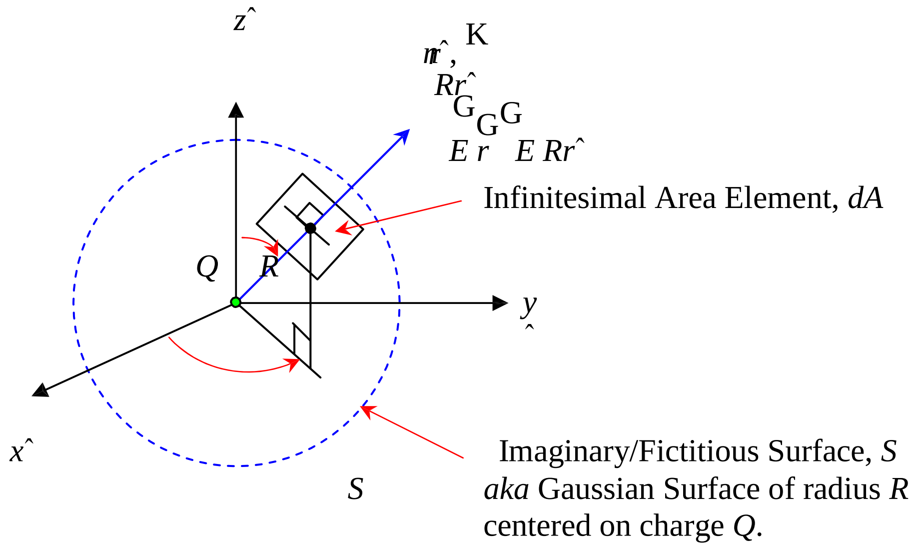
Gauss' Law / Divergence Theorem

Consider an imaginary / fictitious surface enclosing / surrounding e.g. a point charge (or a small charged conducting object). For simplicity, use an imaginary sphere of radius R centered on charge Q at origin:



Area element dA is a VECTOR quantity: $d\vec{A} = dA\,\hat{n} = dA\,\hat{r}$. By convention, $\hat{n}$ is outward-pointing unit normal vector at area element dA. In this particular case (because of spherical symmetry of problem): $\hat{n} = \hat{r}$

FLUX OF ELECTRIC FIELD LINES (through surface S): $\Phi_E = \int_S \vec{E}(\vec{r})\cdot d\vec{A}$

Φ_E = "measure" of "number of E-field "lines" passing through surface S, (SI Units: Volt-meters).

TOTAL ELECTRIC FLUX (Φ_E^{TOT}) associated with any closed surface S, is a measure of the (total) charge enclosed by surface S.

n.b. charge outside of surface S will contribute *nothing* to total electric flux Φ_E (since E-field lines pass through one portion of the surface S and out another – no net flux!)

Consider our point charge Q at origin. Calculate the flux of $\vec{E}$ passing through a sphere of radius r: (see above picture)

$$\Phi_E = \oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \frac{Q}{4\pi\varepsilon_o}\oint_S \frac{1}{r^2}\,\hat{r}\cdot \underbrace{r^2\sin\theta\, d\theta\, d\varphi\,\hat{r}}_{\substack{d\vec{A}\\ \text{infinitesimal vector}\\ \text{area element for}\\ \text{sphere of radius r}}}$$

n.b. Vector area element of sphere of radius, r is $\boxed{d\vec{A} = dA\,\hat{r} = r^2\sin\theta\, d\theta\, d\varphi\,\hat{r}}$ in spherical-polar coordinates.

Thus: $\Phi_E = \frac{Q}{4\pi\varepsilon_o}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \underbrace{\hat{r}\cdot\hat{r}}_{=1} \sin\theta\, d\theta\, d\varphi = \frac{2\pi Q}{4\pi\varepsilon_o}\underbrace{\int_{\theta=0}^{\pi}\sin\theta\, d\theta}_{=2}$

$= \frac{2Q}{2\varepsilon_o} = \frac{Q}{\varepsilon_o}$

<u>Gauss' Law (in Integral Form)</u>: $\boxed{\Phi_E = \oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_o}}$

Electric flux through closed surface S = (electric charge enclosed by surface S)/ε_o

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If $\exists$ (= there exists) lots of <u>discrete</u> charges q_i (ALL <u>enclosed</u> by imaginary / fictitious / Gaussian surface S), we know from principle of superposition that:

$\vec{E}_{NET}(\vec{r}) = \sum_{i=1}^{N} \vec{E}_i(\vec{r})$

Then: $\Phi_E^{NET} = \oint_S \vec{E}_{NET}(\vec{r})\cdot d\vec{A} = \sum_{i=1}^{N}\oint_S \vec{E}_i(\vec{r})\cdot d\vec{A} = \sum_{i=1}^{N}\frac{q_i}{\varepsilon_o} = \frac{1}{\varepsilon_o}\sum_{i=1}^{N} q_i = \frac{Q_{encl}}{\varepsilon_o}$

If $\exists$ volume charge density $\rho(\vec{r})$, then: $Q_{encl} = \int_v \rho(\vec{r})\, d\tau$

Then using the <u>DIVERGENCE THEOREM:</u>

$\Phi_E = \oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \int_v \vec{\nabla}\cdot\vec{E}(\vec{r})\, d\tau = \frac{Q_{encl}}{\varepsilon_o} = \frac{1}{\varepsilon_o}\int_v \rho(\vec{r})\, d\tau$

This relation holds for *<u>any</u>* volume v $\Rightarrow$ the *<u>integrands</u>* of $\int_v d\tau$ *<u>must</u>* be equal, i.e.:

<u>Gauss' Law (in Differential Form)</u>: $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r}) = \rho(\vec{r})/\varepsilon_o}$

The DIVERGENCE OF $\vec{E}(\vec{r})$: $\vec{\nabla}\cdot\vec{E}(\vec{r})$

Calculate $\vec{\nabla}\cdot\vec{E}(\vec{r})$ directly from $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_o}\displaystyle\int_{v\ \text{all space}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\vec{r}\,')\,d\tau'$

n.b. now extended over *all* space!

Remember that $\vec{\mathscr{r}}$ is NOT a constant!

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

field point P ($\vec{r}$), source point S ($\vec{r}\,'$)

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \vec{\nabla}\cdot\left(\frac{1}{4\pi\varepsilon_o}\int_{v\ \text{all space}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\vec{r}\,')\,d\tau'\right) = \frac{1}{4\pi\varepsilon_o}\int_{v\ \text{all space}} \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\rho(\vec{r}\,')\,d\tau'$$

Now: $\vec{\nabla}\cdot\left(\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{\mathscr{r}})$ (3D Dirac fcn.) (see equation 1.100, Griffiths p. 50)

Thus: $\vec{\nabla}\cdot\left(\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{\mathscr{r}})$ or: $\vec{\nabla}\cdot\left(\dfrac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right) = 4\pi\,\delta^3(\vec{r}-\vec{r}\,')$

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{v\ \text{all space}} 4\pi\,\delta^3(\vec{r}-\vec{r}\,')\rho(\vec{r}\,')\,d\tau' = \frac{\rho(\vec{r})}{\varepsilon_o}$$

Gauss' Law in Differential Form:

$$\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_o}}$$

Gauss' Law in Integral Form:

$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \dfrac{\rho(\vec{r})}{\varepsilon_o}$, thus: $\displaystyle\int_V \vec{\nabla}\cdot\vec{E}\,d\tau = \int_v \left(\frac{\rho}{\varepsilon_o}\right)d\tau = \frac{1}{\varepsilon_o}\int_v \rho\,d\tau = \frac{1}{\varepsilon_o}Q_{encl}$

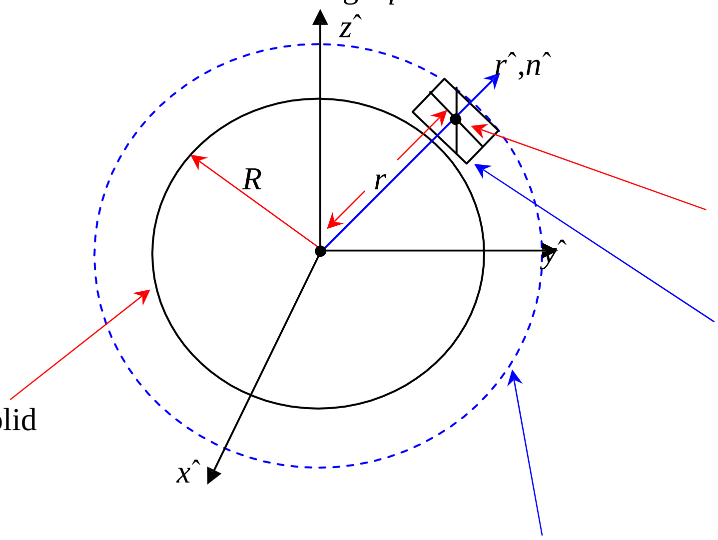
Now apply/use the Divergence Theorem on the volume integral associated with $\vec{\nabla}\cdot\vec{E}(\vec{r})$:

$$\int_v \vec{\nabla}\cdot\vec{E}\,d\tau = \oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \frac{1}{\varepsilon_o}\int_v \rho\,d\tau = \frac{1}{\varepsilon_o}Q_{encl}$$

Thus we obtain: $\boxed{\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_o}}$ <u>Gauss' Law in Integral Form</u>

APPLICATIONS OF GAUSS' LAW - very explicit, detailed derivation -

Griffiths Example 2.2: Find / determine the electric field intensity $\vec{E}(\vec{r})$ outside a uniformly charged solid sphere of radius R and total charge q:



Gauss' Law: $\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \frac{1}{\varepsilon_o} Q_{encl} = \frac{1}{\varepsilon_o} q = \frac{q}{\varepsilon_o}$

$\vec{E}(\vec{r}) = E(r)\hat{r}$ $d\vec{A} = dA\hat{n} = dA\hat{r}$ (for Gaussian sphere)

n.b. by symmetry of sphere: $\vec{E}_{sphere}(r \geq R) = E(r)\hat{r}$ i.e. E must be radial!!

$\vec{E}(\vec{r})\cdot d\vec{A} = E(r)\hat{r}\cdot dA\hat{r} = E(r)dA\underbrace{(\hat{r}\cdot\hat{r})}_{1} = E(r)dA$

NOTE: $|\vec{E}(\vec{r})| = E(r)$

n.b. Here again, by symmetry, the magnitude of $\vec{E}$ is constant (for all θ,φ)/for any fixed r!!! (on the Gaussian spherical surface).

$\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \oint_S E(r)dA = q/\varepsilon_o$

$E(r)\oint_S dA = E(r)\,4\pi r^2 = q/\varepsilon_o$

$E(r) = \frac{q/\varepsilon_o}{4\pi r^2} = \frac{q}{4\pi\varepsilon_o r^2}$ or: $\boxed{\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_o r^2}\hat{r} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\hat{r}}$

= Electric field outside a charged sphere of radius R at radial distance $r > R$ from center of sphere.

n.b. the electric field (for $r > R$) for charged sphere is equivalent / identical to that of a point charge q located at the origin!!!

GAUSS' LAW AND SYMMETRY

Use of (Geometrical / Reflection) symmetry (and any / all kinds of symmetry arguments in general) can be extremely powerful in terms of simplifying seemingly complicated problems!!

Learn skill of recognizing symmetries and applying symmetry arguments to solve problems!

Examples of use of Geometrical Symmetries and Gauss' Law

a) Charged sphere – use concentric Gaussian sphere and spherical coordinates
b) Charged cylinder – use coaxial Gaussian cylinder and cylindrical coordinates
c) Charged box / Charged plane – use appropriately co-located Gaussian "pillbox" (rectangular box) and rectangular coordinates
d) Charged ellipse – use concentric Gaussian ellipse and elliptical coordinates
e) Charged planar equilateral triangle } Think about
f) Charged pyramid } these!!

APPLICATIONS OF GAUSS' LAW (CONTINUED) - very explicit detailed derivation

Griffiths Example 2.3 Consider a long cylinder (e.g. plastic rod) of length L and radius S that carries a volume charge density ρ that is proportional to the distance from the axis s of the cylinder / rod – i.e.

$$\rho(s) = ks \quad \left(\frac{coulombs}{meter^3}\right)$$

$$k = \text{proportionality constant} \quad \left(\frac{coulombs}{meter^4}\right)$$

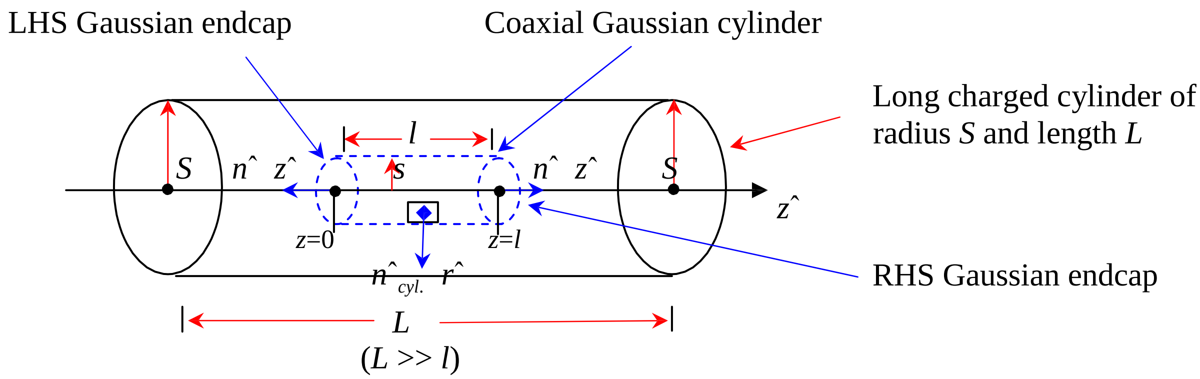
a) Determine the electric field $\vec{E}(\vec{r})$ inside this long cylinder / charged plastic rod
- Use a coaxial Gaussian cylinder of length l and radius s: (with $l << L$)

Gauss' Law $$\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_o}$$

Enclosed charge: $Q_{encl} = \int_v \rho\, d\tau = \int_v ks\, s\, ds\, d\varphi\, dz$ integral over Gaussian surface

$$Q_{encl} = \int_{s'=0}^{s'=s}\int_{\varphi=0}^{2\pi}\int_{z=0}^{z=l} ks\, s\, ds\, d\varphi\, dz = 2\pi k l \int_{s'=0}^{s'=s} s'^2\, ds'$$

$$Q_{encl} = \frac{2}{3}\pi k l s^3$$



Cylindrical Symmetry $\vec{E}(\vec{r}) = \vec{E}(r)\,\hat{r}$ (i.e. $\vec{E}$ points radially outward, ⊥ to z-axis.)

$$\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \int_{\substack{cylindrical\\ portion of\\ Gaussian\\ surface}} \vec{E}(\vec{r})\cdot d\vec{A}_{cyl.} + \int_{\substack{LHS\ endcap\\ portion of\\ Gaussian\\ surface}} \vec{E}(\vec{r})\cdot d\vec{A}_{LHS\ endcap} + \int_{\substack{RHS\ endcap\\ portion of\\ Gaussian\\ surface}} \vec{E}(\vec{r})\cdot d\vec{A}_{RHS\ endcap}$$

Again, from <u>cylindrical symmetry</u> (here):

$\vec{E}(\vec{r}) = \left|\vec{E}(\vec{r})\right|$ = <u>constant</u> on cylindrical Gaussian surface – i.e. fixed $r = |\vec{r}| = s$

What are $d\vec{A}_{cyl.}$, $d\vec{A}_{LHS\ endcap}$, and $d\vec{A}_{RHS\ endcap}$???

$d\vec{A}_{cyl.} = \underbrace{s\,d\varphi\,d\ell}_{}\,\hat{r}$ ($\hat{n}_{cyl.} = \hat{r}$)

infinitesimal surface area element of Gaussian cylinder

$d\vec{A}_{LHS\ endcap} = -s\,ds\,d\varphi\,\hat{z}$ ($\hat{n}_{LHS\ endcap} = -\hat{z}$)

$d\vec{A}_{RHS\ endcap} = +s\,ds\,d\varphi\,\hat{z}$ ($\hat{n}_{RHS\ endcap} = +\hat{z}$)

$$\oint_{\substack{S\\ Gaussian\\ cylinder}} \vec{E}(\vec{r})\cdot d\vec{A} = \int_{\substack{Cyl.\\ Gaussian\\ surface}} E(r)\,\underbrace{\hat{r}\cdot\hat{r}}_{\hat{r}\cdot\hat{r}=1}\,s\,d\ell\,d\varphi - \int_{\substack{LHS\\ Gaussian\\ endcap}} \cancel{E(r)\,\hat{r}\cdot\hat{z}\,s\,ds\,d\varphi}_{\hat{r}\cdot\hat{z}=0} + \int_{\substack{RHS\\ Gaussian\\ endcap}} \cancel{E(r)\,\hat{r}\cdot\hat{z}\,s\,ds\,d\varphi}_{\hat{r}\cdot\hat{z}=0}$$

<u>Note(s):</u>

$\vec{E}(\vec{r}) = \left|\vec{E}(\vec{r})\right|$ = constant on cylindrical Gaussian surface (fixed $r = s$)

$\vec{E}(\vec{r}) = E(r)\,\hat{r}$ by symmetry of charged cylinder

On LHS and RHS endcaps $\vec{E}(\vec{r})$ is ***not*** constant, because r is changing there - (but $\vec{E}$ still points in $\hat{r}$ direction! However, note that $\hat{r}\cdot\hat{r} = 1$ and $\hat{r}\cdot\hat{z} = 0$ ⇒ Gaussian endcap terms do <u>*not*</u> contribute!!!

$$\oint_{\substack{S\\ Gaussian\\ cylinder}} \vec{E}(\vec{r})\cdot d\vec{A} = \int_{\substack{cylindrical\\ Gaussian\\ surface}} \overbrace{\vec{E}(\vec{r})}^{\text{Constant here}}\,s\,d\ell\,d\varphi = E(r=s)\int_{z=0}^{z=l} d\ell\int_{0}^{2\pi} d\varphi = E(r=s)\,l\,2\pi = 2\pi s l E(r)$$

Putting this all together now: $\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \dfrac{Q_{encl}}{\varepsilon_o}$ where (here): $Q_{encl} = \dfrac{2}{3}\pi k l s^3$

$$2\pi s l E(r) = \frac{2\pi l k s^3}{3\varepsilon_o}$$ or: inside $$\vec E_{in}(\vec r) = \frac{ks^2}{3\varepsilon_o}\hat r \quad s = r \le S$$ n.b. $r \equiv s$ as used in Griffith's book, page 73

b) Find ELECTRIC FIELD $\vec E(\vec r)$ outside of this long cylinder / charged plastic rod

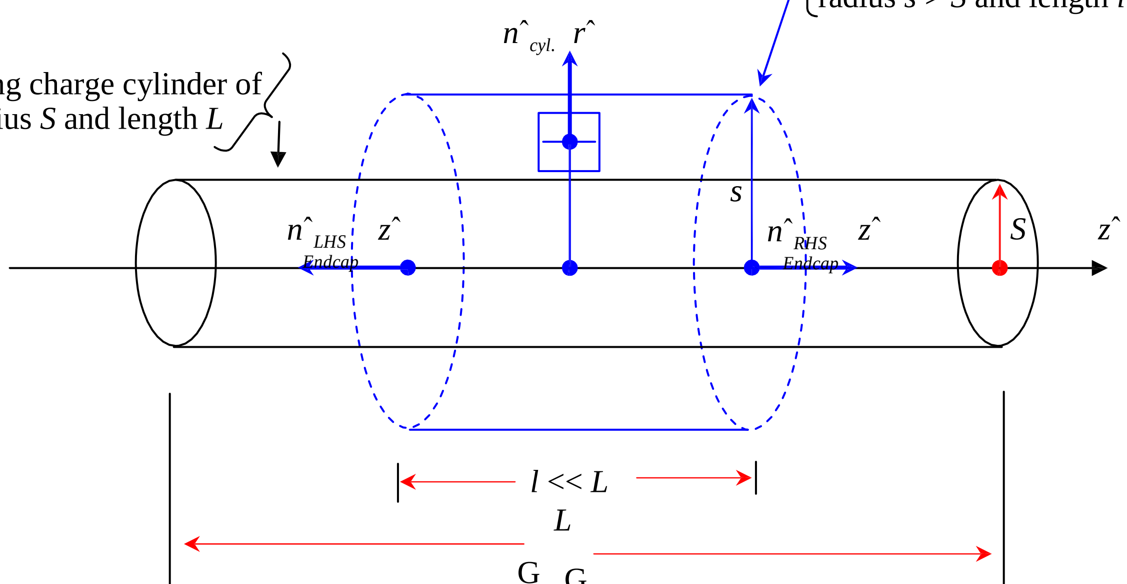
Again, use Coaxial Gaussian cylinder of length l (<< L) and radius s (> S):

Gauss' Law: $\oint_S \vec E(\vec r)\cdot d\vec A = \frac{Q_{encl}}{\varepsilon_o}$

Enclosed charge (for $s > S$): $Q_{encl} = \frac{2}{3}\pi k l S^3$

coaxial Gaussian cylinder radius $s > S$ and length $l << L$

Long charge cylinder of radius S and length L



Again, from symmetry of long cylinder $\vec E(\vec r) = E(r)\hat r$ constant (radial) direction!!

$r = s$ (fixed radius)

$$\oint_S \vec E(\vec r)\cdot d\vec A = \int_{\substack{cylindrical\\Gaussian\\surface}} \vec E(\vec r)\cdot d\vec A_{cyl} + \int_{\substack{LHS\\Gaussian\\endcap}} \vec E(\vec r)\cdot d\vec A_{LHS\,endcap} + \int_{\substack{RHS\\Gaussian\\endcap}} \vec E(\vec r)\cdot d\vec A_{RHS\,endcap}$$

$d\vec A_{cyl} = s\,d\varphi\,dl\,\hat r$

$d\vec A_{\substack{LHS\\endcap}} = -s\,ds\,d\varphi\,\hat z = -s\,ds\,d\varphi\,\hat z = -dA_{LHS}\,\hat z = -\left|d\vec A_{endcap}\right|\hat z$

$\left|d\vec A_{cyl}\right|\hat r = dA_{cyl}\,\hat r$

$d\vec A_{\substack{RHS\\endcap}} = +s\,ds\,d\varphi\,\hat z = s\,ds\,d\varphi\,\hat z = dA_{RHS}\,\hat z = \left|d\vec A_{endcap}\right|\hat z$

Now: $\hat r\cdot\hat r = 1$ and $\hat r\cdot\hat z = 0$

Then:

$$\mathrm{V}_S \; \vec{E}\, r \,\mathrm{i}dA \quad \vec{E}\, r\, \hat{r}\,\mathrm{i}dA_{cyl}\, \hat{r} \quad \vec{E}\, r\, \hat{r}\,\mathrm{i}\, dA_{LHS}\; \hat{z} \quad \vec{E}\, r\, \hat{r}\,\mathrm{i}\, dA_{RHS}\; \hat{z}$$

(cylindrical Gaussian surface; LHS Gaussian endcap $= 0$; RHS endcap $= 0$)

$$E\, r \quad {}_{z0}^{zl}\,{}_{0}^{2}\; s dl d \;\; 2\, slE\, r$$

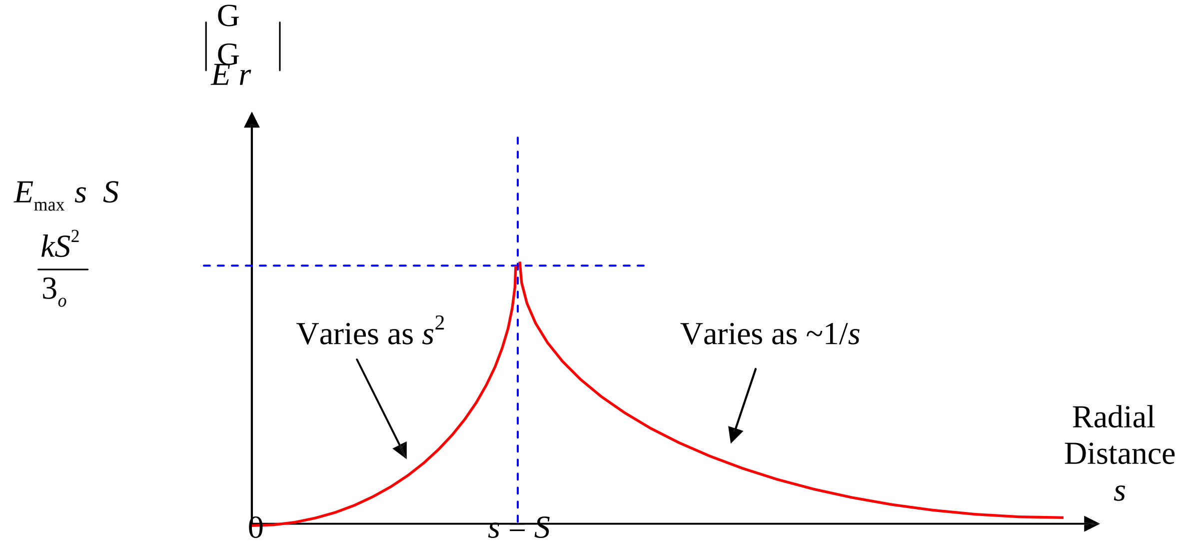
Electric field outside charged rod ($s = r > S$) : $\vec{E}_{out}\, r \quad \frac{2 \; kl\, S^3}{3 i 2\, sl_o}\, \hat{r} \quad \frac{kS^3}{3 s_o}$

ELECTRIC FIELD (INSIDE/OUTSIDE) vs. radial distance s	LONG CHARGED CYLINDER (radius S, $(s) = ks$)

Inside ($s < S$): $\vec{E}_{in}\, r \quad \frac{ks}{3_o}\, \hat{s}$

Outside ($s > S$): $\vec{E}_{out}\, r \quad \frac{kS^3}{3\;_o}\, \frac{1}{s}\, \hat{s} \quad \hat{s}\;\; \hat{r}$

Make a plot of $|\vec{E}\, r|$ vs. radial distance s:



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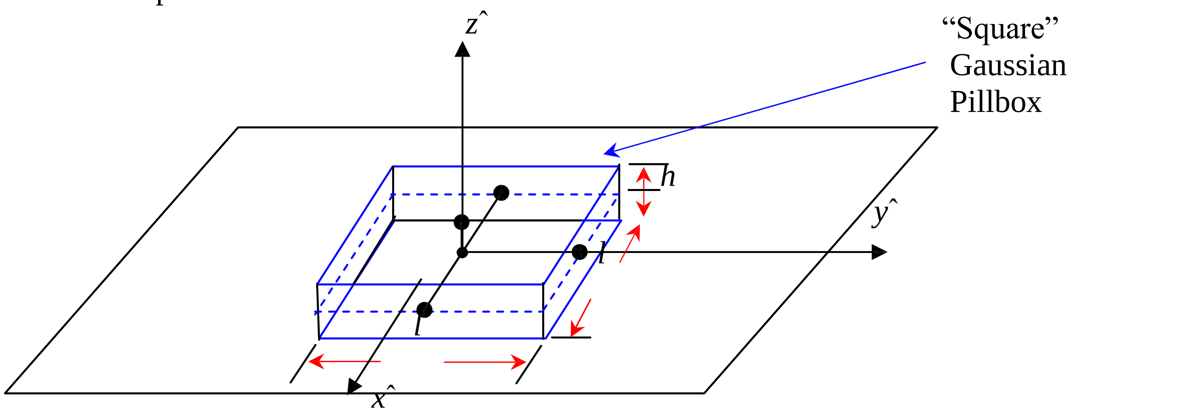
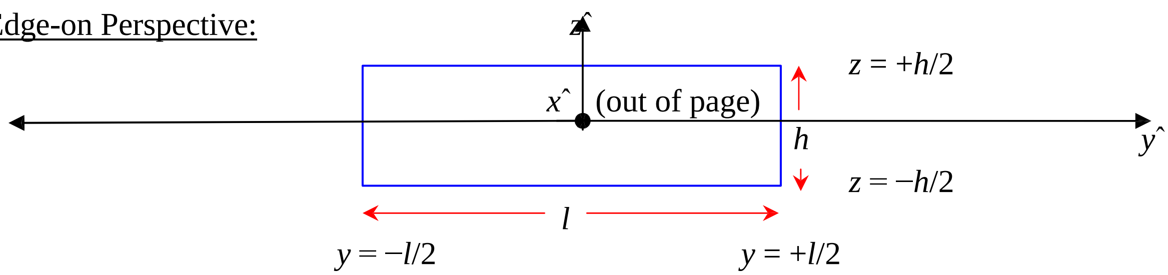
APPLICATIONS OF GAUSS' LAW - very explicit / detailed derivation –

<u>Griffiths Example 2.4</u>: An <u>infinite plane</u> carries uniform charge (coulombs / meter[2]).
Find the electric field a distance $z = z_o$ above (or below) the plane.

Use Gaussian Pillbox
centered on -plane:



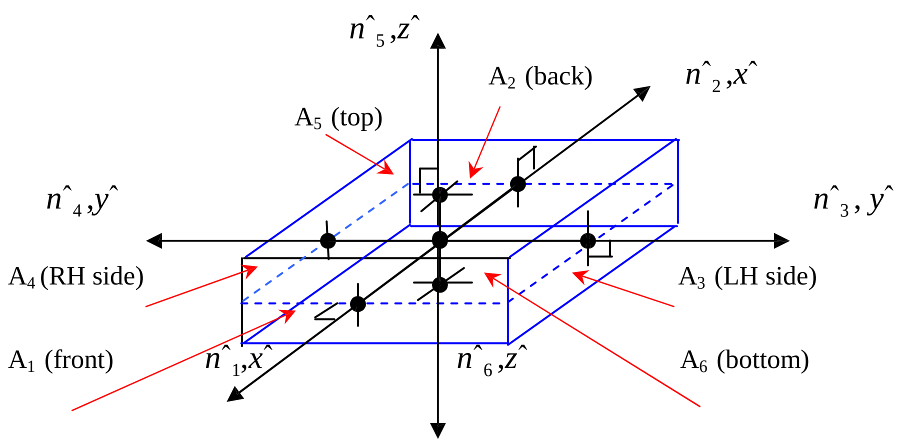
<u>Edge-on Perspective:</u>



Again, from the symmetry associated with -plane,

$\vec{E}(\vec{r})\ \vec{E}(r)\,\hat{z}\ \vec{E}(z)\,\hat{z}$ (above plane), $E(z)\,\hat{z}$ (below plane)

The Gaussian Pillbox has 6 sides – and thus has six outward unit normal vectors: :



Then:

$$\Phi_S = \oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \int_{A_1} \vec{E}(\vec{r})\cdot d\vec{A}_1 + \int_{A_2} \vec{E}(\vec{r})\cdot d\vec{A}_2 + \int_{A_3} \vec{E}(\vec{r})\cdot d\vec{A}_3 + \int_{A_4} \vec{E}(\vec{r})\cdot d\vec{A}_4 + \int_{A_5} \vec{E}(\vec{r})\cdot d\vec{A}_5 + \int_{A_6} \vec{E}(\vec{r})\cdot d\vec{A}_6$$

$$d\vec{A}_1 = dydz\,\hat{x} \qquad d\vec{A}_2 = -dydz\,\hat{x} = dydz(-\hat{x})$$
$$d\vec{A}_3 = dxdz\,\hat{y} \qquad d\vec{A}_4 = -dxdz\,\hat{y} = dxdz(-\hat{y})$$
$$d\vec{A}_5 = dxdy\,\hat{z} \qquad d\vec{A}_6 = -dxdy\,\hat{z} = dxdy(-\hat{z})$$

for $z > 0$: $\vec{E}(\vec{r}) = E(z)\,\hat{z}$

for $z < 0$: $\vec{E}(\vec{r}) = E(-z)(-\hat{z}) = -E(z)\,\hat{z}$

Again, by symmetry (of plane) n.b. $E(z)$ = constant (at least for fixed z).

Now because $\vec{E}(\vec{r}) = \pm E(z)\,\hat{z}$ for $\begin{Bmatrix} z > 0 \\ z < 0 \end{Bmatrix}$ respectively, we must break up integrals over z into two separate regions: $\int_{z=-h/2}^{h/2} dz = \int_{z=-h/2}^{0} dz + \int_{z=0}^{h/2} dz$

Then:

$$\Phi = \oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \int_{y=-\ell/2}^{\ell/2}\int_{z=-h/2}^{h/2} \vec{E}(\vec{r})\cdot dydz\,\hat{x} + \int_{y=-\ell/2}^{\ell/2}\int_{z=-h/2}^{h/2} \vec{E}(\vec{r})\cdot dydz(-\hat{x})$$
$$+ \int_{x=-\ell/2}^{\ell/2}\int_{z=-h/2}^{h/2} \vec{E}(\vec{r})\cdot dxdz\,\hat{y} + \int_{x=-\ell/2}^{\ell/2}\int_{z=-h/2}^{h/2} \vec{E}(\vec{r})\cdot dxdz(-\hat{y})$$
$$+ \int_{x=-\ell/2}^{\ell/2}\int_{y=-\ell/2}^{\ell/2} \vec{E}(\vec{r})\cdot dxdy\,\hat{z} + \int_{x=-\ell/2}^{\ell/2}\int_{y=-\ell/2}^{\ell/2} \vec{E}(\vec{r})\cdot dxdy(-\hat{z})$$

$$\Phi_S = \oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \int_{y=-\ell/2}^{\ell/2}\Big[\int_{z=-h/2}^{0} -E(z)\,\hat{z}\cdot\hat{x}\,\cancel{dydz} + \int_{z=0}^{h/2} E(z)\,\hat{z}\cdot\hat{x}\,\cancel{dydz}\Big] \quad \text{side } A_1 \text{ (front)}$$
$$+ \int_{y=-\ell/2}^{\ell/2}\Big[\int_{z=-h/2}^{0} -E(z)\,\hat{z}\cdot(-\hat{x})\,\cancel{dydz} + \int_{z=0}^{h/2} E(z)\,\hat{z}\cdot(-\hat{x})\,\cancel{dydz}\Big] \quad \text{side } A_2 \text{ (back)}$$
$$+ \int_{x=-\ell/2}^{\ell/2}\Big[\int_{z=-h/2}^{0} -E(z)\,\hat{z}\cdot\hat{y}\,\cancel{dxdz} + \int_{z=0}^{h/2} E(z)\,\hat{z}\cdot\hat{y}\,\cancel{dxdz}\Big] \quad \text{side } A_3 \text{ (RHS)}$$
$$+ \int_{x=-\ell/2}^{\ell/2}\Big[\int_{z=-h/2}^{0} -E(z)\,\hat{z}\cdot(-\hat{y})\,\cancel{dxdz} + \int_{z=0}^{h/2} E(z)\,\hat{z}\cdot(-\hat{y})\,\cancel{dxdz}\Big] \quad \text{side } A_4 \text{ (LHS)}$$
$$+ \underbrace{\int_{x=-\ell/2}^{\ell/2}\int_{y=-\ell/2}^{\ell/2} -E(z)\,\hat{z}\cdot(-\hat{z})\,dxdy\Big|_{z=-h/2}}_{\text{side } A_6 \text{ (bottom)}} + \underbrace{\int_{x=-\ell/2}^{\ell/2}\int_{y=-\ell/2}^{\ell/2} E(z)\,\hat{z}\cdot\hat{z}\,dxdy\Big|_{z=h/2}}_{\text{side } A_5 \text{ (top)}}$$

Now: $\hat{z}\cdot\hat{x} = 0 \quad \hat{z}\cdot\hat{y} = 0 \quad \hat{x}\cdot\hat{z} = 0 \quad \hat{y}\cdot\hat{z} = 0$ etc.

And: $\hat{x}\cdot\hat{x} = 1 \quad \hat{y}\cdot\hat{y} = 1 \quad \hat{z}\cdot\hat{z} = 1$

Because $\hat{x} \perp \hat{y} \perp \hat{z}$, no contributions to $\oint_S \vec{E}\cdot d\vec{A}$ (here) from 4 sides of Gaussian Pillbox (i.e. A_1, A_2, A_3 and A_4)

Only remaining / non-zero contributions are from bottom and top surfaces of Gaussian Pillbox because $\hat{n}_5 = \hat{z}$ and $\hat{n}_6 = -\hat{z}$ which are ∥ (or anti-parallel) to $\vec{E}(z) = E(z)\hat{z}$

Thus, we only have (here):

$$\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \int_{x=-\ell/2}^{x=\ell/2}\int_{y=-\ell/2}^{y=\ell/2} E(-z)\left(-\hat{z}\cdot\hat{z}\right)dxdy \quad \text{side } A_6 \text{ (bottom)}$$

$$+ \int_{x=-\ell/2}^{x=\ell/2}\int_{y=-\ell/2}^{y=\ell/2} E(z)\left(\hat{z}\cdot\hat{z}\right)dxdy \quad \text{side } A_5 \text{ (top)}$$

These integrals are not over z, and $E(z)$ = constant for z = fixed = z_o

⇒ can pull $E(z)$ outside integral, $-\hat{z}\cdot\hat{z} = -1$, $\hat{z}\cdot\hat{z} = 1$ etc.

$$\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = -E(-z)\int_{x=-\ell/2}^{x=\ell/2}\int_{y=-\ell/2}^{y=\ell/2} dxdy \quad \text{side } A_6 \text{ (bottom)}$$

$$+ E(z)\int_{x=-\ell/2}^{x=\ell/2}\int_{y=-\ell/2}^{y=\ell/2} dxdy \quad \text{side } A_5 \text{ (top)}$$

$$= -E(-z)\ell^2 + E(z)\ell^2 = 2E(z)\ell^2$$

But: $\ell^2 = \ell\cdot\ell$ = surface area of top and bottom surfaces of Gaussian Pillbox

Now: $\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \dfrac{Q_{encl}}{\varepsilon_o}$ What is Q_{encl} (by Gaussian Pillbox)?

$$Q_{encl} = \sigma\left(\frac{Coulombs}{meter^2}\right)\cdot\ell^2\left(meters^2\right) = \sigma\ell^2\ Coulombs$$

$$\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_o} = 2E(z)\ell^2 = \sigma\ell^2/\varepsilon_o$$

or: $$\boxed{E(z) = \frac{1}{2}\sigma/\varepsilon_o = \frac{\sigma}{2\varepsilon_o}}$$

Vectorially: $$\boxed{\vec{E}(z) = \frac{\sigma}{2\varepsilon_o}\begin{cases}+\hat{z}, & for\ z>0\\ -\hat{z}, & for\ z<0\end{cases}}$$

NOTE: $|\vec{E}(z)|$ = constant!!

No z – dependence for charged ∞ plane!

$\vec{E}(\vec{r})$ from - plane (slight return):

<u>Note</u> that in the initial process of setting up the Gaussian Pillbox, if we'd shrunk the height h of the Pillbox to be infinitesimally small, i.e. $h \rightarrow h$ and then took the limit $h \rightarrow 0$, the contributions to $\oint_S \vec{E}(\vec{r}) \cdot d\vec{A}$ from (infinitesimally small) <u>sides</u> of (A_1, A_2, A_3 and A_4) Gaussian Pillbox would (formally) have <u>vanished</u> (i.e. = 0) independently of whether integrand $\vec{E}(\vec{r}) \cdot d\vec{A}$ vanished on these sides (or not). Only top and bottom surfaces contribute to $\oint_S \vec{E}(\vec{r}) \cdot d\vec{A}$ then (here).

So using this "trick" of the shrinking Pillbox at a surface / boundary very often can be useful, to <u>simplify</u> doing the problem.

This explicitly shows that (sometimes) there <u>*is*</u> more than one way to <u>correctly</u> do / solve a problem – equivalent methods <u>*may*</u> exist.

→ It is very important, conceptually-speaking to have a (very) clear / good understanding of how to do these Gauss' Law-type problems the "long' way <u>and</u> the "short" way!

The Curl of $\vec{E}(\vec{r})$: $\vec{\nabla}\times\vec{E}(\vec{r})$

First, study / consider simplest possible situation: point charge *at origin*: $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\hat{r}$

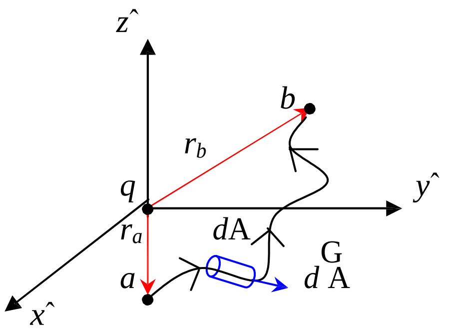
(note: $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = \vec{r}$ here because $\vec{r}\,' = 0$ - charge q located at origin!!!)

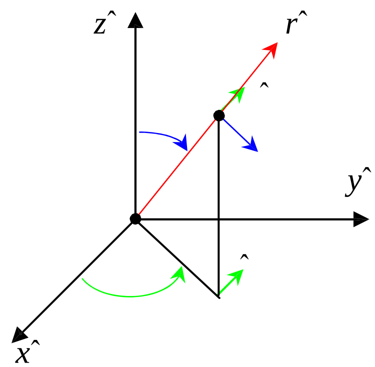
Thus (here), $\vec{E}(\vec{r})$ is <u>radial</u> (i.e. in $\hat{r}$ direction) due to spherical symmetry of problem (rotational invariance), thus static $\vec{E}$-field has <u>no</u> rotation/swirl/whirl ⇒ no curl! (Read Griffith's Ch. 1 on curl)

$\vec{\nabla}\times\vec{E}(\vec{r}) \equiv 0$ (<u>*must*</u> = 0)

Let's calculate:

Line integral $\int_a^b \vec{E}(\vec{r})\cdot d\vec{\ell}$ as shown in figure below:





In spherical coordinates: $d\vec{\ell} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\varphi\hat{\varphi}$

$$\vec{E}(\vec{r})\cdot d\vec{\ell} = \left(\frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\hat{r}\right)\cdot\left(dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\varphi\hat{\varphi}\right)$$

<u>Again</u>:

$\hat{r}\cdot\hat{r} = 1$	$\hat{r}\cdot\hat{\theta} = 0$	$\hat{r}\cdot\hat{\varphi} = 0$
$\hat{\theta}\cdot\hat{\theta} = 1$	$\hat{\theta}\cdot\hat{r} = 0$	$\hat{\theta}\cdot\hat{\varphi} = 0$
$\hat{\varphi}\cdot\hat{\varphi} = 1$	$\hat{\varphi}\cdot\hat{r} = 0$	$\hat{\varphi}\cdot\hat{\theta} = 0$

$\hat{r}$, $\hat{\theta}$, and $\hat{\varphi}$ are mutually orthogonal basis vectors (form *<u>ortho-normal</u>* basis)

$$\vec{E}(\vec{r})\cdot d\vec{\ell} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}dr$$

<u>Thus</u>: $$\int_a^b \vec{E}(\vec{r})\cdot d\vec{\ell} = \frac{1}{4\pi\varepsilon_o}\int_a^b\frac{q}{r^2}dr = -\frac{1}{4\pi\varepsilon_o}\frac{q}{r}\Bigg|_{r_a}^{r_b} = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r_a}-\frac{q}{r_b}\right) = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

r_a = distance from origin ⇒ to point <u>a</u>. r_b = distance from origin ⇒ to point <u>b</u>.

The line integral $\oint_C \vec{E}(\vec{r})\cdot d\vec{\ell}$ around a *<u>closed</u>* contour C is zero!

i.e. $\boxed{\oint_C \vec{E}(\vec{r})\cdot d\vec{\ell} = 0}$ This is not a trivial result! (Not true ∀ vectors!!)

(But *is* true for *static* $\vec{E}$-fields)

Use Stokes' Theorem (See Griffiths, Ch. 1.3.5, p. 34 and Appendix A-5)

$$\boxed{\int_S \left(\nabla\times\vec{E}(\vec{r})\right)\cdot d\vec{A} = \oint_C \vec{E}(\vec{r})\cdot d\vec{\ell} = 0}$$

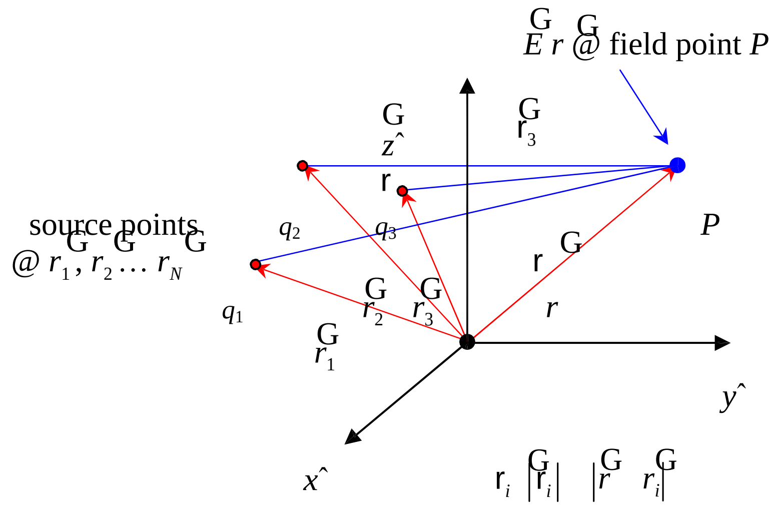
arbitrary closed surface *S* arbitrary closed contour *C* (on *S*)

Since $\int_S \left(\nabla\times\vec{E}(\vec{r})\right)\cdot d\vec{A} = 0$ must be / is true for *arbitrary* closed surface *S*, this can *only* be true for all ∀ closed surfaces *S* IFF (if and only if): $\boxed{\nabla\times\vec{E}(\vec{r}) = 0}$

Can use the Principle of Superposition to show that:

$$\vec{E}_{TOT}(\vec{r}) = \sum_{i=1}^{N}\vec{E}_i(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i$$ ← i = 1,2,3. . . *N* discrete charges, and $\vec{\mathscr{r}}_i \equiv \vec{r} - \vec{r}_i$

$$= \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) + \vec{E}_3(\vec{r}) + \dots + \vec{E}_N(\vec{r})$$



Then: $\nabla\times\vec{E}_{TOT}(\vec{r}) = \nabla\times\sum_{i=1}^{N}\vec{E}_i(\vec{r}) = \sum_{i=1}^{N}\nabla\times\vec{E}_i(\vec{r})$

$$= \sum_{i=1}^{N}\nabla\times\left(\frac{1}{4\pi\varepsilon_o}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i\right) = 0$$ n.b. all individual terms = 0 !!!

or: $\nabla\times\vec{E}_{TOT}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N} q_i\,\nabla\times\left(\frac{1}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i\right) = 0$

It can be shown that $\boxed{\nabla\times\vec{E}(\vec{r}) = 0}$

FOR ANY STATIC
CHARGE DISTRIBUTION

STATIC = NO TIME DEPENDENCE / VARIATION

$\vec{\nabla}\times\vec{E}(\vec{r}) = 0$ HOLDS FOR:

Static Discrete/Point Charges	$q(\vec{r})$	All *Static* Charge Distributions
Static Line Charges	$\lambda(\vec{r})$	
Static Surface Charges	$\sigma(\vec{r})$	
Static Volume Charges	$\rho(\vec{r})$	

Again, this *not* trivial (we'll see why, soon. . .)

One other (very important) point about the mathematical & geometrical nature of vector fields:

The nature of a (physically-realizable) vector field $\vec{A}(\vec{r})$ is fully specified if *both* its divergence $\vec{\nabla}\cdot\vec{A}(\vec{r})$ and its curl $\vec{\nabla}\times\vec{A}(\vec{r})$ are known.

This is a consequence of the so-called Helmholtz theorem – see/read Appendix *B* of Griffiths book.

The Helmholtz theorem also has an important corollary:

Any differentiable vector function $\vec{A}(\vec{r})$ that goes to zero faster than $1/r$ as $r\to\infty$ can be expressed as the gradient of a scalar plus the curl of a vector:

$$\vec{A}(\vec{r}) = -\vec{\nabla}\left(\frac{1}{4\pi}\int_v \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}\,')}{\mathscr{r}}\,d\tau'\right) + \vec{\nabla}\times\left(\frac{1}{4\pi}\int_v \frac{\vec{\nabla}'\times\vec{A}(\vec{r}\,')}{\mathscr{r}}\,d\tau'\right)$$

For the case of electrostatics: $\vec{\nabla}\cdot\vec{E}(\vec{r}) = \rho(\vec{r})/\varepsilon_o$ and $\vec{\nabla}\times\vec{E}(\vec{r}) = 0$

Thus:

$$\vec{E}(\vec{r}) = -\vec{\nabla}\left(\frac{1}{4\pi}\int_v \frac{\vec{\nabla}'\cdot\vec{E}(\vec{r}\,')}{\mathscr{r}}\,d\tau'\right) + \vec{\nabla}\times\left(\frac{1}{4\pi}\int_v \frac{\vec{\nabla}'\times\vec{E}(\vec{r}\,')}{\mathscr{r}}\,d\tau'\right) = -\vec{\nabla}\left(\frac{1}{4\pi\varepsilon_o}\int_v \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau'\right) = -\vec{\nabla}V(\vec{r})$$

i.e. $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ with $V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_v \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau'$ = Electrostatic Potential

SI Units: Volts

This result is valid e.g. in electrostatics for localized (i.e. finite spatial extent) charge distributions.

For infinite-expanse charge distributions (n.b. these are unphysical/artificial!), we must appeal to (more sophisticated) mathematical formalisms than the Helmholtz theorem…