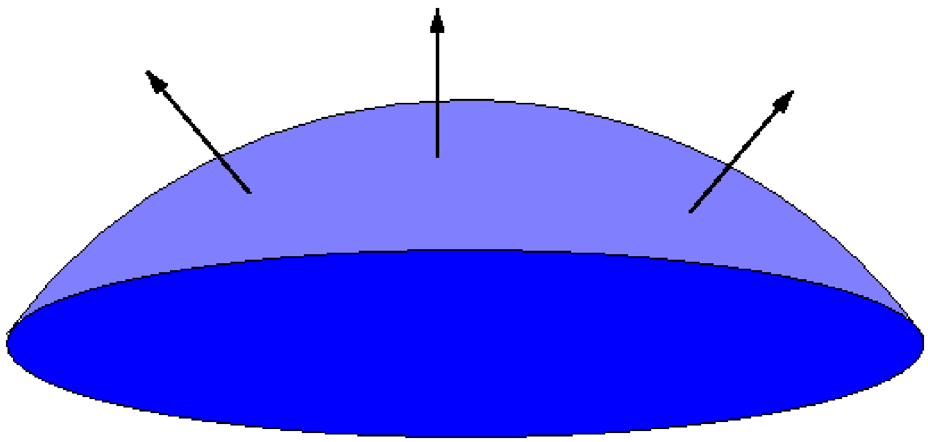
$$D(x,y,z) := \begin{pmatrix} 8\cdot x\cdot y\cdot z^4 \\ 4\cdot x^2\cdot z^4 \\ 16\cdot x^2\cdot y\cdot z^3 \end{pmatrix}\cdot 10^{-12}$$

$$\varepsilon_0 := 8.854 10^{-12}$$

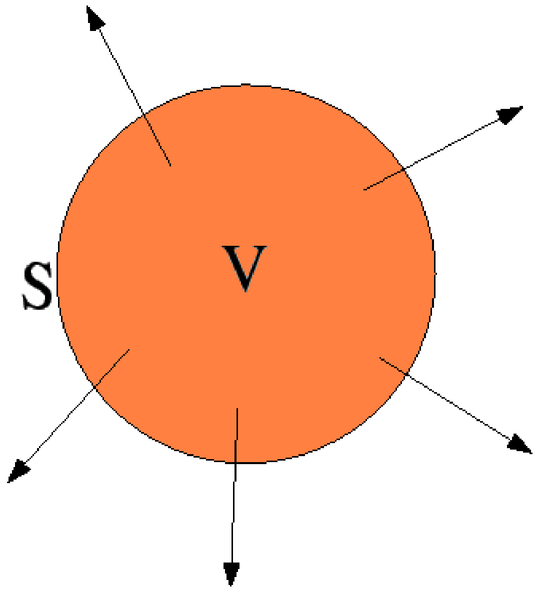
$$P := \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad E := \frac{D(2,-1,3)}{\varepsilon_0} \qquad E = \begin{pmatrix} -146.375 \\ 146.375 \\ -195.166 \end{pmatrix}$$

Divergence

Divergence is the outflow of flux from a small closed surface area (per unit volume) as volume shrinks to zero.



open surface.



closed surface

-Water leaving a bathtub

- -Closed surface (water itself) is essentially incompressible
- -Net outflow is zero

-Air leaving a punctured tire

- -Divergence is positive, as closed surface (tire) exhibits net outflow

Mathematical definition of divergence

$$\mathrm{div}(\mathbf{D}) = \lim_{\Delta v \to 0} \int \frac{\mathbf{D}}{\Delta v} \, d\mathbf{S}$$

Surface integral as the volume element (Δv) approaches zero

D is the vector flux density

$$\mathrm{div}(\mathbf{D}) = \left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z} \right)$$

- *Cartesian*

Divergence in Other Coordinate Systems

Cylindrical

$$\operatorname{div}(\mathbf{D}) = \frac{1}{\rho} \cdot \frac{\delta}{\delta\rho}\left(\rho \cdot D_\rho\right) + \frac{1}{\rho} \cdot \frac{\delta D_\phi}{\delta\phi} + \frac{\delta D_z}{\delta z}$$

Spherical

$$\operatorname{div}(\mathbf{D}) = \frac{1}{r^2} \cdot \frac{\delta\left(D_r \cdot r^2\right)}{\delta r} + \frac{1}{r \cdot \sin(\theta)} \cdot \frac{\delta\left(D_\theta \cdot \sin(\theta)\right)}{\delta\theta} + \frac{1}{r \cdot \sin(\theta)} \cdot \frac{\delta D_\phi}{\delta\phi}$$

Divergence at origin for given *vector flux density* **A**

$$\mathbf{A} = \begin{pmatrix} e^{-x} \cdot \sin(y) \\ -e^{-x} \cdot \cos(y) \\ 2 \cdot z \end{pmatrix}$$

$$\operatorname{div}(\mathbf{A}) = \frac{\delta}{\delta x}\left(e^{-x} \cdot \sin(y)\right) + \frac{\delta}{\delta y} \cdot \left(-e^{-x} \cdot \cos(y)\right) + \frac{\delta}{\delta z} \cdot (2 \cdot z)$$

$$\operatorname{div}(\mathbf{A}) = -e^{-x} \cdot \sin(y) + e^{-x} \cdot \sin(y) + 2$$

3-6: Maxwell's First Equation

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = Q$$

Gauss' Law…

$$\frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

…per unit volume

Volume shrinks to zero

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

Electric flux per unit volume is equal to the volume charge density

Maxwell's First Equation

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

$$\mathrm{div}(\mathbf{D}) = \rho_v$$

Sometimes called the point form of Gauss' Law

Enclosed surface is reduced to a single point

3-7: ∇ and the Divergence Theorem

∇ ➔ *del* operator

What is del?

$$\nabla \quad = \frac{\delta(\mathbf{a_x})}{\delta x} + \frac{\delta(\mathbf{a_y})}{\delta y} + \frac{\delta(\mathbf{a_z})}{\delta z}$$

∇'s Relationship to Divergence

$$\text{div}(\mathbf{D}) = \nabla \cdot \mathbf{D}$$

True for all coordinate systems

$$\nabla \bullet \mathbf{A} = [\partial/\partial x\, x + \partial/\partial y\, y + \partial/\partial z\, z] \bullet [A_x\, x + A_y\, y + A_z\, z]$$

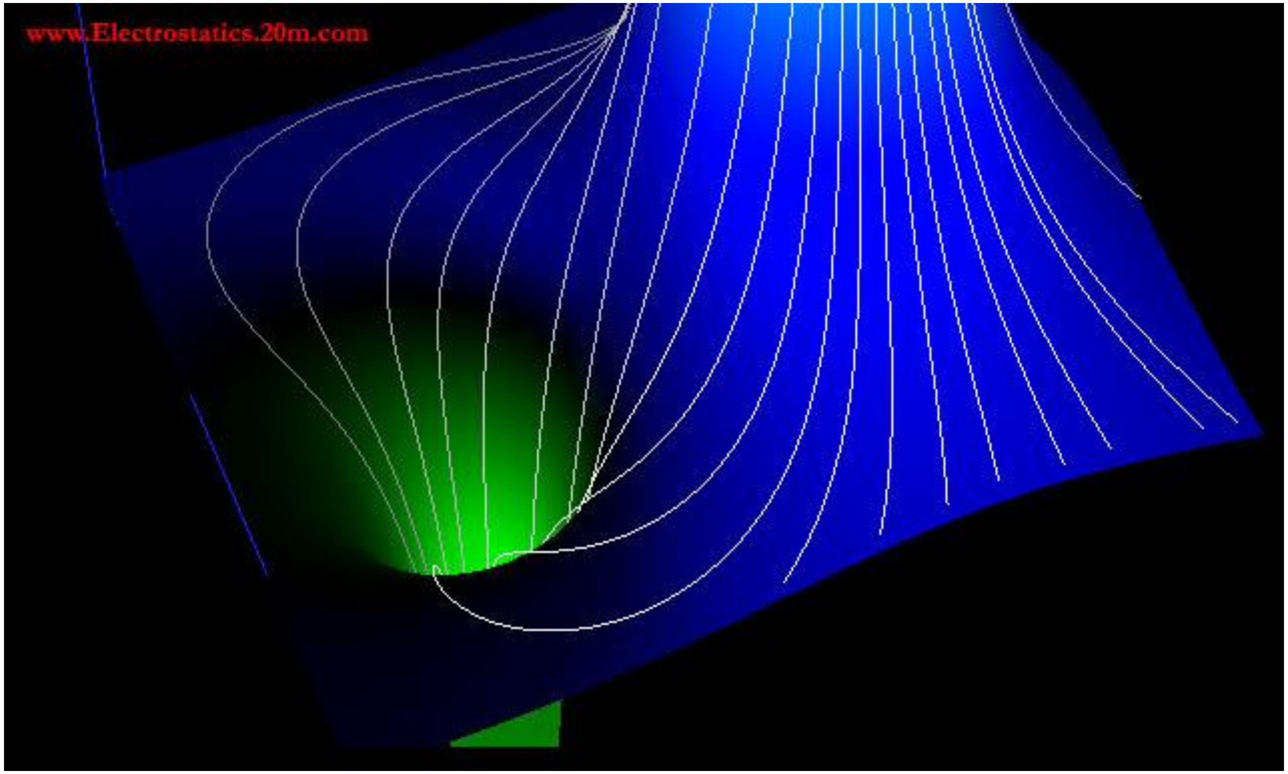
$$= (\partial A_x/\partial x) + (\partial A_y/\partial y) + (\partial A_z/\partial z)\ ;$$

Other ∇ Relationships

Gradient – results from ∇ operating on a function

$$\nabla f = (\partial f/\partial x)\, \mathbf{x} + (\partial f/\partial y)\, \mathbf{y} + (\partial f/\partial z)\, \mathbf{z}\,.$$

Represents direction of greatest change



Curl – cross product of ∇ and

$$\nabla \times A = [\partial/\partial x\, \mathbf{x} + \partial/\partial y\, \mathbf{y} + \partial/\partial z\, \mathbf{z}] \times [A_x \mathbf{x} + A_y \mathbf{y} + A_z \mathbf{z}] =$$

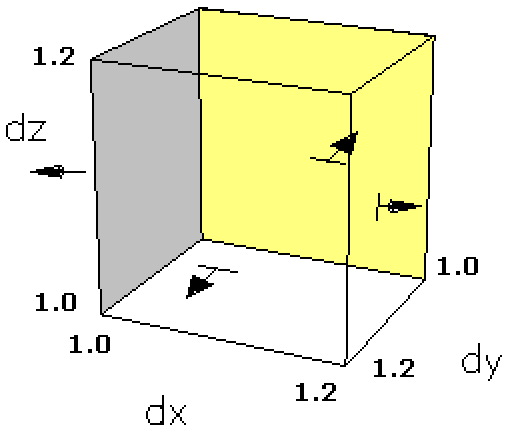
$$(\partial A_y/\partial x)(\mathbf{z}) + (\partial A_z/\partial x)(-\mathbf{y}) + (\partial A_x/\partial y)(-\mathbf{z}) + (\partial A_z/\partial y)(\mathbf{x}) + (\partial A_x/\partial z)(\mathbf{y}) + (\partial A_y/\partial z)(-\mathbf{x})$$

$$= (\partial A_z/\partial y - \partial A_y/\partial z)\, \mathbf{x} + (\partial A_x/\partial z - \partial A_z/\partial x)\, \mathbf{y} + (\partial A_y/\partial x - \partial A_x/\partial y)\, \mathbf{z}$$

Relates to work in a field

If curl is zero, so is work

Examination of ∇ and flux



Cube defined by 1 < x,y,z < 1.2

$$\mathbf{D} = 2 \cdot x^2 \cdot y \cdot \mathbf{a}_x + 3 \cdot x^2 \cdot y^2 \cdot \mathbf{a}_y$$

Calculation of total flux

$$Q = \int_S \mathbf{D}\, d\mathbf{S} = \int_{vol} \rho_v \, dv = \Phi$$

$x_1 := 1$ $\quad x_2 := 1.2$

$y_1 := 1$ $\quad y_2 := 1.2$

$z_1 := 1$ $\quad z_2 := 1.2$

$$\Phi_{total} = \Phi_{left} + \Phi_{right} + \Phi_{front} + \Phi_{back}$$

$$\Phi_{x1} := \int_{z_1}^{z_2} \int_{y_1}^{y_2} -2 \cdot x_1^2 \cdot y \, dy \, dz \qquad \Phi_{y1} := \int_{z_1}^{z_2} \int_{x_1}^{x_2} -3 \cdot x^2 \cdot y_1^2 \, dx \, dz$$

$$\Phi_{x2} := \int_{z_1}^{z_2} \int_{y_1}^{y_2} 2 \cdot x_2^2 \cdot y \, dy \, dz \qquad \Phi_{y2} := \int_{z_1}^{z_2} \int_{x_1}^{x_2} 3 \cdot x^2 \cdot y_2^2 \, dx \, dz$$

$$\Phi_{total} := \Phi_{x1} + \Phi_{x2} + \Phi_{y1} + \Phi_{y2}$$

$$\Phi_{total} = 0.103$$

Evaluation of $\nabla \cdot \mathbf{D}$ at center of cube

$$\mathrm{div}(\mathbf{D}) = \frac{d}{dx}\left(2 \cdot x^2 \cdot y\right) + \frac{d}{dy}\left(3 \cdot x^2 \cdot y^2\right)$$

$$\mathrm{div}(\mathbf{D}) = 4 \cdot x \cdot y + 6 \cdot x^2 \cdot y$$

$$divD := 4 \cdot (1.1) \cdot (1.1) + 6 \cdot (1.1)^2 \cdot (1.1)$$

$$divD = 12.826$$

Non-Cartesian Example

Inside the cylindrical shell, $3 < \rho < 4$ m, the electric flux density is given as

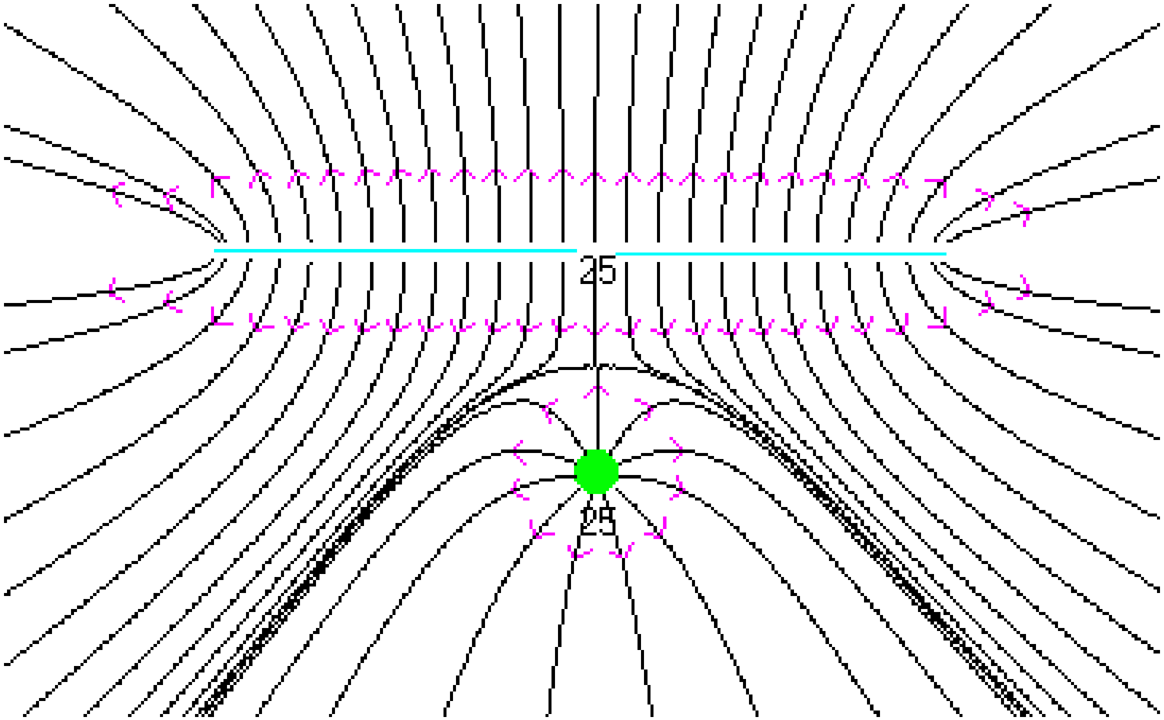
$$\mathbf{D} = 5(\rho - 3)^3\, \mathbf{a}_\rho \ \mathrm{C/m^2}$$

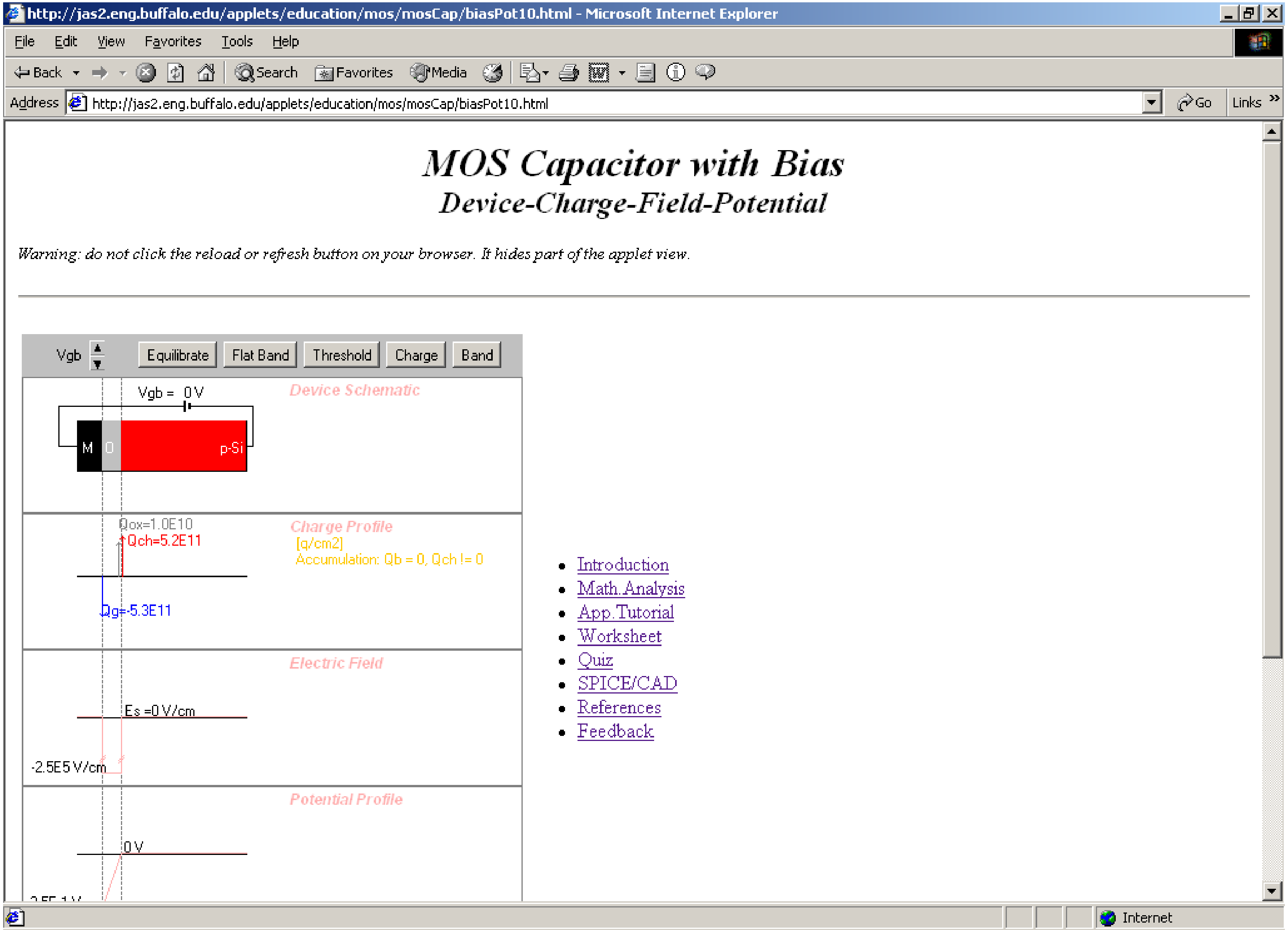
a) What is the volume charge density at $\rho = 4$ m? In this case we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{d}{d\rho}(\rho D_\rho) = \frac{1}{\rho}\frac{d}{d\rho}[5\rho(\rho - 3)^3] = \frac{5(\rho - 3)^2}{\rho}(4\rho - 3)\ \mathrm{C/m^3}$$

Evaluating this at $\rho = 4$ m, we find $\rho_v(4) = \underline{16.25\ \mathrm{C/m^3}}$

Equipotential Surfaces – Free Software

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M
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p-Si
Qox=1.0E10
Qch=5.2E11
Charge Profile
[q/cm2]
Accumulation: Qb = 0, Qch != 0
Qg=-5.3E11
Electric Field
Es =0 V/cm
-2.5E5 V/cm
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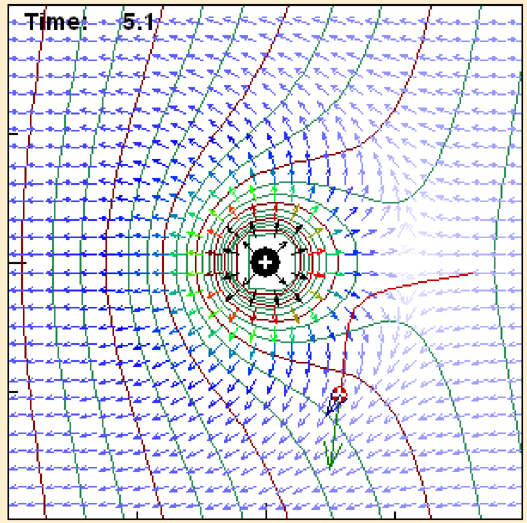
Dynamics



play | pause | <<step | step>> | reset

A positive test charge is placed into a potential field consisting a point charge and a background field in the -x direction. Both the test charge and the fixed charge can be dragged to new positions. Observe the dynamics. Start

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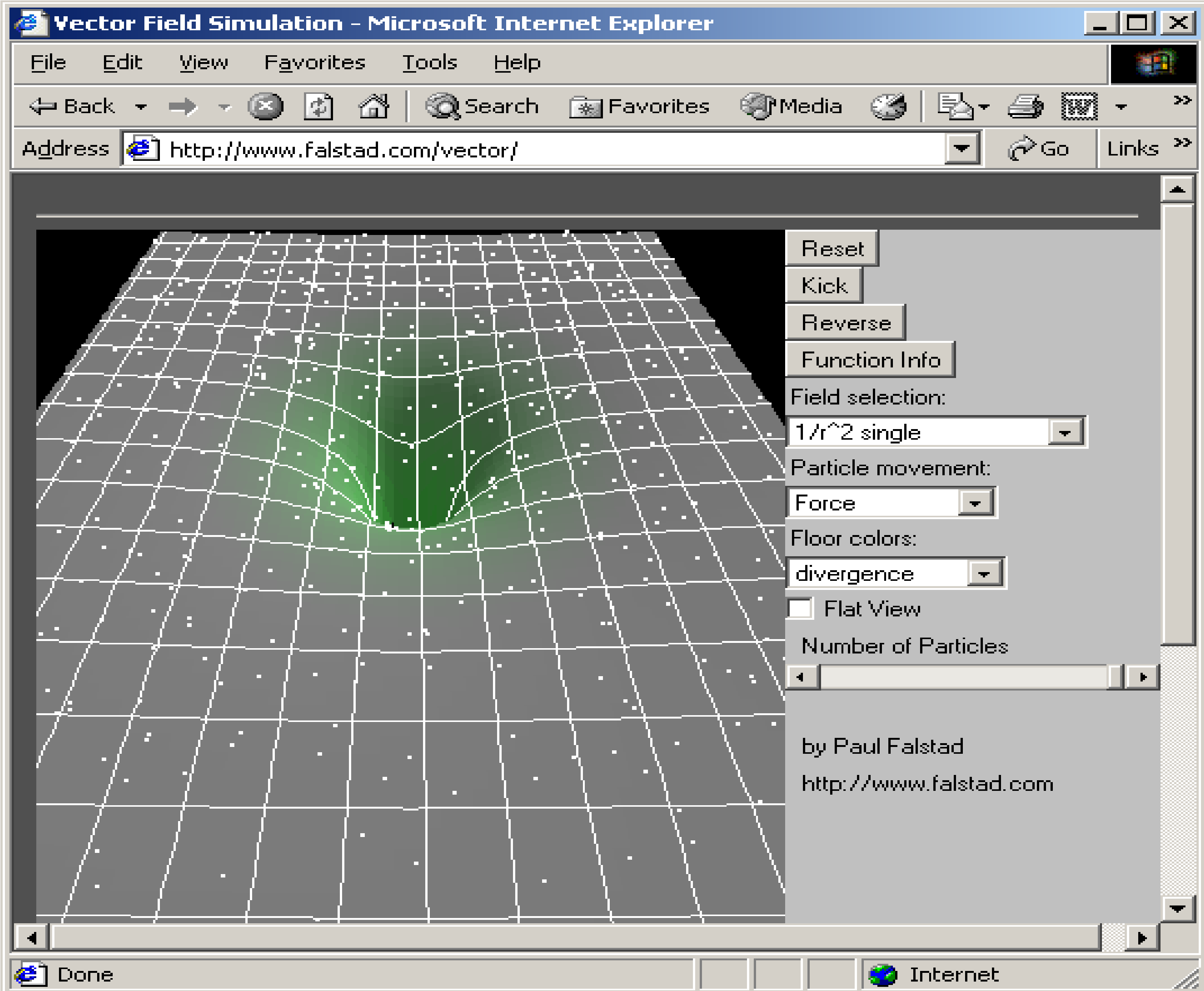
Script Example

```
<script language="JavaScript">
function dynamics(){
        document.EField.setDefault();
        document.EField.setNoDrag(0);
        document.EField.setShowForce(0);
```

Applet started | Internet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Semiconductor Application - Device Charge Field Potential

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Applications of Gauss's Law