Energy and Potential

4.1 Energy to move a point charge through a Field

- Force on Q due to an electric field

$$F_E = QE$$

- Differential work done by an external source moving Q

$$dW = -QE \cdot dL$$

- Work required to move a charge a finite distance

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

4.2 Line Integral

- Work expression without using vectors

$$W = -Q \cdot \int_{\text{initial}}^{\text{final}} E_L \, dL$$

EL is the component of E in the dL direction

$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \qquad \text{(cartesian)}$$

$$d\mathbf{L} = d\rho\,\mathbf{a}\rho + \rho\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\;d\phi\,\mathbf{a}_\phi \qquad \text{(spherical)}$$

- Uniform electric field density

$$W = -QE \cdot L_{BA}$$

Example

$$E(x,y) := \begin{pmatrix} y \\ x \\ 2 \end{pmatrix} \qquad Q := 2 \qquad A := \begin{pmatrix} .8 \\ .6 \\ 1 \end{pmatrix} \qquad B := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Path: $x^2 + y^2 = 1 \qquad z = 1$

Calculate the work to cary the charge from point B to point A.

$$W = -Q \cdot \int_{B_0}^{A_0} E(x,y)_0 \, dx - Q \cdot \int_{B_1}^{A_1} E(x,y)_1 \, dy - Q \cdot \int_{B_2}^{A_2} E(x,y)_2 \, dz$$

Plug path in for x and y in E(x,y)

$$W := -Q \cdot \int_{B_0}^{A_0} E\left(0, \sqrt{1 - x^2}\right)_0 dx - Q \cdot \int_{B_1}^{A_1} E\left(\sqrt{1 - y^2}, 0\right)_1 dy - Q \cdot \int_{B_2}^{A_2} E(0,0)_2 \, dz \qquad W = -0.96$$

Example

$E(x,y) := \begin{pmatrix} y \\ x \\ 2 \end{pmatrix}$ $Q := 2$ $A := \begin{pmatrix} .8 \\ .6 \\ 1 \end{pmatrix}$ $B := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ Path: $y = -3(x-1)$ $z = 1$

(straight line)

Calculate the work to cary the charge from point B to point A.

$$W = -Q\cdot\int_{B_0}^{A_0} E(x,y)_0\,dx - Q\cdot\int_{B_1}^{A_1} E(x,y)_1\,dy - Q\cdot\int_{B_2}^{A_2} E(x,y)_2\,dz$$

Plug path in for x and y in E(x,y)

$$W := -Q\cdot\int_{B_0}^{A_0} E[\,0,-3(x-1)\,]_0\,dx - Q\cdot\int_{B_1}^{A_1} E\left(\frac{-y}{3}+1,0\right)_1 dy - Q\cdot\int_{B_2}^{A_2} E(0,0)_2\,dz \quad W = -0.96$$

- Same amount of work with a different path
- Line integrals are path independent

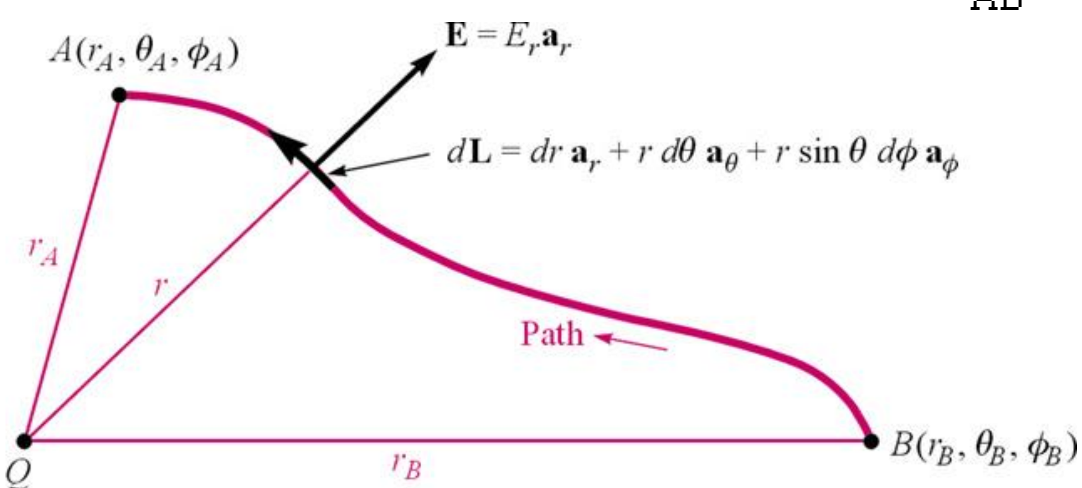
4.3 Potential Difference

- Potential Difference

$$V = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \qquad V_{AB} = -\int_{B}^{A} \mathbf{E} \cdot d\mathbf{L}$$

- Using radial distances from the point charge

$$V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 \cdot r^2}\, dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$



4.3 Potential

- Measure potential difference between a point and something which has zero potential “ground”

$$V_{AB} = V_A - V_B$$

Example – D4.4

$$E(x,y,z) := \begin{pmatrix} 6x^2 \\ 6y \\ 4 \end{pmatrix}$$

a) Find Vmn

$$M := \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} \qquad N := \begin{pmatrix} -3 \\ -3 \\ 2 \end{pmatrix} \qquad V_{MN} := -\int_{N_0}^{M_0} 6x^2\,dx - \int_{N_1}^{M_1} 6y\,dy - \int_{N_2}^{M_2} 4\,dz \qquad V_{MN} = -139$$

b) Find Vm if V=0 at Q(4,-2,-35)

$$Q := \begin{pmatrix} 4 \\ -2 \\ -35 \end{pmatrix} \qquad V_M := -\int_{Q_0}^{M_0} 6x^2\,dx - \int_{Q_1}^{M_1} 6y\,dy - \int_{Q_2}^{M_2} 4\,dz \qquad V_M = -120$$

c) Find Vn if V=2 at P(1, 2, -4)

$$P := \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \qquad V_N := -\int_{P_0}^{N_0} 6x^2\,dx - \int_{P_1}^{N_1} 6y\,dy - \int_{P_2}^{N_2} 4\,dz + 2 \qquad V_N = 19$$

4.4 Potential Field of a Point Charge

- Let V=0 at infinity

$$V = \frac{Q}{4\pi\varepsilon_0 \cdot r}$$

- Equipotential surface:
  - A surface composed of all points having the same potential

Example – D4.5

$Q := 15 \cdot 10^{-9}$ $\quad P_1 := \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$ $\quad \varepsilon_0 := 8.85 \cdot 10^{-12}$

Q is located at the origin

a) Find V1 if V=0 at (6,5,4)

$P_0 := \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$ $\quad V_1 := \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{|P_1|} - \frac{1}{|P_0|}\right)$ $\quad V_1 = 20.677$

b) Find V1 if V=0 at infinity

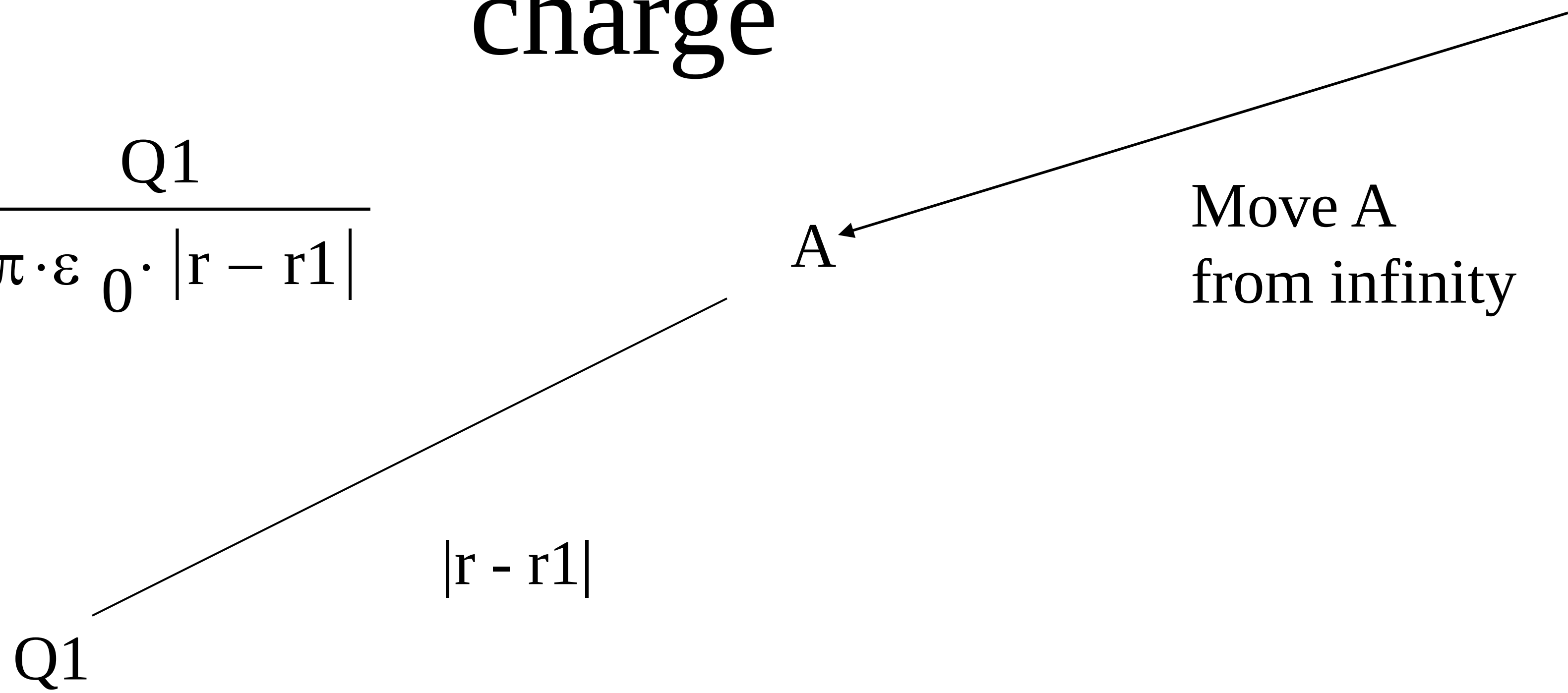
$V_1 := \frac{Q}{4\pi\varepsilon_0} \frac{1}{|P_1|}$ $\quad V_1 = 36.047$

c) Find V1 if V=5 at (2,0,4)

$P_5 := \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ $\quad V_1 := \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{|P_1|} - \frac{1}{|P_5|}\right) + 5$ $\quad V_1 = 10.888$

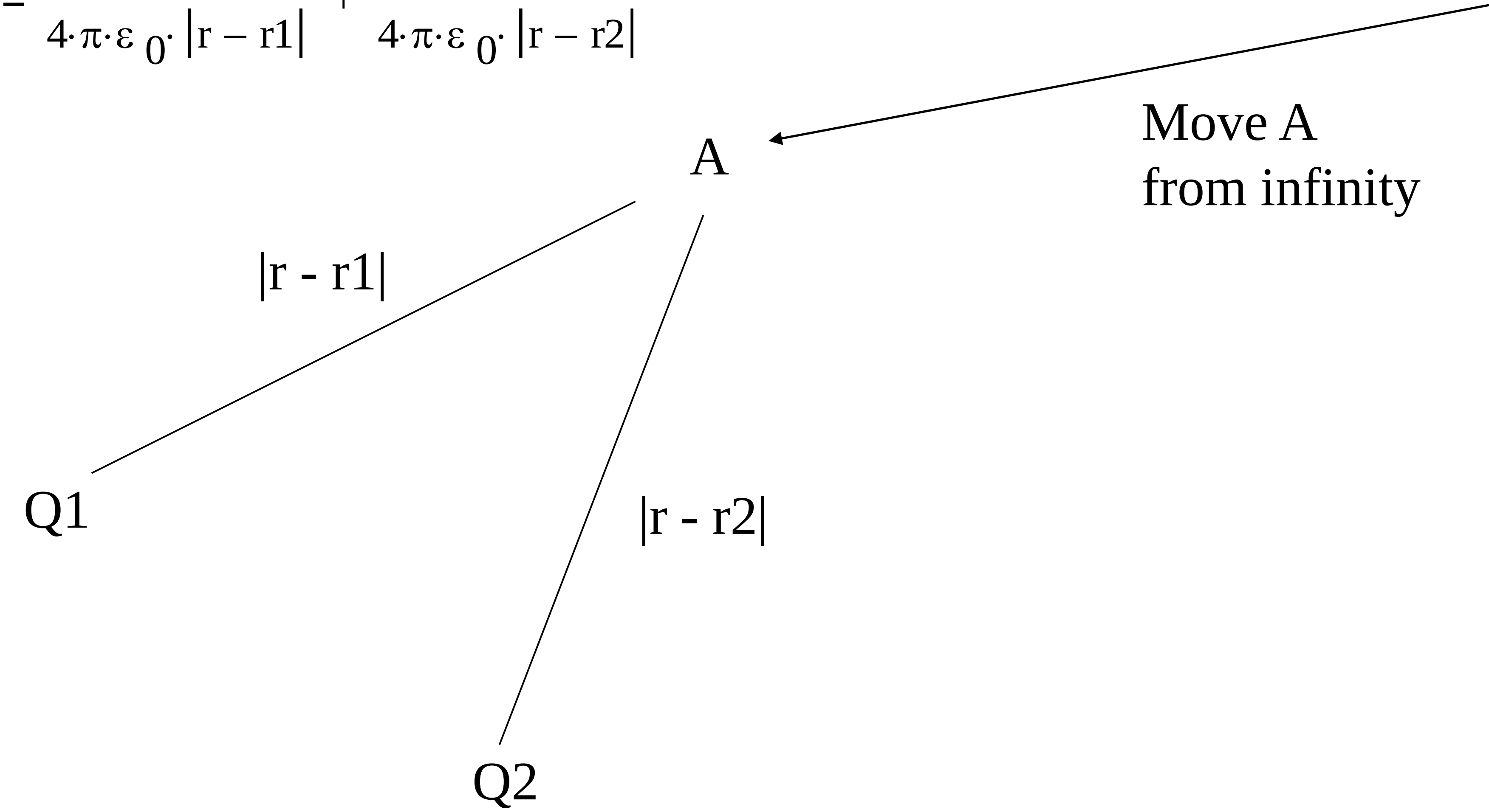
Potential field of single point charge

$$V(r) = \frac{Q1}{4 \cdot \pi \cdot \varepsilon_0 \cdot |r - r1|}$$



Potential due to two charges

$$V(r) = \frac{Q1}{4 \cdot \pi \cdot \varepsilon_0 \cdot |r - r1|} + \frac{Q2}{4 \cdot \pi \cdot \varepsilon_0 \cdot |r - r2|}$$



Potential due to *n* point charges

Continue adding charges

$$V(r) = \frac{Q1}{4\cdot\pi\cdot\varepsilon_0\cdot|r - r1|} + \frac{Q2}{4\cdot\pi\cdot\varepsilon_0\cdot|r - r2|} + \ldots + \frac{Qn}{4\cdot\pi\cdot\varepsilon_0\cdot|r - r_n|}$$

$$V(r) = \sum_{m=1}^{n} \frac{Qm}{4\cdot\pi\cdot\varepsilon_0\cdot|r - r_m|}$$

Potential as point charges become infinite

Volume of charge

$$V(r) = \int \frac{\rho_v(r_{prime})}{4 \cdot \pi \cdot \varepsilon_0 \cdot |r - r_{prime}|} \, dv_{prime}$$

Line of charge

$$V(r) = \int \frac{\rho_L(r_{prime})}{4 \cdot \pi \cdot \varepsilon_0 \cdot |r - r_{prime}|} \, dL_{prime}$$

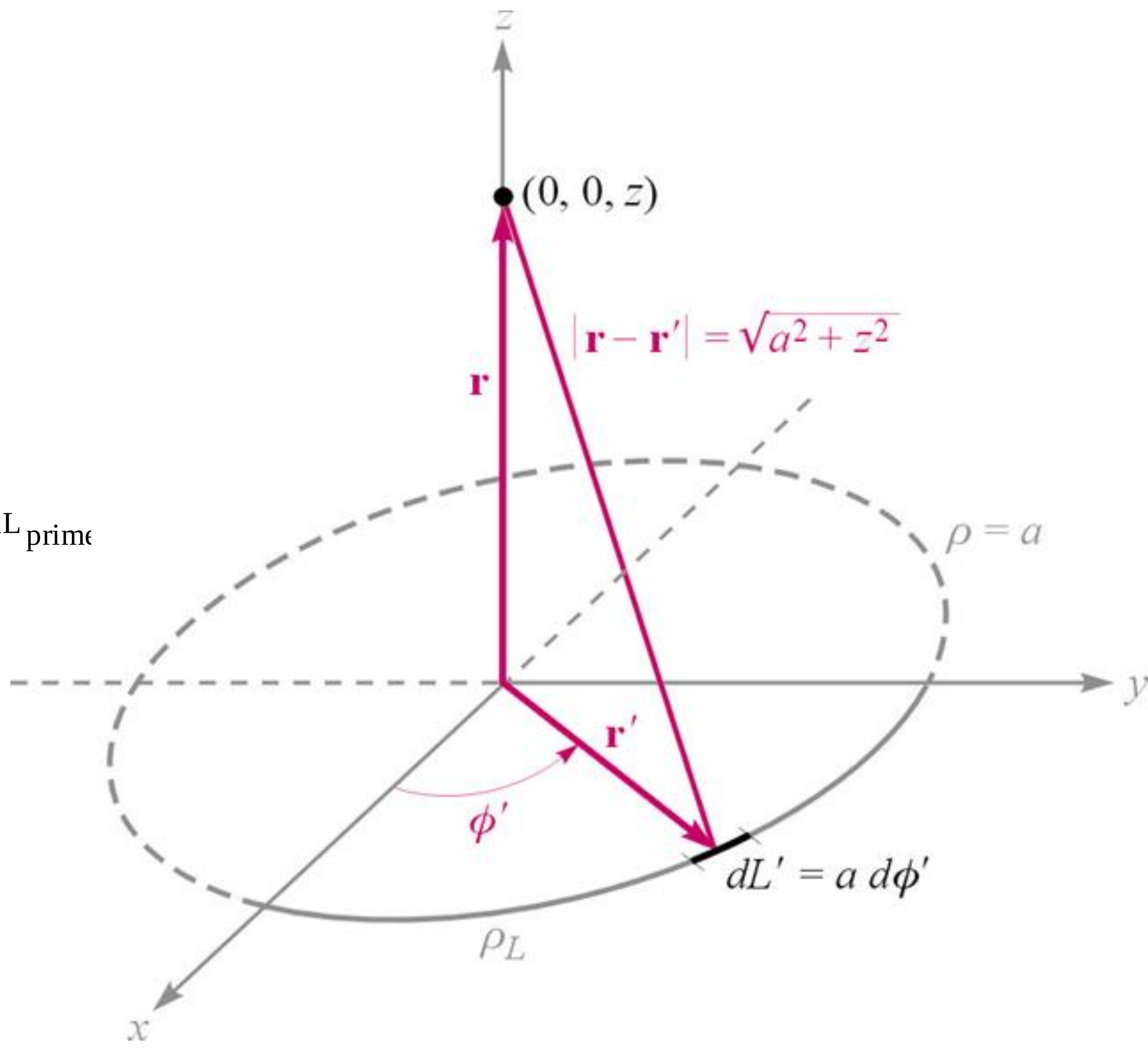
Surface of charge

$$V(r) = \int \frac{\rho_S(r_{prime})}{4 \cdot \pi \cdot \varepsilon_0 \cdot |r - r_{prime}|} \, dS_{prime}$$

Example

Find V on the z axis for a uniform line charge ρL in the form of a ring

$$V(r) = \int \frac{\rho_{L}(r_{prime})}{4\cdot\pi\cdot\varepsilon_{0}\cdot\left|r - r_{prime}\right|}\, dL_{prime}$$



Conservative field

No work is done (energy is conserved) around a closed path

KVL is an application of this

4.6
Potential gradient Relationship between potential and electric field intensity

$$V = - \int E \cdot dL$$

Two characteristics of relationship:

1. The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance

2. This maximum value is obtained when the direction of E is opposite to the direction in which the potential is increasing the most rapidly

Gradient

- The gradient of a scalar is a vector

- The gradient shows the maximum space rate of change of a scalar quantity and the *direction* in which the maximum occurs

- The operation on V by which -**E** is obtained

$$\mathbf{E} = -\ \text{grad}\ V = -\ \nabla V$$

Gradients in different coordinate systems

The following equations are found on page 104 and inside the back cover of the text:

$$\mathrm{grad}V = \frac{\delta V}{\delta x}\cdot a_x + \frac{\delta V}{\delta y}\cdot a_y + \frac{\delta V}{\delta z}\cdot a_z$$

Cartesian

$$\mathrm{grad}V = \frac{\delta V}{\delta \rho}\cdot a_\rho + \frac{1}{\rho}\cdot\frac{\delta V}{\delta \phi}\cdot a_\phi + \frac{\delta V}{\delta z}\cdot a_z$$

Cylindrical

$$\mathrm{grad}V = \frac{\delta V}{\delta r}\cdot a_r + \frac{1}{r}\cdot\frac{\delta V}{\delta \theta}\cdot a_\theta + \frac{1}{r\cdot\sin(\theta)}\cdot\frac{\delta V}{\delta \phi}\cdot a_\phi$$

Spherical

Example 4.3

Given the potential field, $V = 2x^2y - 5z$, and a point P(-4, 3, 6), find the following: potential V, electric field intensity **E**

potential $V_P = 2(-4)^2(3) - 5(6) = 66$ V

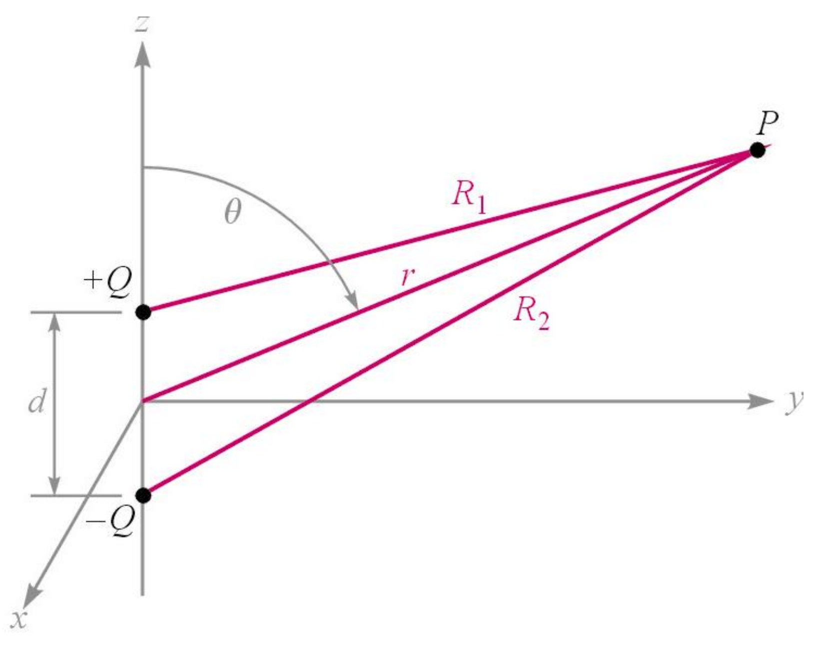
electric field intensity - use gradient operation

$$\mathbf{E} = -4xy\mathbf{a_x} - 2x^2\mathbf{a_y} + 5\mathbf{a_z}$$

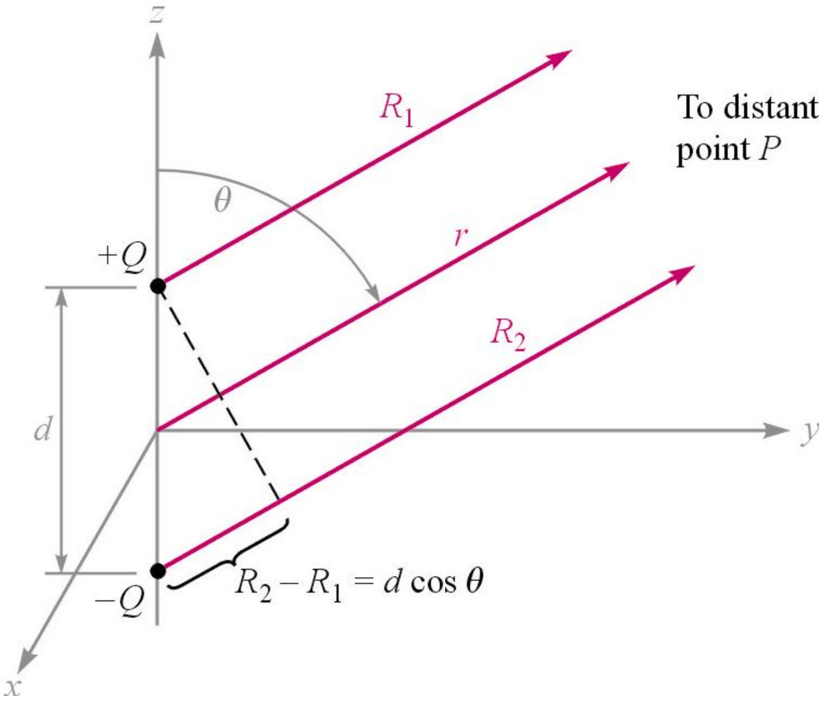
$$\mathbf{E}_P = 48\mathbf{a_x} - 32\mathbf{a_y} + 5\mathbf{a_z}$$

Dipole

The name given to two point charges of equal magnitude and opposite sign, separated by a distance which is small compared to the distance to the point P, at which we want to know the electric and potential fields



Potential



To approximate the potential of a dipole, assume R1 and R2 are parallel since the point P is very distant

$$V = \frac{Q}{4 \cdot \pi \cdot \varepsilon_0} \cdot \left(\frac{1}{R1} - \frac{1}{R2} \right)$$

$$V = \frac{Q \cdot d \cdot \cos(\theta)}{4 \cdot \pi \cdot \varepsilon_0 \cdot r^2}$$

Dipole moment

The dipole moment is assigned the symbol p and is equal to the product of charge and separation

$$\mathbf{p} = Q*\mathbf{d}$$

The dipole moment expression simplifies the potential field equation

Example

An electric dipole located at the origin in free space has a moment **p** = 3*ax - 2*ay + az nC*m. Find V at the points (2, 3, 4) and (2.5, 30°, 40°).

$$p := \begin{pmatrix} 3\cdot 10^{-9} \\ -2\cdot 10^{-9} \\ 1\cdot 10^{-9} \end{pmatrix} \qquad \varepsilon_0 := 8.854 10^{-12}$$

$$P := \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \qquad V := \frac{p}{4\cdot\pi\cdot\varepsilon_0\cdot(|P|)^2}\cdot\frac{P}{|P|} \qquad V = 0.23$$

$$P_{spherical} := \begin{pmatrix} 2.5 \\ 30\cdot\frac{\pi}{180} \\ 40\cdot\frac{\pi}{180} \end{pmatrix}$$

Transform this into rectangular coordinates

$$P_{rectangular} := \begin{pmatrix} 2.5\sin\left(30\cdot\frac{\pi}{180}\right)\cdot\cos\left(40\cdot\frac{\pi}{180}\right) \\ 2.5\sin\left(30\cdot\frac{\pi}{180}\right)\cdot\sin\left(40\cdot\frac{\pi}{180}\right) \\ 2.5\cos\left(30\cdot\frac{\pi}{180}\right) \end{pmatrix} \qquad P_{rectangular} = \begin{pmatrix} 0.958 \\ 0.803 \\ 2.165 \end{pmatrix}$$

$$V := \frac{p}{4\cdot\pi\cdot\varepsilon_0\cdot\left(\left|P_{rectangular}\right|\right)^2}\cdot\frac{P_{rectangular}}{\left|P_{rectangular}\right|} \qquad V = 1.973$$

Potential energy

Bringing a positive charge from infinity into the field of another positive charge requires work. The work is done by the external source that moves the charge into position. If the source released its hold on the charge, the charge would accelerate, turning its potential energy into kinetic energy.

The potential energy of a system is found by finding the work done by an external source in positioning the charge.

Empty universe

Positioning the first charge, Q1, requires no work (no field present)

Positioning more charges does take work

Total positioning work = potential energy of field = W_E =
$Q_2V_{2,1} + Q_3V_{3,1} + Q_3V_{3,2} + Q_4V_{4,1} + Q_4V_{4,2} + Q_4V_{4,3} + ...$

Manipulate this expression to get
$W_E = 0.5(Q_1V_1 + Q_2V_2 + Q_3V_3 + \ldots)$

Where is energy stored?

The location of potential energy cannot be precisely pinned down in terms of physical location - in the molecules of the pencil, the gravitational field, etc?

So where is the energy in a capacitor stored?

Electromagnetic theory makes it easy to believe that the energy is stored in the field itself