Module 3

The Laplace equation $\nabla^2 V = 0$ is solved subjecting to different boundary conditions to get V. Then, $\vec{E} = -\nabla V$

Solutions to Problems on Potential :-

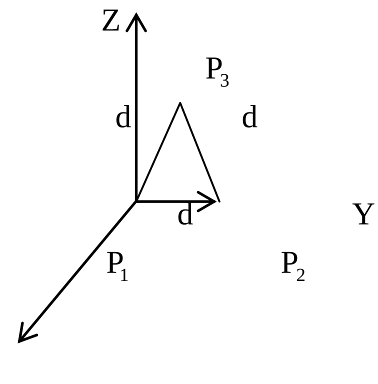
1. Data : $Q_1 = 12$ □C , $Q_2 = 2$ □C , $Q_3 = 3$ □C at the corners of equilateral triangle d m.

To find : $\vec{F}$ on Q_3

Solution :

Let Q_1 , Q_2 and Q_3 lie at P_1, P_2 and P_3 the corners of equilateral triangle of side d meter.

If P_1, P_2 and P_3 lie in YZ plane, with P_1 at origin then



$P_1 = (0,0,0)$ m

$P_2 = (0, d, 0)$ m

$P_3 = (0, 0.5\,d, 0.866\,d)$ m

$\vec{r}_1 = 0$

$\vec{r}_2 = d\,\hat{a}_y$

$\vec{r}_3 = 0.5\,d\,\hat{a}_y + 0.866\,\hat{a}_z$

The force $\vec{F}_3$ is $\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23}$

$$\vec{F}_3 = \frac{Q_3}{4\pi\epsilon_0}\left[\frac{Q_1}{d^2}\,\hat{a}_{13} + \frac{Q_2}{d^2}\,\hat{a}_{23}\right]$$

$$\hat{a}_{13} = \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|} = \frac{0.5\,d\,\hat{a}_y + 0.866\,d\,\hat{a}_z}{d} = 0.5\,\hat{a}_y + 0.866\,\hat{a}_z$$

$$\hat{a}_{23} = \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|} = -0.5\,\hat{a}_y + 0.866\,\hat{a}_z$$

Substituting,

$$\vec{F}_3 = (3 \times 10^{-6})\ 9 \times 10^9 \left[\frac{12 \times 10^{-6}}{d^2}(0.5\,\hat{a}_y + 0.866\,\hat{a}_z) + \frac{2 \times 10^{-6}}{d^2}(-0.5\,\hat{a}_y + 0.866\,\hat{a}_z)\right]$$

$$= \frac{27 \times 10^{-3}}{d^2}\left[\frac{5\,\hat{a}_y + 12.12\,\hat{a}_z}{\sqrt{5^2 + 12.12^2}}\right] 13.11$$

$$\vec{F}_3 = 0.354\,\hat{a}_F\ \text{N} \quad \text{where} \quad \hat{a}_F = (0.38\,\hat{a}_y + 0.924\,\hat{a}_z)$$

2. Data : At the point P, the potential is $V_p = (x^2 + y^2 + z^2)$ V

To find :

(1) $\vec{E}_p$ (2) V_{PQ} given P(1,0.2) and Q (1,1,2) (3) V_{PQ} by using general expression for V

Solution :

$$(1)\ \vec{E}_p = -\nabla V_p = -\left[\frac{\partial V_p}{\partial x}\,\hat{a}_x + \frac{\partial V_p}{\partial y}\,\hat{a}_y + \frac{\partial V_p}{\partial z}\,\hat{a}_z\right]$$

$$= -[\,2\,x\,\hat{a}_x + 2\,y\,\hat{a}_y + 3z^2\,\hat{a}_z\,]\ \text{V/m}$$

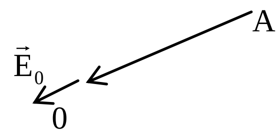
$$(2)\ V_{PQ} = -\int_Q^P \vec{E}_p \cdot \overrightarrow{dl} = \int_1^1 2x\ dx + \int_1^0 2y\ dy + \int_2^2 3z^2\ dz$$

$$= 0 + \left. y^2 \right]^0 + 0 = -1\,\text{V}$$

$$(3)\ V_{PQ} = V_Q - V_P = -1\,\text{V}$$

3. Data : Q = 64.4 nC at A (-4, 2, -3) m

To find : $\vec{E}$ at 0 (0,0,0) m



Solution :

$$\vec{E}_0 = \frac{Q}{4\,\pi \in_0 (AO)^2}\ \hat{a}_{AO}\ \text{N/C}$$

$$= \frac{64.4 \times 10^{-9}}{4\,\pi \times \frac{10^{-9}}{36\,\pi}(AO)^2}\ [\,\hat{a}_{AO}]\ \text{N/C}$$

$$\overrightarrow{AO} = (0+4)\,\hat{a}_x + (0-2)\,\hat{a}_y + (0+3)\,\hat{a}_z = 4\hat{a}_x - 2\,\hat{a}_y + 3\,\hat{a}_z$$

$$\hat{a}_{AO} = \frac{\overrightarrow{AO}}{\left|\overrightarrow{AO}\right|} = \frac{1}{\sqrt{29}}(\overrightarrow{AO}) = (0.743\,\hat{a}_x - 0.37\,\hat{a}_y + 0.56\,\hat{a}_z)$$

$$\vec{E}_0 = \frac{64.4 \times 9}{29}\ \hat{a}_{AO} = 20\ \hat{a}_{AO}\ \text{N/C}$$

4. Q_1 = 100 □C at P_1 (0.03 , 0.08 , - 0.02) m
Q_2 = 0.12 □C at P_2 (- 0.03 , 0.01 , 0.04) m
F_{12} = Force on Q_2 due to Q_1 = ?

Solution :

$$\vec{F}_{12} = \frac{Q_1 \; Q_2}{4\pi \in_0 R_{12}^2} \hat{a}_{12}$$

$$R_{12} = \vec{R}_2 - \vec{R}_1 = (-0.03\,\hat{a}_x + 0.01\,\hat{a}_y + 0.04\,\hat{a}_z) - (0.03\,\hat{a}_x + 0.08\,\hat{a}_y - 0.02\hat{a}_z)$$

$$= (-0.06\,\hat{a}_x - 0.07\,\hat{a}_y + 0.06\,\hat{a}_z) \; ; \; |R_{12}| = 0.11\,\text{m}$$

$$\hat{a}_{12} = (-0.545\,\hat{a}_x - 0.636\,\hat{a}_y + 0.545\,\hat{a}_z)$$

$$\vec{F}_{12} = \frac{100 \text{ x } 10^{-6} \text{ x } 0.121 \text{ x } 10^{-6}}{0.11^2} \text{ x } 9 \text{ x } 10^9 \; \hat{a}_{12}$$

$$\vec{F}_{12} = 9\,\hat{a}_{12} \;\; \text{N}$$

5. $Q_1 = 2 \times 10^{-9}$ C , $Q_2 = -0.5 \times 10^{-9}$ C C

(1) $R_{12} = 4 \times 10^{-2}$ m , $\vec{F}_{12} = ?$

(2) Q_1 & Q_2 are brought in contact and separated by $R_{12} = 4 \times 10^{-2}$ m $\vec{F}^{`}_{12} = ?$

Solution :

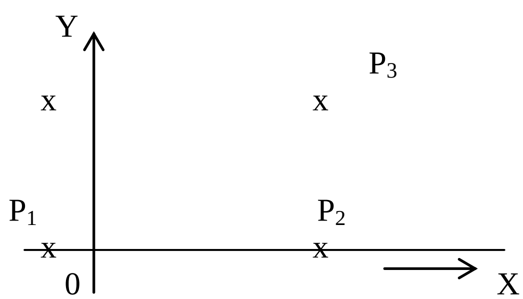
$$\vec{F}_{12} = \frac{2 \times 10^{-9} \times -0.5 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} \times (4 \times 10^{-2})^2} \ \hat{a}_{12} = \frac{-9}{16} \times 10^{-5} \ \hat{a}_{12} = +5.63 \, \mu \text{N (attractive)}$$

(2) When brought into contact $Q^{`}_1 = Q^{`}_2 = \frac{1}{2}(Q_1 + Q_2) = 1.5 \times 10^{-9}$ C

$$\vec{F}^{`}_{12} = \frac{(1.5 \times 10^{-9})^2}{4\pi \times \frac{10^{-9}}{36\pi} \times (4 \times 10^{-2})^2} \ \hat{a}_{12} = \frac{1.5^2}{16} \times 9 \times 10^{-18+13} \ \hat{a}_{12} = 12.66 \, \mu \text{N} \, \hat{a}_{12}$$

(1) $F^{`}_{12} = 12.66 \, \mu$ N (repulsive)

6.



$Q_1 = Q_2 = Q_3 = Q_4 = 20$ □ C
$Q_P = 200$ □ C at P(0,0,3) m

$P_1 = (0, 0 , 0)$ m $P_2 = (4, 0, 0)$ m
$P_3 = (4, 4, 0)$ m $P_4 = (0, 4, 0)$ m
$F_P = ?$

Solution :

$$\vec{F}_p = \vec{F}_{1p} + \vec{F}_{2p} + \vec{F}_{3p} + \vec{F}_{4p}$$

$$\vec{R}_{1p} = 3\hat{a}_z \quad \left| R_{1p} \right| = 3\,\text{m} \quad \hat{a}_{1p} = \hat{a}_z$$

$$\vec{R}_{2p} = -4\hat{a}_x + 3\hat{a}_z \ ; \left| R_{2p} \right| = 5\,\text{m} \quad \hat{a}_{2p} = -0.8\hat{a}_x + 0.6\ \hat{a}_z$$

$$\vec{R}_{3p} = -4\hat{a}_x - 4\hat{a}_y + 3\hat{a}_z \ ; \left| R_{3p} \right| = 6.4\,\text{m} \ ; \quad \hat{a}_{3p} = -0.625\hat{a}_x - 0.625\hat{a}_y + 0.47\ \hat{a}_z$$

$$\vec{R}_{4p} = -4\hat{a}_y + 3\hat{a}_z \ ; \left| R_{4p} \right| = 5\,\text{m} \ ; \quad \hat{a}_{4p} = -0.8\hat{a}_y + 0.6\ \hat{a}_z$$

$$\vec{F}_p = \frac{Q_p}{4\pi \frac{10^{-9}}{36\pi}} \left[\frac{Q_1}{R_{1p}^2}\hat{a}_{1p} + \frac{Q_2}{R_{2p}^2}\hat{a}_{2p} + \frac{Q_3}{R_{3p}^3}\hat{a}_{3p} + \frac{Q_4}{R_{4p}^2}\hat{a}_{4p} \right]$$

$$= 200 \times 10^{-6} \times 9 \times 10^9 \begin{bmatrix} \frac{1}{3^2}\hat{a}_z + \frac{1}{5^2}(-0.8\hat{a}_x + 0.6\hat{a}_z) + \frac{1}{6.4^2}\ (-0.625\hat{a}_x - 0.625\hat{a}_y + 0.47\hat{a}_z) \\ + \frac{1}{5^2}(-0.8\hat{a}_y + 0.6\hat{a}_z) \end{bmatrix} 20 \times 10^{-6}$$

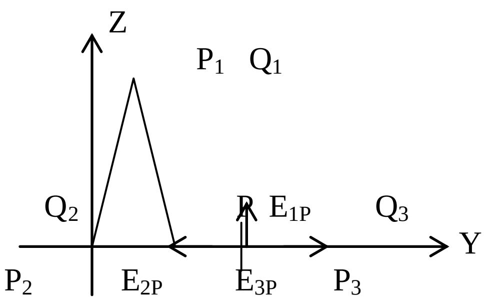
$$= 200\text{x}10^{-6}\text{x}9\text{x}10^9\text{x}10^9\text{x}10^{-6}\ \text{x}\,10^{-2} \begin{bmatrix} \frac{100}{9}\hat{a}_z + \frac{100}{25}(-0.8\hat{a}_x + 0.6\hat{a}_z) + \frac{100}{40.96}(-0.625\hat{a}_x - 0.625\hat{a}_y + 0.47\hat{a}_z) \\ + \frac{100}{25}(-0.8\hat{a}_y + 0.6\hat{a}_z) \end{bmatrix}$$

$$= 0.36 \left[(-3.2 - 1.526)\hat{a}_x + \frac{1}{6.4^2}\ (-1.526 - 3.2)\hat{a}_y + (11.11 + 2.4 + 1.15 + 2.4)\hat{a}_z) \right]$$

$$= (-1.7\hat{a}_x - 1.7\hat{a}_y + 17\hat{a}_z)\ \text{N} \quad = 17.23\hat{a}_p\ \text{N}$$

7. Data : Q_1 , Q_2 & Q_3 at the corners of equilateral triangle of side 1 m.
Q_1 = - 1 □C, Q_2 = -2 □C , Q_3 = - 3 □C
To find : $\vec{E}$ at the bisecting point between Q_2 & Q_3 .

Solution :



P_1 : (0, 0.5, 0.866) m
P_2 : (0, 0, 0) m
P_3 : (0, 1, 0) m
P : (0, 0.5, 0) m

$$\vec{E}_P = \vec{E}_{1P} + \vec{E}_{2P} + \vec{E}_{3P}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Q_1}{R_{1P}^{\ 2}}\hat{a}_{1P} + \frac{Q_2}{R_{2P}^{\ 2}}\hat{a}_{2P} + \frac{Q_3}{R_{3P}^{\ 2}}\hat{a}_{3P}\right]$$

$R_{1P} = -0.866\,\hat{a}_z \qquad |R_{1P}| = 0.866 \qquad \hat{a}_{1P} = -\hat{a}_z$

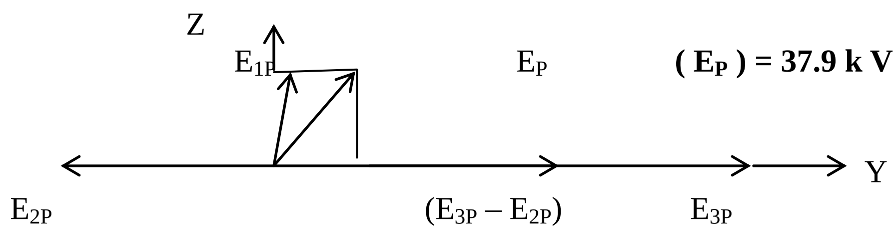
$R_{2P} = +0.5\,\hat{a}_y \qquad |R_{2P}| = 0.5 \qquad \hat{a}_{2P} = \hat{a}_y$

$R_{3P} = -0.5\,\hat{a}_y \qquad |R_{3P}| = 0.5 \qquad \hat{a}_{3P} = -\hat{a}_y$

$$\vec{E}_P = \frac{1}{4\pi\frac{10^{-9}}{36\pi}}\left[\frac{-1\times10^{-6}}{0.866^2}(-\hat{a}_z) + \frac{-2\times10^{-6}}{0.5^2}(-\hat{a}_y) + \frac{-3\times10^{-6}}{0.5^2}(-\hat{a}_y)\right]$$

$$= 9\times10^3\left[1.33\,\hat{a}_z - 8\,\hat{a}_y + 12\,\hat{a}_y\right]$$

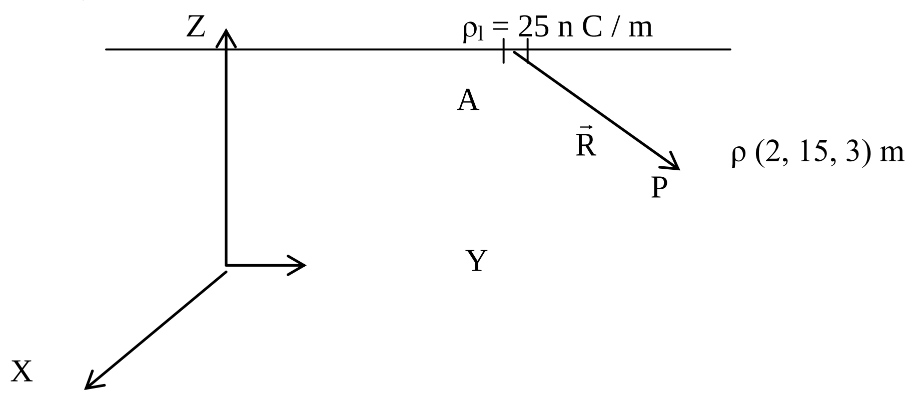
$$= 9\times10^3\left[4\,\hat{a}_y + 1.33\,\hat{a}_z\right] = \left[36\,\hat{a}_y + 12\,\hat{a}_z\right] + 0^3\ \text{V / m} = 37.9\ \angle 18^0\ \text{k V/m}$$



(E_P) = 37.9 k V / m

8. Data P_l = 25 n C /m on (-3, y, 4) line in free space and P : (2,15,3) m
To find : E_P

Solution :



The line charge is parallel to Y axis. Therefore E_{PY} = 0

$$\vec{R} = \overrightarrow{AP} = (2-(-3))\,\hat{a}_x + (3-4)\,\hat{a}_z = (5\,\hat{a}_x - \hat{a}_z)\,;\ = \left| R \right| = 5.1\,m$$

$$\hat{a}_R = \frac{\vec{R}}{|R|} = (0.834\,\hat{a}_x - 0.167\,\hat{a}_z)$$

$$\vec{E}_P = \frac{\rho_l}{2\pi \in_0 R}\ \hat{a}_R = \frac{25\,x}{2\pi\,\frac{10^{-9}}{36\pi}\,x\,5.1}\ \hat{a}_R$$

$$\vec{E}_P = 88.23\,\hat{a}_R V/m$$

9. Data : P_1 (2, 2, 0) m ; P_2 (0, 1, 2) m ; P_3 (1, 0, 2) m
Q_2 = 10 □C ; Q_3 = - 10 □C
To find : E_1 , V_1

Solution :

$$\vec{E}_1 = \vec{E}_{21} + \vec{E}_{21} = \frac{1}{4\pi \in_0}\left[\frac{Q_2}{R_{21}^{\ 2}}\,\hat{a}_{21} + \frac{Q_3}{R_{31}^{\ 2}}\,\hat{a}_{31}\right]$$

$$\vec{R}_{21} = (2\,\hat{a}_x + \hat{a}_y - 2\,\hat{a}_z) \qquad \left|\vec{R}_{21}\right| = 3 \quad \hat{a}_{21} = 0.67\,\hat{a}_x + 0.33\,\hat{a}_y - 0.67\,\hat{a}_z$$

$$\vec{R}_{31} = \hat{a}_x + 2\,\hat{a}_y + 2\,\hat{a}_z \qquad \left|\vec{R}_{31}\right| = 3 \quad \hat{a}_{31} = 0.33\,\hat{a}_x + 0.67\,\hat{a}_y + 0.67\,\hat{a}_z$$

$$\vec{E}_1 = 9\,x\,10^9\left[\frac{10^{-6}}{9}(0.67\,\hat{a}_x + 0.33\,\hat{a}_y - 0.67\,\hat{a}_z) + \frac{10^{-6}}{9}(0.33\,\hat{a}_x + 0.67\,\hat{a}_y + 0.67\,\hat{a}_z)\right]$$

$$= 10^3\,[\,\hat{a}_x + \hat{a}_y\,] = 14.14\,(0.707\,\hat{a}_x + 0.707\,\hat{a}_y)\ V/m$$

$$V_1 = \frac{1}{4\pi \in_0}\left[\frac{Q_2}{R_{21}} + \frac{Q_3}{R_{31}}\right] = 9\,x\,10^9\left[\frac{10^{-6}}{3} + \frac{10^{-6}}{3}\right] = 3000\,V$$

$$\mathbf{\left| E_1 \right| = 14.14\,V/m \qquad V_1 = 3000\,V}$$

10. Data : $Q_1 = 10$ □C at P_1 (0, 1, 2) m ; $Q_2 = -5$ □C at P_2 (-1, 1, 3) m
P_3 (0, 2, 0) m

To find : (1) $\vec{E}_3$ (2) Q at (0, 0, 0) for $E_{3x} = 0$

Solution :

$$(1)\ \vec{E}_3 = \frac{1}{4\pi\epsilon_0}\left[\frac{Q_1}{R_{13}^{\ 2}}\hat{a}_{13} + \frac{Q_2}{R_{23}^{\ 2}}\hat{a}_{23}\right]$$

$$\vec{R}_{13} = (2-1)\,\hat{a}_y + (0-2)\,\hat{a}_z = \hat{a}_y - 2\,\hat{a}_z \qquad |R_{13}| = \sqrt{5}$$

$$\vec{R}_{23} = (0+1)\,\hat{a}_x + (2-1)\,\hat{a}_y + (0-3)\,\hat{a}_z = \hat{a}_x + \hat{a}_y - 3\,\hat{a}_z \qquad |R_{23}| = \sqrt{11}$$

$$\hat{a}_{13} = \frac{\vec{R}_{13}}{|\vec{R}_{13}|} = (0.447\,\hat{a}_y - 0.894\,\hat{a}_z)$$

$$\hat{a}_{23} = \frac{\vec{R}_{23}}{|\vec{R}_{23}|} = 0.3\,\hat{a}_x + 0.3\,\hat{a}_y - 0.9\,\hat{a}_z$$

$$\vec{E}_3 = 9 \times 10^9\left[\frac{10 \times 10^{-6}}{(\sqrt{5})^2}(0.447\,\hat{a}_y - 0.894\,\hat{a}_z) + \frac{-5 \times 10^{-6}}{(\sqrt{11})^2}(0.3\,\hat{a}_x + 0.3\,\hat{a}_y - 0.9\,\hat{a}_z)\right]$$

$$= \left[(8\,\hat{a}_y - 16\,\hat{a}_z) + (-1.23\,\hat{a}_x - 1.23\,\hat{a}_y + 3.68\,\hat{a}_z)\right]$$

$$= \left[-1.23\,\hat{a}_x + 6.77\,\hat{a}_y - 12.32\,\hat{a}_z\right] 10^3 \ \text{V/m}$$

$$(2)\ \vec{E}_3 = 9 \times 10^9\left[\frac{Q_1}{R_{13}^{\ 2}}\hat{a}_{13} + \frac{Q_2}{R_{23}^{\ 2}}\hat{a}_{23} + \frac{Q}{R_{03}^{\ 2}}\hat{a}_{03}\right];\ \vec{R}_{03} = 2\,\hat{a}_y$$

$$\vec{E}_{3x} = -1.23\,\hat{a}_x$$

$\vec{E}_{3x}$ cannot be zero

11. Data : $Q_2 = 121 \times 10^{-9}$ C at P_2 (-0.02, 0.01, 0.04) m
$Q_1 = 110 \times 10^{-9}$ C at P_1 (0.03, 0.08, 0.02) m
P_3 (0, 2, 0) m

To find : $\vec{F}_{12}$

Solution :

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^{\ 2}}\ \hat{a}_{12}\ \text{N}\ ; \quad R_{12} = -0.05\,\hat{a}_x - 0.07\,\hat{a}_y + 0.02\,\hat{a}_z$$

$$\vec{F}_{12} = \frac{121 \times 10^{-9} \times 110 \times 10^{-9}}{4\pi\frac{10^{-9}}{36\pi} \times 7.8 \times 10^{-3}}\ [\hat{a}_{12}] \qquad |R_{12}| = 0.088$$

$$\mathbf{\vec{F}_{12} = 0.015\,\hat{a}_{12}\ N}$$

12. Given V = $(50\, x^2yz + 20y^2)$ volt in free space

Find V_P, $\vec{E}_P$ and $\hat{a}_{np}$ at P (1, 2, -3) m

Solution :

$$V_P = \left[50\,(1)^2\,(2)\,(-3) + 20\,(2)^2\right] = -220\ V$$

$$\vec{E} = -\nabla V = -\frac{\partial}{\partial x} V\,\hat{a}_x - \frac{\partial}{\partial y} V\,\hat{a}_y - \frac{\partial}{\partial z} V\,\hat{a}_z$$

$$\vec{E} = -100\,x\,y\,z\,\hat{a}_x - 50\,x^2 z\,\hat{a}_y - 50\,x^2 y\,\hat{a}_z$$

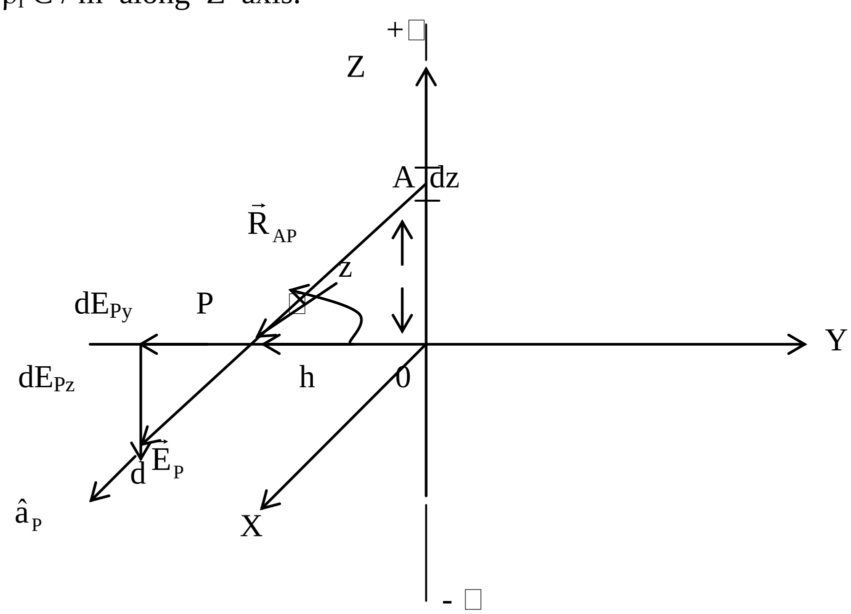
$$\vec{E}_P = -100\,(2)\,(-3)\,\hat{a}_x - 50\,(-3)\,\hat{a}_y - 50\,(2)\,\hat{a}_z$$

$$= 600\,\hat{a}_x + 150\,\hat{a}_y - 100\,\hat{a}_z$$

$$= 626.5\,\hat{a}_P\ V/m\ ;\ \hat{a}_P = 0.957\,\hat{a}_x + 0.234\,\hat{a}_y - 0.16\,\hat{a}_z$$

Additional Problems

A1. Find the electric field intensity $\vec{E}$ at P (0, -h, 0) due to an infinite line charge of density ρ_l C / m along Z axis.



Solution :

Source : Line charge ρ_l C / m. Field point : P (0, -h, 0)

$$\overrightarrow{dE_P} = \frac{dQ}{4\pi\epsilon_0 R^2}\hat{a}_R = \frac{\rho_l\, dz}{4\pi\epsilon_0 R^2}\hat{a}_R \;\; V/m\,; \;\; \vec{R} = \overrightarrow{AP} = -z\,\hat{a}_z - h\,\hat{a}_y$$

$$\left|\vec{R}\right| = \left|\overrightarrow{AP}\right| = \sqrt{z^2 + h^2}$$

$$\hat{a}_R = \frac{\vec{R}}{\left|\vec{R}\right|} = \frac{1}{R}\left[-h\,\hat{a}_y - z\,\hat{a}_z\right]$$

$$\overrightarrow{dE_P} = \frac{\rho_l\, dz}{4\pi\epsilon_0 R^2}\left[-\frac{h}{R}\hat{a}_y - \frac{z}{R}\hat{a}_z\right] = d\,E_{Py}\,\hat{a}_y + d\,E_{Pz}\,\hat{a}_z$$

$$dE_{Py} = -\frac{\rho_l\, dz}{4\pi\epsilon_0 R^2}\frac{h}{R}\hat{a}_y \qquad dE_{Pz} = -\frac{\rho_l\, dz}{4\pi\epsilon_0 R^2}\frac{z}{R}\hat{a}_z$$

Expressing all distances in terms of fixed distance h,
h = R Cos □ or R = h Sec □ ; z = h tan □ , dz = h sec^2 □ d□

$$dE_{Py} = -\frac{\rho_l\, h\,Sec^2\theta\, d\theta}{4\pi\epsilon_0 h^2\, Sec^2\theta}\, x\, Cos\,\theta = -\frac{\rho_l}{4\pi\epsilon_0 h} Cos\,\theta\, d\theta$$

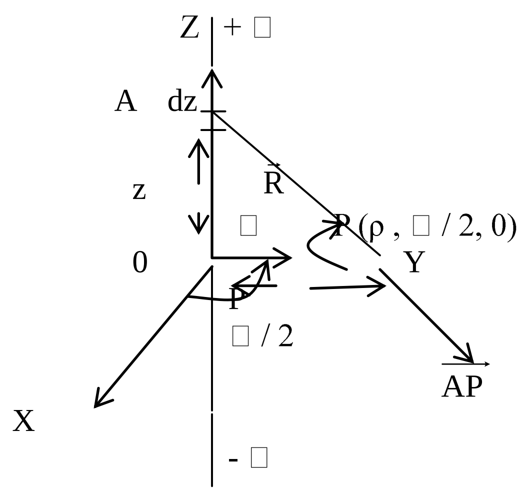
$$E_{Py} = -\frac{\rho_l}{4\pi\epsilon_0 h}\left[+Sin\,\theta\right]_{-\pi/2}^{\pi/2} = -\frac{\rho_l}{4\pi\epsilon_0 h}\, x\, 2 = -\frac{\rho_l}{2\pi\epsilon_0 h}\,\hat{a}_y$$

$$dE_{Pz} = -\frac{\rho_l\, h\,Sec^2\theta\, d\theta}{4\pi\epsilon_0 h^2\, Sec^2\theta}\; x\; \frac{\not{h}\tan\theta}{\not{h}\,Sec\,\theta} = -\frac{\rho_l}{4\pi\epsilon_0 h} Sin\,\theta\, d\theta$$

$$\vec{E}_{Pz} = \frac{\rho_l}{4\pi\epsilon_0 h}\left[Cos\theta\right]_{-\pi/2}^{\pi/2} = 0$$

$$\mathbf{E} = -\frac{\boldsymbol{\rho_l}}{\mathbf{2\pi\epsilon_0 h}}\,\hat{\mathbf{a}}_y \;\; \mathbf{V/m}$$

An alternate approach uses cylindrical co-ordinate system since this yields a more general insight into the problem.



$dQ = \rho_l \overrightarrow{dz}$ is the elemental change at Z.

The field intensity $\overrightarrow{dE_P}$ due to dQ is

$$\overrightarrow{dE_P} = \frac{dQ}{4\pi \in_0 |\vec{R}|^2} \hat{a}_R \ V/m$$

where $\vec{R} = \rho\, \hat{a}_\rho - z\, \hat{a}_z$ and $\hat{a}_R = \frac{1}{R}(\rho\, \hat{a}_\rho - z\, \hat{a}_z)$

$$dQ = \rho_l \overrightarrow{dz} \ C$$

$$\overrightarrow{dE_P} = \frac{\rho_l\, dz}{4\pi \in_0 R^2}\left[\frac{\rho}{R}\, \hat{a}_\rho - \frac{z}{R}\, \hat{a}_z\right] = dE_{P\rho}\, \hat{a}_\rho + dE_{Pz}\, \hat{a}_z$$

(i) $dE_{P\rho} = \frac{\rho_l}{4\pi \in_0 R^2} \rho\, dz$; (ii) $dE_{Pz} = -\frac{\rho_l}{4\pi \in_0 R^2} z\, dz$

Taking $\theta = O\hat{P}A$ as integration variable, and expressing all distances in terms of ρ and θ

$z = \rho \tan\theta$, $dz = \rho\, Sec^2\theta\, d\theta$ and $R = \frac{\rho}{Cos\,\theta} = \rho\, Sec\,\theta$

(i) $dE_{P\rho} = \frac{\rho_l \ x\ \rho\ x\ \rho\, Sec^2\theta}{4\pi \in_0 \rho^3\, Sec^3\theta}\, d\theta = \frac{\rho_l}{4\pi \in_0 \rho}\, Cos\,\theta\, d\theta$

$$E_{P\rho} = \frac{\rho_l}{4\pi \in_0 \rho}[\,Sin\,\theta]_{-\pi/2}^{\pi/2} = \frac{\rho_l}{4\pi \in_0 \rho} \ x\ 2 = \frac{\rho_l}{2\pi \in_0 \rho}$$

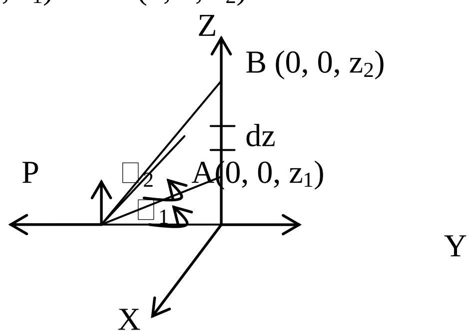
(ii) $dE_{Pz} = \frac{-\rho_l \ x\ \rho\ \tan\theta\ x\ \rho\, Sec^2\theta}{4\pi \in_0 \rho^3\, Sec^3\theta}\, d\theta = \frac{\rho_l}{4\pi \in_0 \rho}(-Sin\,\theta)\, d\theta$

$$E_{Pz} = \frac{\rho_l}{4\pi \in_0 \rho}[\,Cos\,\theta]_{-\pi/2}^{\pi/2} = 0$$

$\therefore\ \vec{E}_P = \frac{\rho_l}{2\pi \in_0 \rho}\, \hat{a}_\rho \ V/m$

Thus, $\vec{E}$ is radial in direction

A2. Find the electric field intensity $\vec{E}$ at (0, -h, 0) due to a line charge of finite length along Z axis between A (0, 0, z_1) and B(0, 0, z_2)



Solution :

$$\overrightarrow{dE_P} = \frac{\rho_l \, dz}{4\pi \in_0 R^2}\left[-\frac{h}{R}\,\hat{a}_y - \frac{z}{R}\,\hat{a}_z\right]$$

$$\vec{E}_P = \int_{z_1}^{z_2} d\,\vec{E}_P = -\frac{\rho_l}{4\pi \in_0 h}\int_{\theta_1}^{\theta_2} \text{Cos}\,\theta\, d\theta\,\hat{a}_y - \frac{\rho_l}{4\pi \in_0 h}\int_{\theta_1}^{\theta_2} \text{Sin}\,\theta\, d\theta\,\hat{a}_z$$

$$= +\frac{\rho_l}{4\pi \in_0 h}(-\text{Sin}\,\theta)_{\theta_1}^{\theta_2}\;\hat{a}_y + \frac{\rho_l}{4\pi \in_0 h}(\text{Cos}\,\theta)_{\theta_1}^{\theta_2}\;\hat{a}_z$$

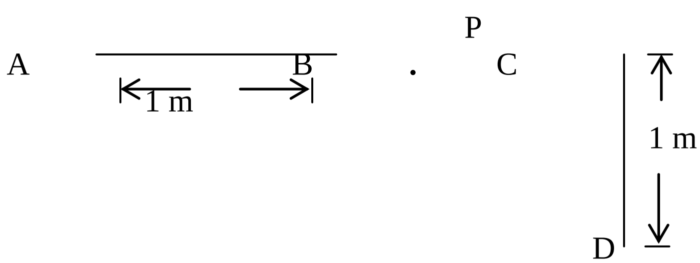
$$\vec{E}_P = +\frac{\rho_l}{4\pi \in_0 h}\left[(\text{Sin}\,\theta_1 - \text{Sin}\,\theta_2)\,\hat{a}_y - (\text{Cos}\,\theta_1 - \text{Cos}\,\theta_2)\,\hat{a}_z\right] \text{ V / m}$$

If the line is extending from $-\infty$ to ∞,

$$\theta_2 = \frac{\pi}{2}\;,\; \theta_1 = -\frac{\pi}{2}$$

$$\mathbf{\vec{E}_P = \frac{-\rho_l}{2\pi \in_0 h}\,\hat{a}_y \text{ V / m}}$$

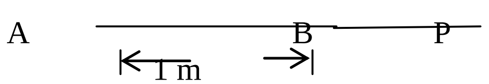
A3. Two wires AB and CD each 1 m length carry a total charge of 0.2 □ C and are disposed as shown. Given BC = 1 m, find $\vec{E}$ at P, midpoint of BC.



Solution :

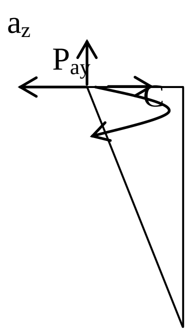
(1)

$\square_1 = 180^0 \quad \square_2 = 180^0$



$$\vec{E}_{P_{AB}} = \frac{\rho_l}{4\pi \in_0 h}\left\{\left[-(\text{Sin}\,\theta_2 - \text{Sin}\,\theta_1)\right]\hat{a}_y + \left[\text{Cos}\,\theta_2 - \text{Cos}\,\theta_1\right]\hat{a}_z\right\} = \frac{0}{0} \text{ (Indeterminate)}$$

(2)



$\square_1 \qquad \square_1 = -\tan^{-1}\dfrac{1}{0.5} = -63.43^0$

$\square_2 = 0$

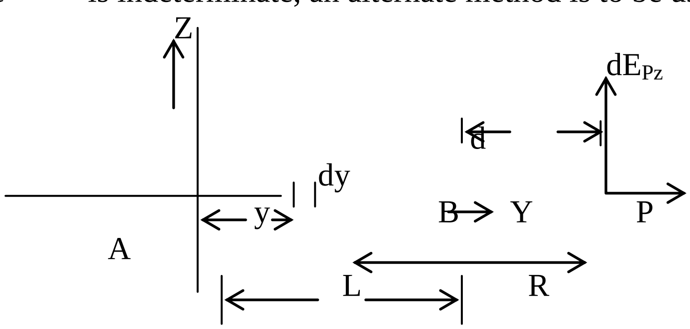
D

$$\vec{E}_{P_{CD}} = \frac{\rho_l}{4\pi \in_0 h}\left[-(\sin\theta_2 - \sin\theta_1)\,\hat{a}_y + (\cos\theta_2 - \cos\theta_1)\,\hat{a}_z\right]$$

$$= \frac{0.2 \times 10^{-6}}{4\pi \dfrac{10^{-9}}{36\pi} 0.5}\left[-(\sin(-63.43))\,\hat{a}_y + (\cos 0 - \cos 63.43)\,\hat{a}_z\right]$$

$$\vec{E}_{P_{CD}} = 3.6 \times 10^3 \left[-0.894\,\hat{a}_y + (1 - 0.447)\,\hat{a}_z\right] = (-3218\,\hat{a}_y + 1989.75\,\hat{a}_z)$$

Since $\vec{E}_{\rho_{AB}}$ is indeterminate, an alternate method is to be used as under :



$$\overrightarrow{dE_P} = \frac{\rho_l\, dy}{4\pi \in_0 R^2}\,\hat{a}_R \quad V/m$$

$$\vec{R} = (L + d - y)\,\hat{a}_R \;;\; \hat{a}_R = \frac{1}{R}(-\hat{a}_y)$$

$$dE_{Py} = \frac{-\rho_l\,\hat{a}_y}{4\pi \in_0 (L + d - y)^2}\, dy$$

$$\text{Let } L + d - y = -t \;;\; -dy = -dt \;;\; y = 0,\, t = \frac{1}{L + d}$$

$$y = L\,;\, t = \frac{1}{d}$$

$$dE_P = \frac{-\rho_l}{4\pi \in_0 t^2}\, dt$$

$$E_P = +\left[\frac{\rho_l}{4\pi \in_0 t}\right]_{\frac{1}{L+d}}^{\frac{1}{d}} = \frac{\rho_l}{4\pi \in_0}\left[\frac{1}{d} - \frac{1}{L + d}\right]$$

$$\therefore \quad \vec{\mathbf{E}}_{\mathbf{P}} = \frac{\boldsymbol{\rho}_{\mathbf{l}}}{\mathbf{4}\,\pi \in_0} \left[\frac{\mathbf{1}}{\mathbf{d}} - \frac{\mathbf{1}}{\mathbf{L}+\mathbf{d}} \right] \mathbf{V / m}$$

$$\vec{E}_{P_{AB}} = \frac{0.2 \text{ x } 10^{-6}}{4\,\pi \, \frac{10^{-9}}{36\,\pi}} \left[\frac{1}{0.5} - \frac{1}{1.5} \right] \hat{a}_y$$

$$\vec{E}_{P_{AB}} = 1800 [\, 2 - 0.67] \ \hat{a}_y = 2400 \ \hat{a}_y \ \text{V/ m}$$

$$\therefore \ \vec{E}_P = \vec{E}_{P_{AB}} + \vec{E}_{P_{CD}} = 2400\,\hat{a}_y - 3218\,\hat{a}_y + 1990\ \hat{a}_z$$

$$= (-820\,\hat{a}_y + 1990\,\hat{a}_z)$$

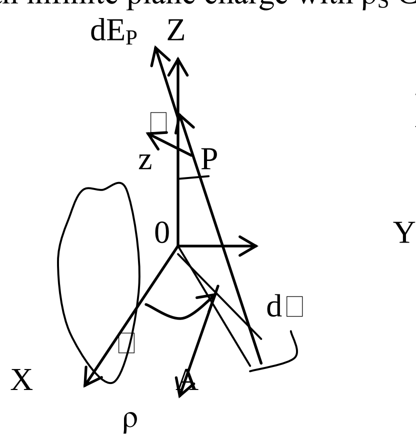
$$= 2152\,\hat{a}_P \ \text{ V / m}$$

where $\hat{a}_P = (-\,0.381\,\hat{a}_y + 0.925\,\hat{a}_z)$

A4. Develop an expression for $\vec{E}$ due to a charge uniformly distributed over an infinite plane with a surface charge density of ρ_S C / m^2.

Solution :

Let the plane be perpendicular to Z axis and we shall use Cylindrical Co-ordinates. The source charge is an infinite plane charge with ρ_S C / m^2 .



$$\overrightarrow{AP} = \vec{R}$$

$$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\overrightarrow{OA} + \overrightarrow{OP}$$

$$\vec{R} = (-\rho\,\hat{a}_\rho + z\,\hat{a}_z)$$

$$\hat{a}_R = \frac{1}{R}(-\rho\,\hat{a}_\rho + z\,\hat{a}_z)$$

The field intensity $\overrightarrow{dE_P}$ due to dQ = ρ_S ds = ρ_S (d□ dρ) is along AP and given by

$$\overrightarrow{dE_P} = \frac{\rho_S \, \rho \, d\phi \, d\rho}{4\pi \in_0 R^2} \hat{a}_R = \frac{\rho_S}{4\pi \in_0 R^3} \left(-\rho \, \hat{a}_\rho + z \, \hat{a}_z \right) d\phi \, \rho \, d\rho$$

Since radial components cancel because of symmetry, only z components exist

$$\therefore \overrightarrow{dE_P} = \frac{\rho_S \, z}{4\pi \in_0 R^3} d\phi \, \rho \, d\rho$$

$$\vec{E}_P = \int_S \overrightarrow{dE_P} = \frac{\rho_S}{4\pi \in_0} \int_0^{2\pi} d\phi \int_0^{\infty} \frac{z \, \rho \, d\rho}{R^3} = \frac{\rho_S}{4\pi \in_0} \; x \, 2\pi \int_0^{\infty} \frac{z \, \rho}{R^3} \, d\rho$$

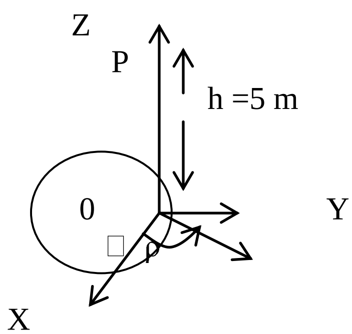
'z' is fixed height of ρ above plane and let $O\hat{P}A = \theta$ be integration variable. All distances are expressed in terms of z and □

ρ = z tan □ , d ρ = z Sec^2 □ d□ ; R = z Sec □ ; ρ = 0, □ = 0 ; ρ = □ , □ = □ / 2

$$\vec{E}_P = \frac{\rho_S}{2\in_0} \int_0^{\infty} \frac{z \, z \tan\theta}{z^3 \, Sec^3 \theta} \, z \, Sec^2 \theta \, d\theta = \frac{\rho_S}{2\in_0} \int_0^{\pi/2} Sin\,\theta \, d\theta = \frac{\rho_S}{2\in_0} [-Cos\,\theta]_0^{\pi/2} \, \hat{a}_z$$

$$= \frac{\rho_S}{2\in_0} \hat{a}_z \text{ (normal to plane)}$$

A5. Find the force on a point charge of 50 □C at P (0, 0, 5) m due to a charge of 500 □□C that is uniformly distributed over the circular disc of radius 5 m.



Solution :
Given : ρ = 5 m, h = 5 m and Q = 500 □□C
To find : f_p & q_p = 50 □C

$$\vec{f}_P = \vec{E}_P \text{ x } q_P \quad \text{where} \quad \vec{E}_P = \frac{\rho_S}{2\in_0}\hat{a}_z$$

$$= \frac{\frac{Q}{A}}{2\in_0}\hat{a}_z = \frac{500\pi \text{ x } 10^{-6}}{2(\pi 5^2) \text{ x } \frac{10^{-9}}{36\pi}}\hat{a}_z$$

$$= \frac{500}{2 \text{ x } 25} \text{ x } 36\pi \text{ x } 10^3\,\hat{a}_z$$

$$= 1131 \text{ x } 10^3\,\hat{a}_z \text{ N / C}$$

$$\vec{f}_P = 1131 \text{ x } 10^3\,\hat{a}_z \text{ x } 50 \text{ x } 10^{-6}$$

$$\vec{f}_P = 56.55\,\hat{a}_z \text{ N}$$