TIME VARYING MAGNETIC FIELDS AND MAXWELL'S EQUATIONS

Introduction

Electrostatic fields are usually produced by static electric charges whereas magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles); time-varying fields or waves are usually due to accelerated charges or time-varying current.

- Stationary charges → Electrostatic fields
- Steady current → Magnetostatic fields
- Time-varying current → Electromagnetic fields (or waves)

Faraday discovered that the induced emf, V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit

This is called Faraday's Law, and it can be expressed as

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt} \qquad 1.1$$

where N is the number of turns in the circuit and ψ is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This is known as Lenz's Law, and it emphasizes the fact that the direction of current flow in the circuit is such that the induced magnetic filed produced by the induced current will oppose the original magnetic field.

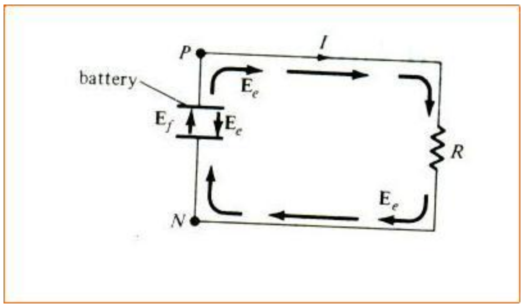


Fig. 1 A circuit showing emf-producing field E_f and electrostatic field E_e

TRANSFORMER AND MOTIONAL EMFS

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single (N = 1), eq. (1.1) becomes

$$V_{emf} = -N\frac{d\Psi}{dt} \qquad 1.2$$

In terms of **E** and **B,** eq. (1.2) can be written as

$$V_{emf} = \oint_L E \cdot dl = -\frac{d}{dt}\int_S B \cdot dS \qquad 1.3$$

where, ψ has been replaced by $\int_S B \cdot dS$ and S is the surface area of the circuit bounded by the closed path L. It is clear from eq. (1.3) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that d**l** and d**S** in eq. (1.3) are in accordance with the right-hand rule as well as Stokes's theorem. This should be observed in Figure 2. The variation of flux with time as in eq. (1.1) or eq. (1.3) may be caused in three ways:

1. By having a stationary loop in a time-varying **B** field
2. By having a time-varying loop area in a static **B** field
3. By having a time-varying loop area in a time-varying **B** field.

A. STATIONARY LOOP IN TIME-VARYING B FIELD (TRANSFORMER EMF)

This is the case portrayed in Figure 2 where a stationary conducting loop is in a time varying magnetic **B** field. Equation (1.3) becomes

$$V_{emf} = \oint_L E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS \qquad 1.4$$

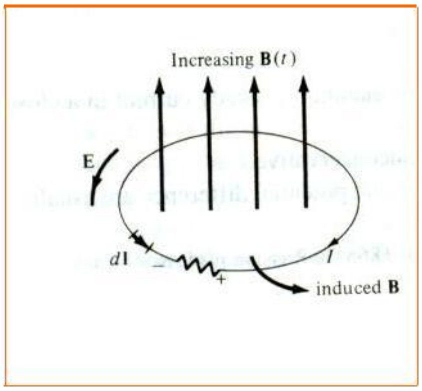


Fig. 2: Induced emf due to a stationary loop in a time varying **B** field.

This emf induced by the time-varying current (producing the time-varying **B** field) in a stationary loop is often referred to as *transformer emf* in power analysis since it is due to transformer action. By applying Stokes's theorem to the middle term in eq. (1.4), we obtain

$$\int_S (\nabla \times E) \cdot dS = -\int_S \frac{\partial B}{\partial t} \cdot dS \qquad 1.5$$

For the two integrals to be equal, their integrands must be equal; that is,

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad 1.6$$

This is one of the Maxwell's equations for time-varying fields. It shows that the time varying E field is not conservative (∇ x E ≠ 0). This does not imply that the principles of energy conservation are violated. The work done in taking a charge about a closed path in a time-varying electric field, for example, is due to the energy from the time-varying magnetic field.

B. MOVING LOOP IN STATIC B FIELD (MOTIONAL EMF)

When a conducting loop is moving in a static **B** field, an emf is induced in the loop. We recall from eq. (1.7) that the force on a charge moving with uniform velocity **u** in a magnetic field **B** is

$$\mathbf{F_m} = Q\mathbf{u} \times \mathbf{B} \qquad 1.7$$

We define the *motional electric field* E_m as

$$E_m = \frac{F_m}{Q} = u \times B \qquad 1.8$$

If we consider a conducting loop, moving with uniform velocity **u** as consisting of a large number of free electrons, the emf induced in the loop is

$$V_{emf} = \oint_L E_m \cdot dl = \oint_L (u \times B) \cdot dl \qquad 1.9$$

This type of emf is called *motional emf or flux-cutting emf* because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators.

C. MOVING LOOP IN TIME-VARYING FIELD

This is the general case in which a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining equation 1.4 and 1.9 gives the total emf as

$$V_{emf} = \oint_L E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS + \oint_L (u \times B) \cdot dl \qquad 1.10$$

$$\nabla \times E_m = \nabla \times (u \times B) \qquad 1.11$$

or from equations 1.6 and 1.11.

$$\nabla \times E = -\frac{\partial B}{\partial t} + \nabla \times (u \times B) \qquad 1.12$$

DISPLACEMENT CURRENT

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad 1.13$$

But the divergence of the curl of any vector field is identically zero.

Hence,

$$\nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J \qquad 1.14$$

The continuity of current requires that

$$\nabla \cdot J = -\frac{\partial \rho_v}{\partial t} \neq 0 \qquad 1.15$$

Thus eqs. 1.14 and 1.15 are obviously incompatible for time-varying conditions. We must modify eq. 1.13 to agree with eq. 1.15. To do this, we add a term to eq. 1.13, so that it becomes

$$\nabla \times H = J + J_d \qquad 1.16$$

where J_d is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J + \nabla \cdot J_d \qquad 1.17$$

In order for eq. 1.17 to agree with eq. 1.15,

$$\nabla \cdot J_d = -\nabla \cdot J = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot D) = \nabla \cdot \frac{\partial D}{\partial t} \qquad 1.18$$

or

$$J_d = \frac{\partial D}{\partial t} \qquad 1.19$$

Substituting eq. 1.19 into eq. 1.15 results in

$$\nabla \times H = J + \frac{\partial D}{\partial t} \qquad 1.20$$

This is Maxwell's equation (based on Ampere's circuit law) for a time-varying field. The term $J_d = \partial D/\partial t$ is known as *displacement current density and* J is the conduction current density ($J = \sigma E$)[3].

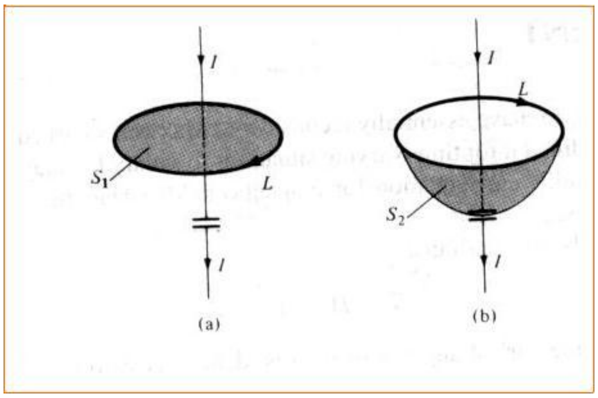


Fig. 3 Two surfaces of integration showing the need for J_d in Ampere's circuit law

The insertion of J_d into eq. 1.13 was one of the major contribution of Maxwell. Without the term J_d, electromagnetic wave propagation (radio or TV waves, for example) would be impossible. At low frequencies, J_d is usually neglected compared with J. however, at radio frequencies, the two terms are comparable. At the time of Maxwell, high-frequency sources were not available and eq. 1.20 could not be verified experimentally.

Based on displacement current density, we define the displacement current as

$$I_d = \int J_d \cdot dS = \int \frac{\partial D}{\partial t} \cdot dS \qquad 1.21$$

We must bear in mind that displacement current is a result of time-varying electric field. A typical example of such current is that through a capacitor when an alternating voltage source is applied to its plates.

PROBLEM: A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 sin 10^3 t V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\,\varepsilon_0$.

Solution:

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\frac{dV}{dt}$$

Hence,

$$I_d = J_d \cdot S = \frac{\varepsilon S}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

which is the same as the conduction current, given by

$$I_c = \frac{dQ}{dt} = S\frac{d\rho_s}{dt} = S\frac{dD}{dt} = \varepsilon S\frac{dE}{dt} = \frac{\varepsilon S}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

$$I_d = 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{5\times 10^{-4}}{3\times 10^{-3}} \cdot 10^3 \times 50 \cos 10^3 t$$

$= 147.4 \cos 10^3 t$ nA

EQUATION OF CONTINUITY FOR TIME VARYING FIELDS

Equation of continuity in point form is

$$\nabla \cdot J = -\rho_v$$

where,

J = conduction current density (A/M^2)

P_υ = volume charge density (C/M^3), $\rho_v = \frac{\partial \rho_v}{\partial t}$

∇ = vector differential operator (1/m)

$$\nabla = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}$$

Proof: Consider a closed surface enclosing a charge *Q*. There exists an outward flow of current given by

$$I = \oint_S J \cdot dS$$

This is equation of continuity in **integral form.**

From the principle of conservation of charge, we have

$$I = \oint_S J \cdot dS = \frac{-dQ}{dt}$$

From the divergence theorem, we have

$$I = \oint_S J \cdot dS = \int_v (\nabla \cdot J) d\upsilon$$

Thus, $$\int_\upsilon (\nabla \cdot J) d\upsilon = \frac{-dQ}{dt}$$

By definition, $Q = \int_{\upsilon} \rho_\upsilon \, d\upsilon$

where, ρ_υ = volume charge density (C/m^3)

So, $\int_{\upsilon} (\nabla \cdot J) d\upsilon = \int \frac{\partial \rho_\upsilon}{\partial t} d\upsilon = \int - \dot{\rho}_\upsilon \, d\upsilon$

where $\dot{\rho}_\upsilon = \frac{\partial \rho_\upsilon}{\partial t}$

The volume integrals are equal only if their integrands are equal.

Thus, $\nabla \cdot \mathbf{J} = - \dot{\rho}_\upsilon$

MAXWELL'S EQUATIONS FOR STATIC EM FIELDS

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S D \cdot dS = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S B \cdot dS = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$	$\oint_L E \cdot dl = -\frac{\partial}{\partial t} \int_s B \cdot dS$	Faraday's Law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial D}{\partial t}$	$\oint_L H \cdot dl = \int_s J \cdot dS$	Ampere's circuit law

MAXWELL'S EQUATIONS FOR TIME VARYING FIELDS

These are basically four in number.

Maxwell's equations in **differential form** are given by

$$\nabla \times H = \frac{\partial D}{\partial t} + J$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

∂t

$\nabla . D = \rho_\upsilon$

$\nabla . B = 0$

Here,

H = magnetic field strength (A/m)

D = electric flux density, (C/m^2)

(∂D/∂t) = displacement electric current density

(A/m^2) J = conduction current density (A/m^2)

E = electric field (V/m)

B = magnetic flux density wb/m^2 or Tesla

(∂B/∂t) = time-derivative of magnetic flux density (wb/m^2 -

sec) B is called magnetic current density (V/m^2) or Tesla/sec

P_υ = volume charge density (C/m^3)

Maxwell's equations for time varying fields in **integral form** are given by

$$\oint_L H \cdot dL = \int_S \left(\dot{D} + J \right) \cdot dS$$

$$\oint_L E \cdot dL = -\int_S \dot{B} \cdot dS$$

$$\oint_S D \cdot dS = \int_\upsilon \rho_\upsilon \, d\upsilon$$

$$\oint_S B \cdot dS = 0$$

MEANING OF MAXWELL'S EQUATIONS

1. The first Maxwell's equation states that the magnetomotive force around a closed path is equal to the sum of electric displacement and, conduction currents through any surface bounded by the path.

2. The second law states that the electromotive force around a closed path is equal to the inflow of magnetic current through any surface bounded by the path.

3. The third law states that the total electric displacement flux passing through a closed surface (Gaussian surface) is equal to the total charge inside the surface.

4. The fourth law states that the total magnetic flux passing through any closed surface is zero.

MAXWELL'S EQUATIONS FOR STATIC FIELDS

Maxwell's Equations for static fields are:

$$\nabla \times H = J \leftrightarrow \oint_L H \cdot dL = \int_S J \cdot dS$$

$$\nabla \times E = 0 \leftrightarrow \oint_L E \cdot dL = 0$$

$$\nabla \cdot D = \rho_\upsilon \leftrightarrow \oint_S D \cdot dS = \oint_\upsilon \rho_\upsilon \, d\upsilon$$

$$\nabla \cdot B = 0 \leftrightarrow \oint_S B \cdot dS = 0$$

As the fields are static, all the field terms which have time derivatives are zero, that is, $\frac{\partial D}{\partial t} = 0$, $\frac{\partial B}{\partial t} = 0$.

PROOF OF MAXWELLS EQUATIONS

1. From Ampere's circuital law, we have

$$\nabla \times H = J$$

Take dot product on both sides

$$\nabla \cdot \nabla \times H = \nabla \cdot J$$

As the divergence of curl of a vector is zero,

$$RHS = \nabla \cdot J = 0$$

But the equation of continuity in point form is

$$\nabla \cdot J = \frac{-\partial \rho_\upsilon}{\partial t} = -\rho_\upsilon$$

This means that if $\nabla \times H = J$ is true, it is resulting in $\nabla \cdot J = 0$.

As the equation of continuity is more fundamental, Ampere's circuital law should be modified. Hence we can write

$$\nabla \times H = J + F$$ Take

dot product on both sides

$$\nabla \cdot \nabla \times H = \nabla \cdot J + \nabla \cdot F$$

that is, $\nabla \cdot \nabla \times H = 0 = \nabla \cdot J + \nabla \cdot F$

Substituting the value of $\nabla . J$ from the equation of continuity in the above expression, we get

$$\nabla \cdot F + (-\rho_{\upsilon}) = 0$$

or, $\nabla \cdot F = -\rho_{\upsilon}$

The point form of Gauss's law is

$$\nabla \cdot D = \rho_{\upsilon}$$

or, $\nabla \cdot D = \rho_{\upsilon}$

From the above expressions, we get

$$\nabla \cdot F = \nabla \cdot D$$

The divergence of two vectors are equal only if the vectors are identical,

that is, $F = D$

So, $\nabla \times H = D + J$

Hence proved.

2. According to Faraday's law,

$$emf = \frac{-d\phi}{dt}$$

ϕ = magnetic flux, (wb)

and by definition,

$$emf = \oint_L E \cdot dL$$

$$\oint_L E \cdot dL = \frac{-d\phi}{dt}$$

But $$\phi = \int_S B \cdot dS$$

$$\oint_L E \cdot dL = -\int_S \frac{\partial B}{\partial t} \cdot dS$$

$$= -\int_S B \cdot dS, \qquad B = \frac{\partial B}{\partial t}$$

Applying Stoke's theorem to LHS, we get

$$\oint_L E \cdot dL = -\int_S (\nabla \times E) \cdot dS$$

$$\int_S (\nabla \times E) \cdot dS = \int_S - B \cdot dS$$

Two surface integrals are equal only if their integrands are equal,

that is, $\nabla \times E = - B$

Hence proved.

3. From Gauss's law in electric field, we have

$$\oint_S D \cdot dS = Q = \int_\upsilon \rho_\upsilon \, d\upsilon$$

Applying divergence theorem to LHS, we get

$$\oint_S D \cdot dS = \int_\upsilon (\nabla \cdot D) d\upsilon = \int_\upsilon \rho_\upsilon \, d\upsilon$$

Two volume integrals are equal if their integrands are equal,

that is, $\nabla \cdot D = \rho_\upsilon$

Hence proved.

4. We have Gauss's law for magnetic fields as

$$\oint_S B \cdot dS = 0$$

RHS is zero as there are no isolated magnetic charges and the magnetic flux lines are closed loops.

Applying divergence theorem to LHS, we get

$$\oint_\upsilon \nabla \cdot B \, d\upsilon = 0$$

or,

$\nabla \cdot B = 0$ Hence proved.

PROBLEM 1:

Given $E = 10 \sin(\omega t - \beta y)\, a_y$ V/m, in free space, determine D, B and H.

Solution:

$E = 10 \sin(\omega t - \beta y)\, a_y$, V/m

$D = \epsilon_0 E$, $\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$D = 10\epsilon_0 \sin(\omega t - \beta y)\, a_y$, C/m^2

Second Maxwell's equation is

$\nabla \times E = -B$

That is,

$$\nabla \times E = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

or,

$$\nabla \times E = a_x \left[-\frac{\partial}{\partial z} E_y \right] + 0 + a_z \left[\frac{\partial}{\partial x} E_y \right]$$

As $E_y = 10 \sin(\omega t - \beta z)$ V/m

$$\frac{\partial E_y}{\partial x} = 0$$

Now, $\nabla \times E$ becomes

$$\nabla \times E = -\frac{\partial E_y}{\partial z} a_x$$

$= 10\, \beta \cos(\omega t - \beta z)\, a_x$

$$= -\frac{\partial B}{\partial t}$$

$$B = -\int 10\beta \cos(\omega t - \beta z)\, dt\; a_x$$

or

$$B = \frac{10\beta}{\omega} \sin(\omega t - \beta z)\, a_z\,,\ wb/m^2$$

and

$$H = \frac{B}{\mu_0} = \frac{10\beta}{\mu_0 \omega} \sin(\omega t - \beta z)\, a_z\,,\ A/m$$

PROBLEM 2: If the electric field strength, E of an electromagnetic wave in free space is given by $E = 2\cos\omega\left(t - \frac{z}{\upsilon_0}\right)a_y$ V/m, find the magnetic field, H.

Solution: We have

$\partial B/\partial t = -\nabla \times E$

$$= \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

$$= -\left[a_x\left[-\frac{\partial E_y}{\partial z}\right] + a_y(0) + a_z\left[\frac{\partial E_y}{\partial x}\right]\right]$$

$$= \frac{\partial E_y}{\partial z} a_x$$

$$= \frac{2\omega}{\upsilon_0}\sin\omega\left(t - \frac{z}{\upsilon_0}\right)a_x$$

$$B = \frac{2\omega}{\upsilon_0}\int \sin\omega\left(t - \frac{z}{\upsilon_0}\right)dt\, a_x$$

or, $$B = \frac{-2\omega}{\upsilon_0\omega}\cos\omega\left(t - \frac{z}{\upsilon_0}\right)$$

or, $$H = \frac{B}{\mu_0} = \frac{-2}{\upsilon_0\mu_0}\cos\omega\left(t - \frac{z}{\upsilon_0}\right)$$ $$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi\Omega$$

Thus, $$H = \frac{-2}{\eta_0}\cos\omega\left(t - \frac{z}{\upsilon_0}\right)$$ $$\left[\upsilon_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}}\right]$$

$$H = \frac{-1}{60\pi}\cos\omega\left(t - \frac{z}{\upsilon_0}\right)a_x \quad A/m$$

PROBLEM 3: If the electric field strength of a radio broadcast signal at a TV receiver is given by

$$E = 5.0 \cos(\omega t - \beta y)\, a_z, \text{ V/m},$$

determine the displacement current density. If the same field exists in a medium whose conductivity is given by 2.0 x 10^3 (mho)/cm, find the conduction current density.

Solution:

E at a TV receiver in free space

$$= 5.0 \cos(\omega t - \beta y)\, a_z, \text{ V/m}$$

Electric flux density

$$D = \in_0 E = 5 \in_0 \cos(\omega t - \beta y)\, a_z, \text{ V/m}$$

The displacement current density

$$J_d = D = \frac{\partial D}{\partial t}$$

$$= \frac{\partial}{\partial t}\left[-5 \in_0 \cos(\omega t - \beta y) a_z\right]$$

$$J_d = -5\in_0 \omega \sin(\omega t - \beta y)\, a_z, \text{ V/m}^2$$

The conduction current density,

$$J_c = \sigma E$$

$$\sigma = 2.0 \times 10^3 \text{ (mho) /cm}$$

$$= 2 \times 10^5 \text{ mho /m}$$

$$J_c = 2 \times 10^5 \times 5 \cos(\omega t - \beta y)\, a_z$$

$$\mathbf{J_c = 10^6 \cos(\omega t - \beta y)\, a_z \text{ V/m}^2}$$

UNIFORM PLANE WAVES

In free space (source-less regions where $\rho = \vec{J} = \sigma = 0$), The wave equation for electric field, in free-space is,

$$\nabla^2 \vec{E} = \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \_\_\_\_\_\_\_\_\_\_ (2)$$

The wave equation (2) is a composition of these equations, one each component wise,

ie,

$$\frac{\partial^2 Ex}{\partial x^2} = \mu \in \frac{\partial^2 Ey}{\partial t^2} \_\_\_\_\_\_\_\_\_\_ (2)\ a$$

$$\frac{\partial^2 Ey}{\partial y^2} = \mu \in \frac{\partial^2 Ey}{\partial t^2} \_\_\_\_\_\_\_\_\_\_ (2)\ b$$

$$\frac{\partial^2 Ez}{\partial z^2} = \mu \in \frac{\partial^2 Ez}{\partial t^2} \_\_\_\_\_\_\_\_\_\_ (2)\ c$$

Further, eqn. (1) may be written as

$$\frac{\partial Ex}{\partial x} + \frac{\partial Ey}{\partial y} + \frac{\partial Ez}{\partial z} = 0 \_\_\_\_\_\_\_\_\_\_ (1)\ a$$

For the UPW, $\vec{E}$ is independent of two coordinate axes; x and y axes, as we have assumed.

$$\therefore \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

Therefore eqn. (1) reduces to

$$\frac{\partial E_z}{\partial z} = 0 \_\_\_\_\_\_\_\_\_ (3)$$

ie., there is no variation of E_z in the z direction.

Also we find from 2 (a) that $\frac{\partial^2 Ez}{\partial t^2}$ = 0 ____(4)

These two conditions (3) and (4) require that E_z can be

(1) Zero

(2) Constant in time or

(3) Increasing uniformly with time.

A field satisfying the last two of the above three conditions cannot be a part of wave motion. Therefore E_z can be put equal to zero, (the first condition).

$$\boxed{E_z = 0}$$

The uniform plane wave (traveling in z direction) does not have any field components of $\vec{E}$ & $\vec{H}$ in its direction of travel.

Therefore the UPWs are transverse., having field components (of $\vec{E}$ & $\vec{H}$) only in directions perpendicular to the direction of propagation does not have any field component only the direction of travel.

RELATION BETWEEN $\vec{E}$ & $\vec{H}$ in a uniform plane wave.

We have, from our previous discussions that, for a UPW traveling in z direction, both $\vec{E}$ & $\vec{H}$ are independent of x and y; and $\vec{E}$ & $\vec{H}$ have no z component. For such a UPW, we have,

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x}(=0) & \frac{\partial}{\partial y}(=0) & \frac{\partial}{\partial z} \\ Ex & Ey & Ez(=0) \end{vmatrix} = \hat{i}\left(-\frac{\partial E_y}{\partial_z}\right) + \hat{j}\left(\frac{\partial E_x}{\partial_z}\right) \text{\_\_\_\_\_\_} (5)$$

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x}(=0) & \frac{\partial}{\partial y}(=0) & \frac{\partial}{\partial z} \\ Hx & Hy & Hz(=0) \end{vmatrix} = \hat{i}\left(-\frac{\partial H_y}{\partial_z}\right) + \hat{j}\left(\frac{\partial H_x}{\partial_z}\right) \text{\_\_\_\_\_\_} (6)$$

Then Maxwell's curl equations (1) and (2), using (5) and (6), (2) becomes,

$$\nabla \times \vec{H} = \in \frac{\partial \vec{E}}{\partial t} = \quad \in \frac{\partial Ex}{\partial t}\hat{i} + \in \frac{\partial Ey}{\partial t}\hat{j} = \hat{i}\left(-\frac{\partial Hy}{\partial z}\right) + \hat{j}\left(\frac{\partial Hx}{\partial z}\right) \text{\_\_\_\_\_\_} (7)$$

$$and$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = \quad \mu \frac{\partial Hx}{\partial t}\hat{i} - \mu \frac{\partial Hy}{\partial t}\hat{j} = i\left(-\frac{\partial Ey}{\partial z}\right) + \hat{j}\left(\frac{\partial Ex}{\partial z}\right) \text{\_\_\_\_\_\_} (8)$$

Thus, rewriting (7) and (8) we get

$$-\frac{\partial Hy}{\partial z}\,\hat{i} + \frac{\partial Hx}{\partial z}\,\hat{j} \quad = \quad \in\left(\frac{\partial Ex}{\partial t}\,\hat{i} + \frac{\partial Ey}{\partial t}\,\hat{j}\right) \_\_\_\_\_\_\_ (7)$$

$$-\frac{\partial Ey}{\partial z}\,\hat{i} + \frac{\partial Ex}{\partial z}\,\hat{j} \quad = -\mu\left(\frac{\partial Hx}{\partial t}\,\hat{i} - \frac{\partial Hy}{\partial t}\,\hat{j}\right) \_\_\_\_\_\_\_ (8)$$

Equating $\hat{i}$ th and j th terms, we get

$$-\frac{\partial Hy}{\partial z} = \in \frac{\partial Ex}{\partial t} \qquad \_\_\_\_\_\_\_\_\_ 9\,(a)$$

$$\frac{\partial Hx}{\partial z} = \in \frac{\partial Ey}{\partial t} \qquad \_\_\_\_\_\_\_\_\_ 9\,(b)$$

$$-\frac{\partial Ey}{\partial z}\,\hat{i} - \mu\,\frac{\partial Hx}{\partial t} \qquad \_\_\_\_\_\_\_\_\_ 9(c)$$

$$and$$

$$\frac{\partial Ex}{\partial z} = \mu\,\frac{\partial Hy}{\partial t} \qquad \_\_\_\_\_\_\_\_\_ 9\,(d)$$

$$Let$$

$$Ey = f_1\left(z - \nu_0 t\right); \qquad \nu_0 = \frac{1}{\sqrt{\mu E}}. \quad Then,$$

$$\frac{\partial Ey}{\partial t} = f_1\left(z - \nu_0 t\right)\left(-\nu_0\right). = -\nu_0\, f_1.$$

$$\therefore\ From\ eqn.\,9(c),\ we\ get,$$

$$\frac{\partial Hx}{\partial t} = -\frac{\nu_0}{\mu}\,f' \quad = \quad -\sqrt{\frac{\in_0}{\mu_0}}\;f'_1$$

$$\therefore\ Hx = -\sqrt{\frac{\in}{\mu_0}} \quad \int f_1'\;dz + c.$$

$$Now$$

$$\frac{\partial f_1^{'}}{\partial z} = f_1^{'} \frac{\partial (z - \nu_0 t)}{\partial z} = f_1^{'}$$

$$\therefore H_z = -\sqrt{\frac{\in}{\mu}} \int \frac{\partial f_1}{\partial z} + C$$

$$Now$$

$$\frac{\partial f_1^{'}}{\partial z} = f_1^{'} \frac{\partial (z - \nu_0 t)}{\partial z} = f_1^{'}$$

$$\therefore = -\sqrt{\frac{\in}{\mu}} \int \frac{\partial f_1}{\partial z} dz + c = -\sqrt{\frac{\in}{\mu}} f_1 + c$$

$$Hx = -\sqrt{\frac{\in}{\mu}} Ey + c$$

The constan C indicates that a field independent of Z could be present. Evidently this is not a part of the wave motion and hence is rejected.

Thus the relation between H_X and E_Y becomes,

$$H_x = -\sqrt{\frac{\in}{\mu}} E_y$$

$$\therefore \frac{E_y}{H_x} = -\sqrt{\frac{\mu}{\in}} \text{ ______________ } (10)$$

Similarly it can be shown that

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\in}} \text{ ____________ } (11)$$

In our UPW, $\vec{E} = E_x \hat{i} + E_y \hat{j}$

$$\nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \right)$$

$$\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \_\_\_\_\_\_\_\_\_ (xi)$$

$$But\ \nabla \cdot \vec{E} = \frac{\rho}{\in_0}$$

$$\vec{E}$$

DERIVATION OF WAVE EQUATION FOR A CONDUCTING MEDIUM:

In a conducting medium, □ = $□_0$, □ = $□_0$. Surface charges and hence surface currents exist, static fields or charges do not exist.

For the case of conduction media, the point form of maxwells equations are:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (i)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (ii)$$

$$\nabla \cdot \vec{D} = \nabla \cdot \in \vec{E} = \in \nabla \cdot \vec{E} = 0 \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (iii)$$

$$\nabla \cdot \vec{B} = \nabla \cdot \mu \vec{H} = \mu \nabla \cdot \vec{H} = 0 \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (iv)$$

Taking curl on both sides of equation (i), *we get*

$$\nabla \times \nabla \times \vec{H} = \nabla \times \left(\sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \right)$$

$$= \sigma \, \nabla \times \vec{E} + \in \frac{\partial}{\partial t} \nabla \times \vec{E} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (v)$$

substituting eqn. (ii) *in eqn.* (v), *we get*

$$\nabla \times \nabla \times \vec{H} = \sigma \left(-\mu \frac{\partial \vec{H}}{\partial t} \right) + \in \left(-\mu \frac{\partial^2 \vec{H}}{\partial t^2} \right) \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (vi)$$

But $\quad \nabla \times \nabla \times \vec{H} = \nabla \left(\nabla \cdot \vec{H} \right) - \nabla^2 \vec{H} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (vii)$

$\therefore$ *eqn.* (vi) *becomes*

$$\therefore \nabla \left(\nabla \cdot \vec{H} \right) - \nabla^2 \vec{H} = -\mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (viii)$$

But $\nabla \cdot \vec{H} = \nabla \cdot \frac{\vec{B}}{\mu} = \frac{1}{\mu} \nabla \cdot \vec{B} = \frac{1}{\mu} \cdot 0 = 0$

$\therefore$ *eqn.* $(viii)$ *becomes,*

$$\nabla^2 \vec{H} - \mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (ix)$$

This is the wave equation for the magnetic field $\vec{H}$ in a conducting medium.

Next we consider the second Maxwell's curl equation (ii)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \_\_\_\_\_\_\_\_\_\_ \quad (ii)$$

Taking curl on both sides of equation (ii) we get

$$\nabla \times \nabla \times \vec{E} = -\mu \nabla \times \frac{\partial \vec{H}}{\partial t} = -\mu \frac{\partial \left(\nabla \times \vec{H} \right)}{\partial t} \quad \_\_\_\_\_\_\_\_\_\_ \quad (x)$$

$$But \ \nabla \times \nabla \times \vec{E} = \nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} \ ;$$

Vector identity and substituting eqn. (1) in eqn (2), we get

$$\nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} \left(\sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \right)$$

$$= -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \quad \_\_\_\_\_\_\_\_\_\_ \quad (xi)$$

$$But \ \nabla \cdot \vec{E} = \frac{\rho}{\in_0}$$

(Point form of Gauss law) However, in a conductor, ρ = 0, since there is no net charge within a conductor,

Therefore we get $\nabla \cdot \vec{E} = 0$

Therefore eqn. (xi) becomes,

$$\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \quad \_\_\_\_\_\_\_\_\_\_ \quad (xii)$$

This is the wave equation for electric field $\vec{E}$ in a conducting medium.

<u>Wave equations for a conducting medium:</u>

1. Regions where conductivity is non-zero.
2. Conduction currents may exist.

For such regions, for time varying fields

The Maxwell's eqn. Are:

$$\nabla \times \vec{H} = \vec{J} + \in \frac{\partial \vec{E}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (1)$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (2)$$

$$\vec{J} = \sigma \vec{E} \qquad \sigma : Conductivity\ (\Omega / m)$$

= conduction current density.

Therefore eqn. (1) becomes,

$$\nabla \times \vec{H} = \sigma \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (3)$$

Taking curl of both sides of eqn. (2), we get

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

$$= -\mu \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (4)$$

$$But$$

$$\nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \Box \vec{E}\right) - \nabla^2 \vec{E} \ (vector\ identity)$$

$$u \sin g\ this\ eqn.\ (4)\ becomes\ vector\ identity,$$

$$\nabla\left(\nabla \Box \vec{E}\right) - \nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial^2 \vec{E}}{\partial t^2} \text{ \_\_\_\_\_\_\_\_\_\_ } (5)$$

$$\therefore But\ \nabla \Box \vec{D} = \rho$$

$$\because \in is\ cons\tan t, \quad \nabla \Box \vec{E} = \frac{1}{\in} \nabla \Box \vec{D}$$

Since there is no net charge within a conductor the charge density is zero (there can be charge on the surface), we get.

$$\nabla \Box \vec{E} = \frac{1}{\in} \nabla \Box \vec{D} = 0$$

Therefore using this result in eqn. (5)

we get

$$\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \text{ \_\_\_\_\_\_\_\_\_\_ } (6)$$

This is the wave eqn. For the electric field $\vec{E}$ in a conducting medium.

This is the wave eqn. for $\vec{E}$. The wave eqn. for $\vec{H}$ is obtained in a similar manner.

Taking curl of both sides of (1), we get

$$\nabla \times \nabla \times \vec{H} = \in \nabla \times \frac{\partial \vec{E}}{\partial t} + \sigma \nabla \times \vec{E} \_\_\_\_\_\_\_\_\_\_ (7)$$

$$But \ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \_\_\_\_\_\_\_\_\_\_ (2)$$

$$\therefore (1) \ becomes,$$

$$\nabla \times \nabla \times \vec{H} = -\mu \in \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\sigma \frac{\partial \vec{H}}{\partial t} \_\_\_\_\_\_\_\_\_\_ (8)$$

As before, we make use of the vector identity.

$$\nabla \times \nabla \times \vec{H} = \nabla \left(\nabla \cdot \vec{H} \right) - \nabla^2 \vec{H}$$

in eqn. (8) and get

$$\nabla \left(\nabla \cdot \vec{H} \right) - \nabla^2 \vec{H} = -\mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} \_\_\_\_\_\_\_\_ (9)$$

$$But$$

$$\nabla \cdot \vec{H} = \nabla \cdot \frac{\vec{B}}{\mu} = \frac{1}{\mu} \nabla \cdot \vec{B} = \frac{1}{\mu} \cdot 0 = 0$$

$$\therefore eqn.(9) becomes$$

$$\nabla^2 \cdot \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} \_\_\_\_\_\_\_\_ (10)$$

This is the wave eqn. for $\vec{H}$ in a conducting medium.

<u>Sinusoidal Time Variations:</u>

In practice, most generators produce voltage and currents and hence electric and magnetic fields which vary sinusoidally with time. Further, any periodic variation can be represented as a weight sum of fundamental and harmonic frequencies.

Therefore we consider fields having sinusoidal time variations, for example,

$E = E_m \cos \omega t$

$E = E_m \sin \omega t$

Here, w = $2\pi f$, f = frequency of the variation.

Therefore every field or field component varies sinusoidally, mathematically by an additional term. Representing sinusoidal variation. For example, the electric field $\vec{E}$ can be represented as

$$\vec{E}(x, y, z, t)\, as$$

$$ie., \tilde{\vec{E}}(\vec{r}, t); \vec{r}(x, y, z)$$

Where $\tilde{\vec{E}}$ is the time varying field.

The time varying electric field can be equivalently represented, in terms of corresponding phasor quantity $\tilde{\vec{E}}$ (r) as

$$\tilde{\vec{E}}(\vec{r}, t) = R_e\left[\vec{E}(r)e^{j\omega t}\right] \_\_\_\_\_\_\_\_\_\_ (11)$$

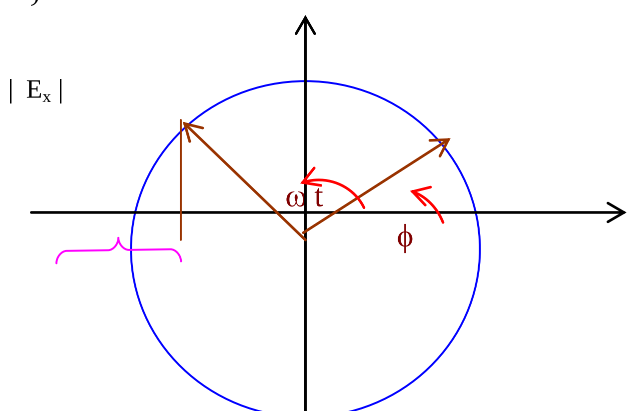
The symbol 'tilda' placed above the E vector represents that $\vec{E}$ is time – varying quantity.

<u>The phasor notation:</u>

We consider only one component at a time, say E_x.

The phasor E_x is defined by

$$\tilde{E}_x(\vec{r}, t) = R_e\{E_x(r)\ \ e^{j\omega t}\} \_\_\_\_\_\_\_\_\_\_ (12)$$



$E_x(\vec{r})$ denotes E_x as a function of space (x,y,z). In general $E_x(\vec{r})$ is complex and hence can be represented as a point in a complex and hence can be represented as a point in a complex plane. (see fig) Multiplication by e^{jwt} results in a rotation through an angle wt measured from the angle □. At t increases, the point E_x e^{jwt} traces out a circle with center at the origin. Its projection on the real axis varies sinusoidally with time & we get the time-harmonically varying electric field $\tilde{E}x$ (varying sinusoidally with time). We note that the phase of the sinusoid is determined by □, the argument of the complex number E_x.

Therefore the time varying quantity may be expressed as

$$\tilde{E}_x = R_e\{|E_x|\, e^{j\phi}\ e^{j\omega t}\} \_\_\_\_\_\_\_\_ (13)$$

$$= |E_x| \cos(\omega t + \phi) \_\_\_\_\_\_\_\_ (14)$$

<u>Maxwell's eqn. in phasor notation:</u>

In time – harmonic form, the Maxwell's first curl eqn. is:

$$\nabla \times \tilde{\vec{H}} = \tilde{\vec{J}} + \frac{\partial \tilde{\vec{D}}}{\partial t} \text{ \_\_\_\_\_\_\_\_ } (15)$$

using phasor notation, this eqn. becomes,

$$\nabla \times R_e\left(\vec{H}e^{j\omega t}\right) = \frac{\partial}{\partial t} R_e\left[\vec{D}e^{j\omega t}\right] + R_e\left[\vec{J}e^{j\omega t}\right] \text{ \_\_\_\_\_\_\_\_ } (16)$$

The diff. Operator ∇ & R_e part operator may be interchanged to get,

$$R_e\left(\nabla \times \vec{H}e^{j\omega t}\right) = R_e\left[\frac{\partial}{\partial t}\left(\vec{D}e^{j\omega t}\right) + R_e\left[\vec{J}e^{j\omega t}\right]\right.$$

$$= R_e\left[j\omega\vec{D} \quad e^{j\omega t}\right] + R_e\left[\vec{J}e^{j\omega t}\right]$$

$\therefore$

$$R_e\left[\left(\nabla \times \vec{H} - j\omega\vec{D} - \vec{J}\right)e^{j\omega t}\right] = 0$$

This relation is valid for all t. Thus we get

$$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D} \text{ \_\_\_\_\_\_\_\_\_\_ } (17)$$

This phasor form can be obtained from time-varying form by replacing each time derivative by

$$jw\left(ie., \quad \frac{\partial}{\partial t} \text{is to be replaced by} \int \omega\right)$$

For the sinusoidal time variations, the Maxwell's equation may be expressed in phasor form as:

$$(17) \quad \nabla \times \vec{H} = \vec{J} + j\omega\vec{D} \qquad \oint_L \vec{H} \cdot d\vec{L} = \int_S \left(J + j\omega\vec{D}\right) \cdot d\vec{s}$$

$$(18) \quad \nabla \times \vec{E} = -j\omega\vec{B} \qquad \oint_L \vec{E} \cdot d\vec{l} = \int_S -j\omega\vec{B} \cdot d\vec{s}$$

$$(19) \quad \nabla \cdot \vec{D} = \rho \qquad \oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_V \, d_V$$

$$(20) \quad \nabla \cdot \vec{B} = 0 \qquad \oint_S \vec{B} \cdot d\vec{s} = 0$$

The continuity eqn., contained within these is,

$$\nabla \cdot \vec{J} = -j\omega\rho \qquad \oint_S \vec{J} \cdot d\vec{s} = -\int_{vol} j\omega\rho \, dv \text{ \_\_\_\_\_\_\_\_ } (21)$$

The constitutive eqn. retain their forms:

$$\vec{D} = \in \vec{E}$$
$$\vec{B} = \mu\vec{H}$$
$$\vec{J} = \sigma\vec{E} \qquad \text{\_\_\_\_ } (22)$$

For sinusoidal time variations, the wave equations become

$$\left\{\nabla^2 \vec{E} = -\omega^2 \mu \in \vec{E} \qquad (\,for\ electric\ \ field\,)\ \right\}$$

$$\left\{\nabla^2 \vec{H} = -\omega^2 \mu \in \vec{H} \qquad (\,for\ electric\ \ field\,)\right\} \text{\_\_\_\_\_\_\_\_\_\_ (23)}$$

Vector Helmholtz eqn.

In a conducting medium, these become

$$\nabla^2 \vec{E} + \left(\omega^2 \mu \in - j\omega\mu\sigma\right) \vec{E} = 0$$

$$\nabla^2 \vec{H} + \left(\omega^2 \mu \in - j\omega\mu\sigma\right) \vec{H} = 0 \qquad \text{\_\_\_\_\_\_\_\_ (24)}$$

Wave propagation in a loss less medium:

In phasor form, the wave eqn. for VPW is

$$\left.\begin{aligned}\frac{\partial^2 \vec{E}}{\partial x^2} &= -\omega^2 \mu \in \vec{E} \\ &= -\beta^2 \vec{E}\end{aligned}\right\} ;\ \frac{\partial^2 E_y}{\partial x^2} = -\beta^2 E_y \text{\_\_\_\_\_\_\_\_\_\_ (25)}$$

$$\therefore E_y = C_1 \quad e^{-j\beta x} + C_2 e^{j\beta x} \text{\_\_\_\_\_\_\_\_\_\_ (26)}$$

C_1 & C_2 are arbitrary constants.

The corresponding time varying field is

$$\begin{aligned}\tilde{E}_y(x,t) &= R_e\left[E_y(x)e^{j\omega t}\right] \\ &= R_e\left[C_1\ e^{j(\omega t - \beta z)} + C_2\ e^{j(\omega t + \beta z)}\ \right] \text{\_\_\_\_\_\_\_\_\_ (27)} \\ &= C_1 \cos(\omega t - \beta z) + C_2\ \cos(\omega t + \beta z) \text{\_\_\_\_\_\_\_\_\_ (28)}\end{aligned}$$

When C_1 and C_2 are real.

Therefore we note that, in a homogeneous, lossless medium, the assumption of sinusoidal time variations results in a space variation which is also sinusoidal.

Eqn. (27) and (28) represent sum of two waves traveling in opposite directions.

If $C_1 = C_2$, the two traveling waves combine to form a simple standing wave which does not progress.

If we rewrite eqn. (28) with E_y as a f_n of (x-□t),

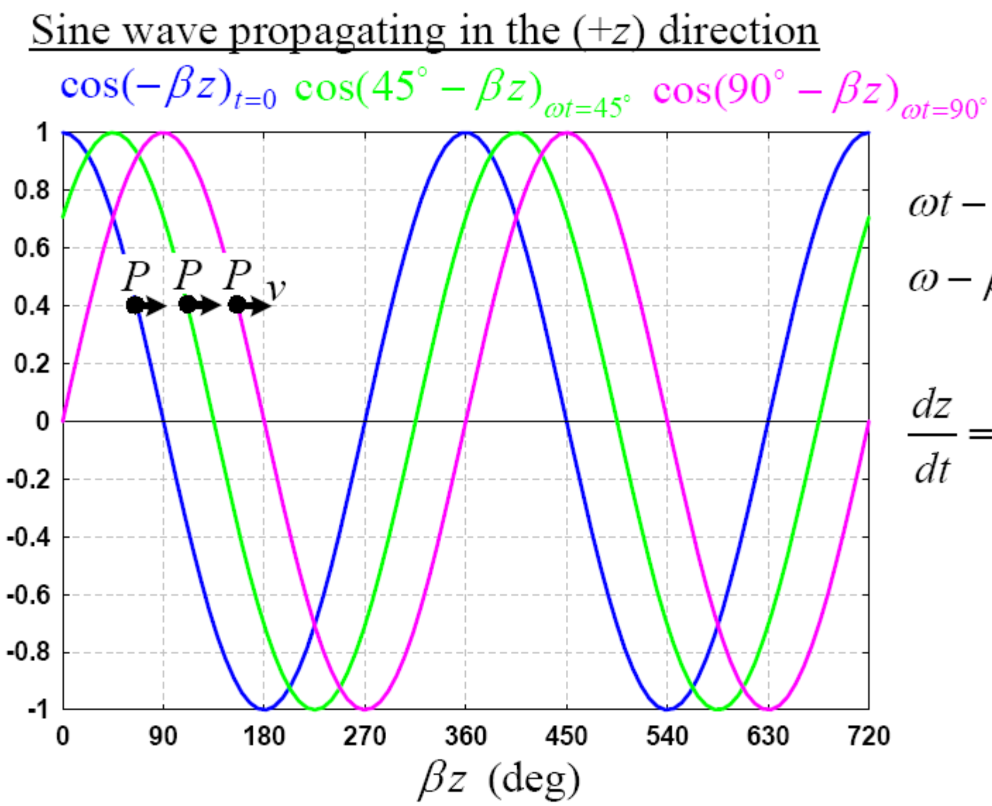
we get □ = $\dfrac{\omega}{\rho}$

Let us identify some point in the waveform and observe its velocity; this point is $(\omega t - \beta x) = a$ constant

Then $$\upsilon = \frac{dx}{dt} = \frac{\omega}{\beta} \qquad \because \frac{\partial x}{\partial t} = \frac{\partial^{\left(\frac{'a' - \omega t}{}\right)}}{\beta} = \frac{\omega}{\beta}$$

This velocity is called phase velocity, the velocity of a phase point in the wave.

β is called the phase shift constant of the wave.



$$\omega t - \beta z = \xi_P = con$$

$$\omega - \beta \frac{dz}{dt} = 0$$

$$\frac{dz}{dt} = \boxed{v = \frac{\omega}{\beta}}, \quad \frac{\text{m}}{\text{s}}$$

Wavelength: These distance over which the sinusoidal waveform passes through a full cycle of 2π radians

ie.,

$$\beta\lambda = 2\pi$$

$$\beta = \frac{2\pi}{\lambda} \quad or \quad \lambda = \frac{2\pi}{\beta}$$

But

$$\beta = \frac{\omega}{\upsilon} \qquad \therefore \lambda = \frac{2\pi\upsilon}{\omega} = \frac{\upsilon}{f}$$

or

$$\upsilon = f\lambda; \qquad f \ in \ H_z$$

$$\upsilon : \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \in}} = \upsilon_o$$

Wave propagation in a conducting medium

We have,

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$$

$$\gamma^2 = -\omega^2 \mu \in + j\omega\mu\sigma$$

$$= j\omega\mu(\sigma + j\omega \in)$$

Where

□ is called the propagation constant is, in general, complex.

Therefore, □ = □ + j□

□ = Attenuation constant

□ = phase shift constant.

The eqn. for UPW of electric field strength is

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \gamma^2 \vec{E}$$

One possible solution is

$$\vec{E}(x) = E_0 e^{-\gamma x}$$

Therefore in time varying form, we get

$$\tilde{\vec{E}}(x,t) = R_e\left[E\, e^{-\gamma x}\, e^{j\omega t}\right]$$

$$= e^{-\alpha x}\, R_e\left[E_0 e^{jwt}\right]$$

This eqn. shown that a up wave traveling in the +x direction and attenuated by a factor $e^{-\alpha x}$.

The phase shift factor

$$\beta = \frac{2\pi}{\lambda}$$

$$and\ velocity = f\lambda = \frac{\omega}{\beta}$$

□ = Real part of □ = RP $\sqrt{j\omega\mu(\sigma + j\omega t)}$

$$\omega\sqrt{\frac{\mu \in}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2 \in^2}} - 1\right)}$$

$$\beta = \omega\sqrt{\frac{\mu \in}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2 \in^2}} + 1\right)}$$

=

Conductors and dielectrics:

We have the phasor form of the 1st Maxwell's curl eqn.

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega \in \vec{E} = J_c + J_{disp}$$

where $J_c = \sigma \vec{E} =$ conduction current density (A/m^2)

$J_{disp} = j\omega \in \vec{E} =$ displacement current density (A/m^2)

$$\therefore \left| \frac{J_{cond}}{J_{disp}} \right| = \frac{\sigma}{\omega \in}$$

We can choose a demarcation between dielectrics and conductors;

$$\frac{\sigma}{\omega \in} = 1$$

* $\frac{\sigma}{\omega \in} > 1$ is conductor. Cu: $3.5*10^8$ @ 30 GHz

* $\frac{\sigma}{\omega \in} < 1$ is dielectric. Mica: 0.0002 @ audio and RF

* For good conductors, □ & □ are independent of freq.

* For most dialectics, □ & □ are function of freq.

* $\frac{\sigma}{\omega \in}$ is relatively constant over frequency range of interest

Therefore dielectric " constant "

* $\frac{\sigma}{\omega \in}$ dissipation factor D

if D is small, dissipation factor is practically as the power factor of the dielectric.

PF = sin□

□ = $\tan^{-1}$D

PF & D difference by <1% when their values are less than 0.15.

Example 11.1

1. Express

$$E_y = 100 \cos\left(2\pi 10^8 t - 0.5z + 30^0\right) v/m \; as \; a \; phasor$$

$$E_y = R_e \left[100 \, e^{j 2\pi \times 10^6 t - 0.5z + 30^0} \right]$$

Drop R_e and suppress e^{jwt} term to get phasor

Therefore phasor form of E_{ys} = $100e^{-0.5z+30^0}$

Whereas E_y is real, E_{ys} is in general complex.

Note: 0.5z is in radians; 30^0 in degrees.

Example 11.2

Given

$$\vec{E}_s = 100\ \angle 30^0\,\hat{a}x + 20\ \angle -50^0\,\hat{a}y + 40\ \angle 210^0\,\hat{a}z, V/m$$

find its time varying form representation

Let us rewrite $\vec{E}_s$ as

$$\vec{E}_s = 100e^{j30^0}\,\hat{a}x + 20e^{-j50^0}\,\hat{a}y + 40e^{j210^0}\,\hat{a}z. V/m$$

$$\therefore \tilde{\vec{E}} = R_e\left[E_s \quad e^{j\omega t}\right]$$

$$= R_e\left[100e^{j\left(\omega t + 30^0\right)} + 20e^{j\left(\omega t - 50^0\right)} + 40e^{j\left(\omega t + 210^0\right)}\right] V/m$$

$$\tilde{\vec{E}} = 100\ \cos\left(\omega t + 30^0\right) 20\ \cos\left(\omega t - 50^0\right) + 40\ \cos\left(\omega t + 210^0\right) V/m$$

None of the amplitudes or phase angles in this are expressed as a function of x,y or z.

Even if so, the procedure is still effective.

2. Consider

$$H_s = 20e^{-(0.1 + j20)z}\,\hat{a}x\ A/m$$

$$\tilde{\vec{H}}(t) = R_e\left[20e^{-(0.1+j20)z}\,\hat{a}x \quad e^{j\omega t}\right]$$

$$= 20e^{-0.1z}\cos\left(\omega t - 20z\right)\hat{a}x\ A/m$$

$$E_x = E_x\left(x, y, z\right)$$

$$Note: \qquad consider\ \frac{\partial E_x}{\partial t} = \frac{\partial}{\partial t} R_e\left[E_x\left(x, y, z\right) \quad e^{j\omega t}\right]$$

$$= R_e\left[j\omega E_x e^{j\omega t}\right]$$

Therefore taking the partial derivative of any field quantity wrt time is equivalent to multiplying the corresponding phasor by j □.

Example

Given

$$\vec{E}_{0s} = \left(500\angle -40^0\,\hat{a}y + \left(200 - j600\right)\hat{a}z\right) e^{-j0.4x}\ V/m$$

$$Find\ (a)\ \omega$$

$$(b)\ \vec{E}\ at\ (2,3,1)\ at\ t = 0$$

$$(c)\ \vec{E}\ at\ (2,3,1)\ at\ t = 10\ ns.$$

$$(d)\ \vec{E}\ at\ (3,4,2)\ at\ t = 20\ ns.$$

Q. From given data,

$$\beta = 0.4 = \omega\sqrt{\mu_0 \in_0}$$

$$\therefore \omega = \frac{0.4\times3\times10^8}{\sqrt{4\pi\times10^{-7}\times\frac{10^{-9}}{36\pi^{-9}}}} = 120\times10^6$$

$$f = 19.1\times10^6\,Hz$$

R. Given,

$$\vec{E}_s = \left(500\angle -40^0\ \hat{a}y + (200 - j600)\,\hat{a}z\right)e^{-j0.4x}$$

$$= 500e^{-j40}\ e^{-j0.4x}\ \hat{a}y + 632.456e^{-j71.565^0}\ e^{-j0.4x}\ \hat{a}z$$

$$= 500e^{-j\left(0.4x+40^0\right)}\hat{a}y + 632.456e^{-j\left(0.4x+71.565^0\right)}\ \hat{a}z$$

$$\vec{E}(t) = 500\ R_e\left[e^{+j\omega t}\ e^{-j\left(0.4x+40^0\right)}\ \hat{a}y + 632.456\ e^{j\omega t}\ e^{-j\left(0.4x+71.565^0\right)}\ \hat{a}z\right]$$

$$= 500\ \cos\left(\omega t - 0.4x - 40^0\right)\hat{a}y + 632.456\ \cos\left(\omega t - 0.4x - 71.565\right)\hat{a}z$$

$$\vec{E}\ at\ (2,3,1)t = 0 = 500\ \cos\left(-0.4x - 40^0\right)\hat{a}y + 632.456\left(-0.4x - 71.565\right)\hat{a}z$$

$$= 36.297\ \hat{a}y - 291.076\ \hat{a}z\ V/m$$

c)

$$\vec{E}\ at\ (t = 10\ ns)\ at\ (2,3,1)$$

$$= 500\ \cos\left(120\times10^6\times10\times10^{-9} - 0.4\times2 - 40^0\right)\hat{a}y$$

$$+ 632.456\ \cos\left(120\times10^6\times10\times10^{-9} - 0.4\times2 - 71.565^0\right)\hat{a}z$$

$$= 477.823\ \hat{a}y + 417.473\ \hat{a}z\ V/m$$

d)

at t = 20 ns,

$$\vec{E}\ at\ (2,3,1)$$

$$= 438.736\ \hat{a}y + 631.644\ \hat{a}z\ V/m$$

D 11.2:

Given $\vec{H}_s = \left(2\angle -40^0\ \hat{a}x - 3\angle 20\ \hat{a}y\right) e^{-j0.07z} \quad A/m$ for a UPW traveling in free space. Find

(a) □ (b) H_x at p(1,2,3) at t = 31 ns. (c) $\left|\vec{H}\right|$ at t=0 at the origin.

(a) we have p = 0.07 $(e^{-j\beta z}\ term)$

$$\therefore \omega\sqrt{\mu \in} = 0.07$$

$$\omega = \frac{0.07}{\sqrt{\mu \in}} = 0.07\times 3\times 10^8 = 21.0\times 10^6\ rad/\sec$$

$$= 21.0\times 10^6\ rad/\sec$$

(b)

$$\vec{H}(t) = R_e\left\{\left[2\,e^{-j40^0}\ e^{-j0.07z}\ \hat{a}x - 3\,e^{j20^0}\ e^{-j0.07z}\ \hat{a}y\ \right]e^{j\omega t}\right\}$$

$$= 2\cos\left(\omega t - 0.07z - 40^0\right)\hat{a}x - 3\cos\left(\omega t - 0.07z + 20^0\right)\hat{a}y$$

$$H_x(t) = 2\cos\left(\omega t - 0.07z - 40^0\right)$$

$$H_x(t)\ at\ p(1,2,3)$$

$$= 2\cos\left(2.1\times 10^6 t - 0.21 - 40^0\right)$$

$$At\ t = 31n\sec; = 2\cos\left(2.1\times 10^6\times 31\times 10^{-9} - 0.21 - 40^0\right)$$

$$= 2\cos\left(651\times 10^{-3} - 0.21 - 40^0\right)$$

$$= 1.9333 \quad A/m$$

(c)

$$\vec{H}(t)\ at\ t = 0 = 2\cos\left(-0.07z - 0.7\right)\hat{a}x - 3\cos\left(-0.7z + 0.35\right)\hat{a}y$$

$$\vec{H}(t) = 2\cos\left(0.7\right)\hat{a}x - 3\cos\left(0.3\right)\hat{a}y$$

$$= 1.53\hat{a}x - 2.82\hat{a}y$$

$$= 3.20666 \quad A/m$$

In free space,

$$E(z,t) = 120\sin(\omega t - \beta z)\hat{a}y \quad V/m$$

$$find \quad H(z,t)$$

$$we\ have \quad \frac{E_y}{H_x} = -\eta = -120\pi$$

$$\therefore H_x = -\frac{E_y}{120\pi} = -\frac{120}{120\pi}\sin(\omega t - \beta z)\hat{a}y$$

$$= -\frac{1}{\pi}\sin(\omega t - \beta z)$$

$$\therefore \vec{H}(z,t) = -\frac{1}{\pi}\sin(\omega t - \beta z)\hat{a}x$$

Problem 3. J&B

Non uniform plans waves also can exist under special conditions. Show that the function

$$F = e^{-\alpha z}\sin\frac{\omega}{\upsilon}(x - \upsilon t)$$

satisfies the wave equation $\nabla^2 F = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$

provided the wave velocity is given by

$$\upsilon = e\sqrt{\left(1 + \frac{\alpha^2 c^2}{\omega^2}\right)}$$

Ans:

From the given eqn. for F, we note that F is a function of x and z,

$$\therefore \nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$$

$$\frac{\partial F}{\partial x} = e^{-\alpha z}\frac{\omega}{\upsilon}\cos\frac{\omega}{\upsilon}(x - \upsilon t)$$

$$\frac{\partial^2 F}{\partial x^2} = -e^{-\alpha z}\left(\frac{\omega}{\upsilon}\right)\left(\frac{\omega}{\upsilon}\right)\sin\frac{\omega}{\upsilon}(x - \upsilon t) = -\frac{\omega^2 e^{-\alpha z}}{\upsilon^2}F$$

$$\frac{\partial F}{\partial z} = -e^{-\alpha z}\sin\frac{\omega}{\upsilon}(x - \upsilon t)$$

$$\frac{\partial^2 F}{\partial z^2} = +\alpha^2 e^{-\alpha z}\sin\frac{\omega}{\upsilon}(x - \upsilon t) = \alpha^2 F$$

$$\therefore \nabla^2 F = \left(-\frac{\omega^2}{\upsilon^2} + \alpha^2\right) F$$

$$\frac{dF}{dt} = e^{-\alpha z}\left(\frac{\omega}{\upsilon}\right)(-\upsilon)\cos\frac{\omega}{\upsilon}(x - \upsilon t)$$

$$\frac{d^2F}{dt^2} = -e^{-\alpha z}\left(\frac{\omega}{\upsilon}\right)\frac{\omega}{\upsilon}(-\upsilon)(-\upsilon)\sin(x - \upsilon t)$$

$$= -\omega^2 F$$

The given wave equation is

$$\nabla^2 F = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$$

$$\therefore \left(\alpha^2 - \frac{\omega^2}{\upsilon^2}\right) F = \frac{1}{c^2}\left(-\omega^2\right) F$$

$$\therefore \alpha^2 - \frac{\omega^2}{\upsilon^2} = -\frac{\omega^2}{c^2}$$

$$\alpha^2 + \frac{\omega^2}{c^2} = \frac{\omega^2}{\upsilon^2}$$

$$\upsilon^2 = \frac{\omega^2}{\alpha^2 + \frac{\omega^2}{c^2}}$$

$$\therefore \upsilon^2 = \frac{\omega^2 c^2}{\alpha^2 c^2 + \omega^2} = \frac{c^2}{1 + \frac{\alpha^2 c^2}{\omega^2}}$$

$$or\ \upsilon = \frac{c}{\sqrt{1 + \frac{\alpha^2 c^2}{\omega^2}}}$$

Example

The electric field intensity of a uniform plane wave in air has a magnitude of 754 V/m and is in the z direction. If the wave has a wave length = 2m and propagating in the y direction.

Find

(i) Frequency and when the field has the form $A\cos(\omega t - \beta z)$.

(ii) Find an expression for $\vec{H}$.

In air or free space,

$$\upsilon = c = 3\times10^{8}\ m/\sec$$

(i)

$$f = \frac{e}{\lambda} = \frac{3\times10^{8}}{2m} m/\sec = 1.5\times10^{8} Hz = 150MHz$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{2m} = 3.14\ rad/m$$

$$\therefore E_z = 754\cos\left(2\pi\times150\times10^{6}t - \pi y\right)$$

(ii)

For a wave propagating in the +y direction,

$$\frac{E_z}{H_z} = \eta = -\frac{E_x}{H_z}$$

For the given wave,

$$E_z = 754\,V/m; \quad E_x = 0$$

$$\therefore H_x = 754\times\eta = \frac{754}{120\pi} = \frac{754}{377}\ A/m$$

$$\therefore \vec{H} = 2\cos\left(2\pi\times150\times10^{6}t - \pi y\right)\hat{a}x \quad A/m$$

Example

find □ for copper having □ = 5.8*10^{7} (□/m) at 50Hz, 3MHz, 30GHz.

$$\delta=\sqrt{\frac{2}{\omega\mu\sigma}}=\sqrt{\frac{1}{\pi f\mu\sigma}}$$

$$=\sqrt{\frac{1}{\pi}\times\frac{1}{4\pi\times10^{-7}}\times\frac{1}{5.8\times10^{7}}\times\frac{1}{f}}$$

$$=\sqrt{\frac{1}{4\pi^{2}\times5.8}\times\frac{1}{f}}=\sqrt{\frac{1}{23.2\pi^{2}f}}=\frac{66\times10^{-3}}{\sqrt{f}}$$

$$(i)=\frac{66\times10^{-3}}{\sqrt{50}}=9.3459\times10^{-3}m$$

$$(ii)=\frac{66\times10^{-3}}{\sqrt{3\times10^{6}}}=3.8105\times10^{-5}m$$

$$(iii)=\frac{66\times10^{-3}}{\sqrt{3\times10^{6}}}=3.8105\times10^{-7}m$$

Wave Propagation in a loss less medium:

Definition of uniform plane wave in Phasor form:

In phasor form, the uniform plane wave is defined as one for which the equiphase surface is also an equiamplitude surface, it is a uniform plane wave.

For a uniform plane wave having no variations in x and y directions, the wave equation in phasor form may be expressed as

$$\frac{\partial^{2}\vec{E}}{\partial Z^{2}}=-\omega^{2}\mu\in\vec{E}\qquad 0r\qquad \frac{\partial^{2}\vec{E}}{\partial Z^{2}}=-\beta^{2}\vec{E}\ \_\_\_\_\_\_\_\_\_\_\ (i)$$

where $\beta=\omega\sqrt{\mu\in}$. Let us consider eqn.(i) for, the E_y component, we get

$$\frac{\partial^{2}E_{y}}{\partial Z^{2}}=-\beta^{2}E_{y}$$

E_y has a solution of the form,

$$E_{y}=C_{1}e^{-j\beta z}+C_{2}e^{+j\beta z}\ \_\_\_\_\_\_\_\_\_\_\ (2)$$

Where C_1 and C_2 are arbitrary complex constants. The corresponding time varying form of E_y is

$$\tilde{E}_{y}(z,t)=R_{e}\left\{E_{y}(z)\quad e^{j\omega t}\right\}$$

$$=R_{e}\left[\left(C_{1}\quad e^{-j\beta z}+C_{2}\quad e^{j\beta z}\right)e^{j\omega t}\right]\ \_\_\_\_\_\_\_\_\ (3)$$

If C_1 and C_2 are real, the result of real part extraction operation is,

$$\therefore E_y(z,t) = C_1 \cos(\omega t - \beta z) + C_2 \cos(\omega t + \beta z) \_\_\_\_\_\_\_\_ (4)$$

From (3) we note that, in a homogeneous lossless medium, sinusoidal time variation results in space variations which is also sinusoidal.

Equations (3) and (4) represent sum of two waves traveling in opposite directions.

If $C_1 = C_2$, the two wave combine to form a standing wave which does not progress.

Phase velocity and wavelength:

The wave velocity can easily obtained when we rewrite E_y as a function and $(z \pm \upsilon t)$, as in eqn. (4). This shows that

$$\upsilon = \frac{\omega}{\beta} \_\_\_\_\_\_\_\_\_\_(5)$$

In phasor form, identifying a some reference point on the waveform and observing its velocity may obtain the same result. For a wave traveling in the +Z direction, this point is given by $\omega t - \beta z = a$ constant.

$$\therefore \upsilon = \frac{dz}{dt} = \frac{\omega}{\beta}$$, as in eqn. (5)

This velocity of some point on the sinusoidal waveform is called the phase velocity. □ is called the phase-shift constant and is a measure of phase shift in radians per unit length.

Wavelength: Wavelength is defined as that distance over which the sinusoidal waveform passes through a full cycle of 2□ radius.

ie.,

$$\beta\lambda = 2\pi$$

$$\therefore \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu \in}} = \frac{2\pi}{2\pi f\sqrt{\mu \in}} = \frac{\upsilon}{f}; \upsilon = \frac{1}{\sqrt{\mu \in}} \_\_\_\_\_\_\_\_\_(7)$$

$$\therefore \upsilon = f\lambda, \qquad f \text{ in } Hz \qquad \_\_\_\_\_\_\_\_\_(8)$$

For the value of □ given in eqn. (1), the phase velocity is,

$$\upsilon = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu \in}} = \frac{1}{\sqrt{\mu \in}} = \upsilon_0 \_\_\_\_\_\_\_\_\_(9)$$

$$\upsilon_0 = C \quad ; \quad C = 3 \times 10^8 \ m/\text{sec}$$

Wave propagation in conducting medium:

The wave eqn. written in the form of Helmholtz eqn. is

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0 \_\_\_\_\_\_\_\_(10)$$

$$where \quad \gamma^2 = \left(-\omega^2 \mu \in - j\omega\mu\sigma\right) = j\omega\mu(\sigma + j\omega) \_\_\_\_\_\_\_\_(11)$$

□, the propagation constant is complex = □ + j□ ________(12)

We have, for the uniform plane wave traveling in the z direction, the electric field $\vec{E}$ must satisfy

$$\frac{\partial^2 \vec{E}}{\partial Z^2} = \gamma^2 \vec{E} \quad \_\_\_\_\_\_\_\_\_(13)$$

This equation has a possible solution

$$\vec{E}(Z) = E_0 e^{-\gamma Z} \quad \_\_\_\_\_\_\_\_(14)$$

In time varying form this is becomes

$$\tilde{\vec{E}}(z,t) = R_e\left\{E_0 \quad e^{-\gamma Z} \quad e^{j\omega t}\right\} \_\_\_\_\_\_\_\_(15)$$

$$= e^{-\alpha z} \; R_e\left\{E_0 \quad e^{j(\omega t - \beta z)}\right\} \_\_\_\_\_\_\_\_\_(16)$$

This is the equation of a wave traveling in the +Z direction and attenuated by a factor $e^{-\alpha Z}$. The phase shift factor and the wavelength phase, velocity, as in the lossless case, are given by

$$\beta = \frac{2\pi}{\lambda} \qquad\qquad \upsilon = f\lambda = \frac{\omega}{\beta}$$

The propagation constant

We have, $\gamma = \sqrt{j\omega\mu(\sigma + j\omega \in)}$ ________(11)

$$\therefore \gamma^2 = (\alpha + j\beta)^2 = \alpha^2 + 2j\alpha\beta - \beta^2 = j\omega\mu\sigma - \omega^2\mu \in \_\_\_\_\_\_\_\_\_(17)$$

$$\therefore \alpha^2 - \beta^2 = -\omega^2\mu \in; \quad \beta^2 = \alpha^2 + \omega^2\mu \in \_\_\_\_\_\_\_\_\_(18)$$

$$\alpha\beta = \omega\mu\sigma$$

$$\therefore \alpha = \frac{\omega\mu\sigma}{2\beta} \_\_\_\_\_\_\_\_\_(19)$$

Therefore (19) in (18) gives:

$$\beta^2 = \left(\frac{\omega\mu\sigma}{4\beta}\right)^2 + \omega^2\mu \in$$

$$4\beta^4 - 4\beta^2\omega^2\mu \in -\omega^2\mu^2\sigma^2 = 0$$

$$\beta^4 - \beta^2\omega^2\mu \in -\frac{\omega^2\mu^2\sigma^2}{4} = 0$$

$$\beta^2 = \frac{\omega^2\mu \in \pm\sqrt{\omega^4\mu^2\sigma^2 + \omega^2\mu^2\sigma^2}}{2}$$

$$= \frac{\omega^2\mu \in \pm\omega^2\mu \in \sqrt{\left(1+\frac{\omega^2\sigma^2}{\in^2}\right)}}{2}$$

$$= \frac{\omega^2\mu \in}{2}\left(1\pm\sqrt{1+\frac{\omega^2\sigma^2}{\omega^2 \in^2}}\right)$$

$$\therefore \beta = \omega\sqrt{\frac{\mu \in}{2}\sqrt{\left(1+\frac{\sigma^2}{\omega^2 \in^2}\right)}+1} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} (20)$$

$$and$$

$$\alpha = \omega\sqrt{\frac{\mu \in}{2}\sqrt{\left(1+\frac{\sigma^2}{\omega^2 \in^2}\right)}-1} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} (21)$$

We choose some reference point on the wave, the cosine function,(say a rest). The value of the wave ie., the cosine is an integer multiple of 2 at erest.

$$\therefore k_0 z = 2m\pi$$ at m^th erest.

Now let us fix our position on the wave as this m^th erest and observe time variation at this position, nothing that the entire cosine argument is the same multiple of 2 for all time in order to keep track of the point.

ie., $$\omega t - k_0\beta_0 z = 2m\pi = \omega\left(t - z/c\right)$$

Thus at t increases, position z must also increase to satisfy eqn. ( ). Thus the wave erest (and the entire wave moves in a +ve direction) with a speed given by the above eqn. Similarly, eqn. ( ) having a cosine argument $\left(\omega t + \beta_0 z\right)$ describes a wave that moves in the negative direction (as + increases z must decrease to keep the argument constant). These two waves are called the traveling waves.

Let us further consider only +ve z traveling wave:

We have

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x}=0 & \frac{\partial}{\partial y}=0 & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix}$$

$$\nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s$$

$$i\left(-\frac{\partial E_y}{\partial z}\right) + j\frac{\partial E_x}{\partial z} + \hat{k}_0 = -j\omega\mu\left(iH_0 x + j + b_y\right)$$

$$\therefore \frac{\partial E_{xs}}{\partial z} = -j\omega\mu H_{0y}$$

$$\therefore H_{oy} = -\frac{1}{j\omega\mu}\left(E_{z0} \quad e^{-jk_0 z}\right) = E_{x0}\sqrt{\frac{\in_0}{\mu_0}} \quad e^{-j\beta_0 z}$$

$$\therefore H_y\left(z,t\right) = E_{x0}\sqrt{\frac{\in_0}{\mu_0}}\cos\left(\omega t - \beta_0 z\right)$$

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\in}} = \eta \quad ; \quad \eta = \eta_0 = 377\Omega = 120\pi\Omega$$

E_y and H_x are in phase in time and space. The UPW is called so because is uniform thought any plane Z = constant.

Energy flow is in +Z direction.

E and H are perpendicular to the direction of propagation; both lie in a plane that is transverse to the direction of propagation. Therefore also called a TEM wave.

11.1. The electric field amplitude of a UPW in the $\hat{a}z$ direction is 250 V/m. If $\vec{E}$ = $E_x\hat{a}x$ and = 1m rad/sec, find (i) f (ii) (iii) period (iv) amplitude of $\vec{H}$.

$$f = \frac{\omega}{2\pi} = \frac{2\pi f}{2\pi} = \frac{10^6}{2\pi} = 159.155\ KHz$$

$$\lambda = \frac{C}{f} = 1.88495\ km$$

$$period = \frac{1}{f} = 6.283\ \mu s$$

$$amplitude\ of\ H_y = \frac{E_x}{H_y} = \eta = 120\pi$$

$$\therefore H_y = \frac{E_x}{120\pi} = \frac{250}{120\pi} = 0.6631\ A/m$$

1. Given $\vec{H}_s = \left(2\angle -40^0\,\hat{a}x - 3\angle 20^0\,\hat{a}y\right)e^{-j0.07z}\,A/m$ for a certain UPW traveling in free space.

Find (i)□, (ii)H_x at p(1,2,3) at t = 31ns and (iii) $\left|\vec{H}\right|$ at t = 0 at the orign.

Wave propagation in dielectrics:

For an isotopic and homogeneous medium, the wave equation becomes

$$\nabla^2 \vec{E}_s = -k^2 \in_s$$

$$k = \omega\sqrt{\mu \in} = k_0\sqrt{\mu_r \in_r} = \beta_0\sqrt{\mu_r \in_r}$$

For E_x component

We have

$$\frac{d^2E_{xs}}{dz^2} = -k^2E_{xs}$$

for E_x comp. Of electric field wave traveling in Z – direction.

k can be complex one of the solutions of this eqn. is,

$$jk = \alpha + j\beta$$

$$E_{xs} = E_{x0}e^{-\alpha z}\ e^{-j\beta z}$$

Therefore its time varying part becomes,

$$E_{xs} = E_{x0}e^{-\alpha z}\ \cos\left(\omega t - \beta z\right)$$

This is UPW that propagates in the +Z direction with phase constant □ but losing its amplitude with increasing $Z\left(e^{-\alpha z}\right)$. Thus the general effect of a complex valued k is to yield a traveling wave that changes its amplitude with distance.

If □ is +ve $\longrightarrow$ □ = attenuation coefficient if □ is +ve wave decays

If □ is -ve $\longrightarrow$ □ = gain coefficient $\longrightarrow$ wave grows

In passive media, □ is +ve □ is measured in repers per meter

In amplifiers (lasers) □ is –ve.

Wave propagation in a conducting medium for medium for time-harmonic fields:

(Fields with sinusoidal time variations)

For sinusoidal time variations, the electric field for lossless medium (□ = 0) becomes

$$\nabla^2 \vec{E} = -\omega^2 \mu \in \vec{E}$$

In a conducting medium, the wave eqn. becomes for sinusoidal time variations:

$$\nabla^2 \vec{E} + \left(\omega^2 \mu \in - j\omega\sigma\right) \vec{E} = 0$$

Problem:

Using Maxwell's eqn. (1) show that

$\nabla . \vec{D} = 0$ in a conductor

if ohm's law and sinusoidal time variations are assumed. When ohm's law and sinusoidal time variations are assumed, the first Maxwell's curl equation is

$$\nabla \times \vec{H} = \sigma \vec{E} + j\omega \in \vec{E}$$

Taking divergence on both sides, we get,

$$\nabla \left(\nabla \times \vec{H}\right) = \sigma \; \nabla \Box \vec{E} + j\omega \in \nabla \Box \vec{E} = 0$$

$$\therefore \nabla \Box \vec{E} \left(\sigma + j\omega \in\right) = 0$$

$$or \; \nabla \Box \vec{D} \left(\frac{\sigma}{\in} + j\omega\right) = 0$$

$\sigma, \in \& \; \omega \; are$

constants and of finite values and $\therefore \neq 0$

$$\nabla \Box \vec{D} = 0$$

Wave propagation in free space:

The Maxwell's equation in free space, ie., source free medium are,

$$\nabla \times \vec{H} = \in_0 \frac{\partial \vec{E}}{\partial t} \vec{H} \text{ \_\_\_\_\_\_\_\_\_\_(1)}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \text{ \_\_\_\_\_\_\_\_\_\_(2)}$$

$$\nabla \cdot \vec{D} = 0 \text{ \_\_\_\_\_\_\_\_\_\_(3)}$$

$$\nabla \cdot \vec{B} = 0 \text{ \_\_\_\_\_\_\_\_\_\_(4)}$$

Note that wave motion can be inferred from the above equation.

How? Let us see,

Eqn. (1) states that if electric field $\vec{E}$ is changing with time at some, point then magnetic field $\vec{H}$ has a curl at that point; thus $\vec{H}$ varies spatially in a direction normal to its orientation direction. Further, if $\vec{E}$ varies with time, then $\vec{H}$ will, in general, also change with time; although not necessarily in the same way.

Next

From (2) we note that a time varying $\vec{H}$ generates $\vec{E}$; this electric field, having a curl, therefore varies spatially in a direction normal to its orientation direction.

We thus have once more a time changing electric field, our original hypothesis, but this field is present a small distance away from the point of the original disturbance.

The velocity with which the effect has moved away from the original disturbance is the velocity of light as we are going to prove later.

UNIFORM PLANE WAVE:

Uniform plane wave is defined as a wave in which (1) both fields $\vec{E}$ and $\vec{H}$ lie in the transverse plane. Ie., the plane whose normal is the direction of propagation; and (2) both $\vec{E}$ and $\vec{H}$ are of constant magnitude in the transverse plane.

Therefore we call such a wave as transverse electro magnetic wave or TEM wave.

The spatial variation of both $\vec{E}$ and $\vec{H}$ fields in the direction normal to their orientation (travel) ie., in the direction normal to the transverse plane.

Differentiating eqn. (7) with respect to Z_1 we get

$$\frac{\partial^2 E_x}{\partial Z^2} = -\mu_0 \frac{\partial}{\partial Z}\left(\frac{\partial Hy}{\partial t}\right) = -\mu_0 \frac{\partial^2 \vec{H}}{\partial t \partial Z} \text{ \_\_\_\_\_\_\_\_\_\_(9)}$$

Differentiating (8) with respect to t_1 we get

$$\frac{\partial^2 H}{\partial t \partial Z} = - \in_0 \frac{\partial^2 E_x}{\partial t^2} \_\_\_\_\_\_\_\_\_\_\_\_ (10)$$

Therefore substituting (10) into (9) gives,

$$\frac{\partial^2 E_x}{\partial t^2} = + \mu_0 \in_0 \frac{\partial^2 E_x}{\partial t^2} \_\_\_\_\_\_\_\_\_\_\_\_ (11)$$

This eqn.(11) is the wave equation for the x-polarized TEM electric field in free space.

The constant $\frac{1}{\sqrt{\mu_0 \in_0}}$ is the velocity of the wave in free space, denoted c and has a value $3 \times 10^8 m/\sec$, on substituting the values, $\mu_0 = 4\pi \times 10^{-7} H/m \; and \in_0 = \frac{10^{-9}}{36\pi}$ Differentiating (10) with respect to Z and differentiating (9) with respect to 't' and following the similar procedure as above, we get

$$\frac{\partial^2 H_y}{\partial Z^2} = \mu_0 \in_0 \frac{\partial^2 H_y}{\partial t^2} \_\_\_\_\_\_\_\_\_\_\_\_ (13)$$

eqn. (11 and (13) are the second order partial differential eqn. and have solution of the form, for instance,

$$E_x(Z,t) = f_1(t - Z/\upsilon) + f_2(t - Z/\upsilon) \_\_\_\_\_\_\_\_\_\_ (14)$$

Let $\vec{E} = E_x \hat{a}x$ (ie., the electric field is polarized (!) in the x- direction !) traveling along Z direction. Therefore variations of $\vec{E}$ occurs only in Z direction.

Form (2) in this case, we get

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x}(=0) & \frac{\partial}{\partial y}(=0) & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\frac{\partial E_x}{\partial z}\hat{j} = -\mu_0 \frac{\partial \vec{H}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}\hat{j} \_\_\_\_\_\_\_\_\_\_ (5)$$

Note that the direction of the electric field $\vec{E}$ determines the direction of $\vec{H}$, we is now along the y direction.

Therefore in a UPW, $\vec{E}$ and $\vec{H}$ are mutually orthogonal. (ie., perpendicular to each other). This in a UPW .

(i) $\vec{E}$ and $\vec{H}$ are perpendicular to each other (mutually orthogonal and

(ii) $\vec{E}$ and $\vec{H}$ are also perpendicular to the direction of travel.

Form eqn. (1), for the UPW, we get

$$\nabla \times \vec{H} = -\frac{\partial H_y}{\partial Z}\hat{a}x = \in_0 \frac{\partial \vec{E}}{\partial t} = t_0 \frac{\partial E_x}{\partial t}\hat{a}x$$

(using the mutually orthogonal property) _______________(6)

Therefore we have obtained so far,

$$\frac{\partial E_x}{\partial Z} = -\mu_0 \frac{\partial H_y}{\partial t} \text{ ____________(7)}$$

$$\frac{\partial H_y}{\partial Z} = -\in_0 \frac{\partial E_x}{\partial t} \text{ ____________(8)}$$

f_1 and f_2 can be any functions who se argument is of the form $t \pm Z/\upsilon$.

The first term on RHS represents a forward propagating wave ie., a wave traveling along positive Z direction.

The second term on RHS represents a reverse propagating wave ie., a wave traveling along negative Z direction.

(Real instantaneous form and phaser forms).

The expression for E_x (z,t) can be of the form

$$\begin{aligned} E_x(z,t) &= E_x(z,t) + E^1{}_x(z,t) \\ &= E_{x0}\cos\left[\omega\left(t - Z/\upsilon_p\right) + \phi_1\right] + E^1{}_{x0}\cos\left[\omega\left(t - Z/\upsilon_p\right) + \phi_2\right] \\ &= E_{x0}\cos\left(\omega t - k_0 z + \phi_1\right) + E^1{}_{x0}\cos\left(\omega t + k_0 z + \phi_2\right) \text{ ________ } (15) \end{aligned}$$

υ_p is called the phase velocity = c in free space k_0 is called the wave number in free space = $\frac{\omega}{c}$ rad/m _________(16)

eqn. (15) is the real instantaneous forms of the electric (field) wave. (experimentally measurable)

□$_0$t and k_0z have the units of angle usually in radians.

□ : radian time frequency, phase shift per unit time in rad/sec.

k_0 : spatial frequency, phase shift per unit distance in rad/m.

k_0 is the phase constant for lossless propagation.

Wavelength in free space is the distance over which the spatial phase shifts by 2□ radians, (time fixed)

ie.,

$$k_0 z = k_0 \lambda = 2\pi$$

$$or \quad \lambda = \frac{2\pi}{k_0}$$

(in free space) _________(17)

Let us consider some point, for instance, the crest or trough or zero crossing (either –ve to +ve or +ve to –ve). Having chosen such a reference, say the crest, on the forward-propagating cosine function, ie., the function $\cos\left(\omega t - k_0 z + \phi_1\right)$. For a erest to occur, the argument of the cosine must be an integer multiple of 2□. Consider the mth erest of the wave from our reference point, the condition becomes,

K_0z = 2m□, m an integer.

This point on the cosine wave we have chosen, let us see what happens as time increases.

The entire cosine argument must have the same multiple of 2 for all times, in order to keep track of the chosen point.

Therefore we get, $\omega t - k_0 z = \omega\left(t - Z/\upsilon\right) = 2m\pi$ _______(18)

As time increases, the position Z must also increase to satisfy (18). The wave erest, and the entire wave, moves in the positive Z-direction with a phase velocity C (in free space).

Using the same reasoning, the second term on the RHS of eqn. (15) having the cosine argument $\left[\omega t + k_0 z\right]$ represents a wave propagating in the Z direction, with a phase velocity C, since as time t increases, Z must decrease to keep the argument constant.

POLARISATION:

It shows the time varying behavior of the electric field strength vector at some point in space.

Consider of a UPW traveling along Z direction with $\tilde{\vec{E}}$ and $\tilde{\vec{H}}$ vectors lying in the x-y plane.

1. If $\tilde{E}y = 0$ and only $\tilde{E}x$ is present, the wave is said to be polarized in the x-direction.
2. If $\tilde{E}x$ = 0 and only $\tilde{E}y$ is present, the wave is said to be polarized in the y-direction.

Therefore the direction of $\vec{E}$ is the direction of polarization

3. If both $\tilde{E}x$ and $\tilde{E}y$ are present and are in phase, then the resultant electric field $\vec{E}$ has a direction that depends on the relative magnitudes of $\tilde{E}x$ and $\tilde{E}y$.

The angle which this resultant direction makes with the x axis is $\tan^{-1} \dfrac{\tilde{E}y}{\tilde{E}x}$; and this angle will be constant with time.

1. Linear polarization:

In all the above three cases, the direction of the resultant vector is constant with time and the wave is said to be linearly polarized.

If $\tilde{E}x$ and $\tilde{E}y$ are not in phase ie., they reach their maxima at different instances of time, then the direction of the resultant electric vector will vary with time. In this case it can be shown that the locus of the end point of the resultant $\tilde{\vec{E}}$ will be an ellipse and the wave is said to be elliptically polarized.

In the particular case where $\tilde{E}x$ and $\tilde{E}y$ have equal magnitudes and a 90^0 phase difference, the locus of the resultant $\tilde{\vec{E}}$ is a circle and the wave is circularly polarized.

Linear Polarisation:

Consider the phasor form of the electric field of a UPW traveling in the Z-direction:

$$\vec{E} \leq (Z) = E_0 e^{-j\beta z}$$

Its time varying or instanious time form is

$$\tilde{\vec{E}}(Z,t) = R_e \left\{ E_0 e^{-j\beta z} e^{j\omega t} \right\}$$

The wave is traveling in Z-direction.

Therefore $\vec{E}_z$ lies in the x-y plane. In general, $\vec{E}_0$ is a complex vector ie., a vector whose components are complex numbers.

Therefore we can write $\vec{E}_0$ as,

$$\vec{E}_0 = \vec{E}_r + j\vec{E}_{0i}$$

Where $\vec{E}_0$ and $\vec{E}_{0i}$ are real vectors having, in general, different directions.

At some point in space, (say z = 0) the resultant time varying electric field is

$$\tilde{\vec{E}}(0,t) = R_e \left\{ \left(\vec{E}_{0r} + j\ \vec{E}_{0i} \right) e^{j\omega t} \right\}$$
$$= \vec{E}_{0r} \cos \omega t - \vec{E}_{0i} \sin \omega t$$

Therefore $\vec{E}$ not only changes its magnitude but also changes its direction as time varies.

Circular Polarisation:

Here the x and y components of the electric field vector are equal in magnitude.

If E_y leads E_x by 90^0 and E_x and E_y have the same amplitudes,

Ie., $|E_x| = |E_y|$, we have, $\tilde{\vec{E}} = (\hat{a}x + j\ \hat{a}y) E_0$

The corresponding time varying version is,

$$\tilde{\vec{E}}(0,t) = [\hat{a}x \cos \omega t - \hat{a}y \sin \omega t] \vec{E}_0$$
$$\therefore E_x = E_0 \cos \omega t$$
$$and\ E_y = E_0 \sin \omega t$$
$$\therefore E_x^2 + E_y^2 = E_0^2$$

Which shows that the end point of $\tilde{\vec{E}}_0(0,t)$ traces a circle of radius E_0 as time progresses.

Therefore the wave is said to the circularly polarized. Further we see that the sense or direction of rotation is that of a left handed screw advancing in the Z-direction (ie., in the direction of propagation). Then this wave is said to be left circularly polarized.

Similar remarks hold for a right-circularly polarized wave represented by the complex vector,

$$\tilde{\vec{E}} = (\hat{a}x + j\ \hat{a}y) E_0$$

It is apparent that a reversal of the sense of rotation may be obtained by a 180^0 phase shift applied either to the x component of the electric field.

Elliptical Polarisation:

Here x and y components of the electric field differ in amplitudes $\left(\tilde{E}_x \neq \tilde{E}_y\right)$.

Assume that E_y leads E_x by 90^0.

Then,

$$E_0 \hat{a}x\ A + j\ \hat{a}y\ B$$

Where A and B are +ve real constants.

Its time varying form is

$$\tilde{\vec{E}}\left(0,t\right) = \hat{a}xA\cos\omega t - \hat{a}yB\sin\omega t$$

$$\therefore \tilde{E}_x = A\cos\omega t$$

$$\tilde{E}_y = -B\sin\omega t$$

$$\therefore \frac{\tilde{E}_{x^2}}{A^2} + \frac{\tilde{E}_{y^2}}{B^2} = 1$$

Thus the end point of the $\tilde{\vec{E}}\left(0,t\right)$ vector traces out an ellipse and the wave is elliptically polarized; the sense of polarization is left-handed.

Elliptical polarization is a more general form of polarization. The polarization is completely specified by the orientation and axial ratio of the polarization ellipse and by the sense in which the end point of the electric field moves around the ellipse.