Module 1
Coulomb's law, Electric Field Intensity, Flux Density & Gauss's Law

Electric field is the region or vicinity of a charged body where a test charge experiences a force. It is expressed as a scalar function of co-ordinates variables. This can be illustrated by drawing 'force lines' and these may be termed as 'Electric Flux' represented by and unit is coulomb (C).

Electric Flux Density $(\vec{D})$ is the measure of cluster of 'electric lines of force'. It is the number of lines of force per unit area of cross section.

i.e., $\vec{D} = \frac{\Psi}{A}\ \text{c/m}^2$ or $\psi = \int_S \vec{D}\ \hat{n}\ ds\ \ C$ where $\hat{n}$ is unit vector normal to surface

Electric Field Intensity $(\vec{E})$ at any point is the electric force on a unit +ve charge at that point.

i.e., $$\vec{E} = \frac{\vec{F}}{q} = \frac{q_1}{4\pi \in_0 r_1^2}\ \hat{a}_1\ \ N/c$$

$$= \frac{1}{\in_0}\left(\frac{q_1}{4\pi r_1^2}\right)\hat{a}_1\ \ N/c = \frac{\vec{D}}{\in_0}\ \ N/c \quad \text{or} \quad \vec{D} = \in_0\ \vec{E}\ \ C$$

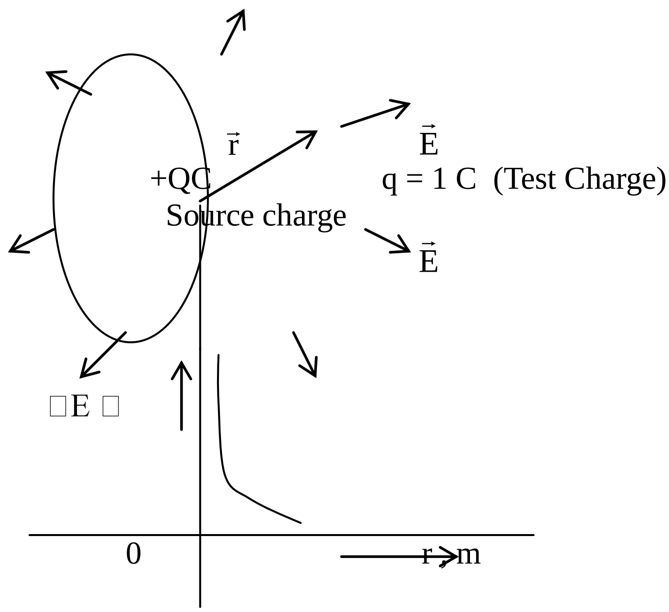
in vacuum

In any medium other than vacuum, the field Intensity at a point distant r m from + Q C is

$$\vec{E} = \frac{Q}{4\pi \in_0 \in_r r^2}\ \hat{a}_r\ \ N/c\ \ (\text{or}\ \ V/m)$$

and $\vec{D} = \in_0 \in_r \vec{E}\ \ C$ or $\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r\ \ C$

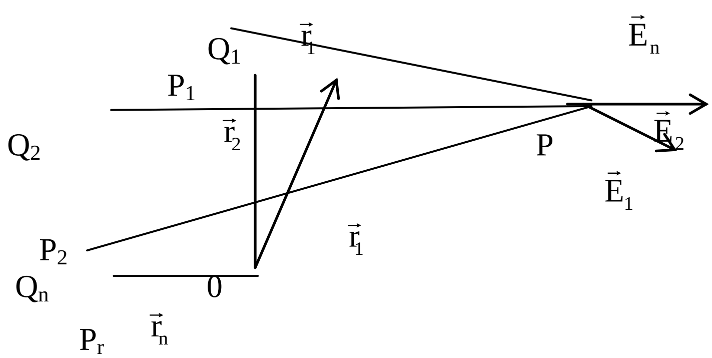
Thus $\vec{D}$ is independent of medium, while $\vec{E}$ depends on the property of medium.



Electric Field Intensity $\vec{E}$ for different charge configurations

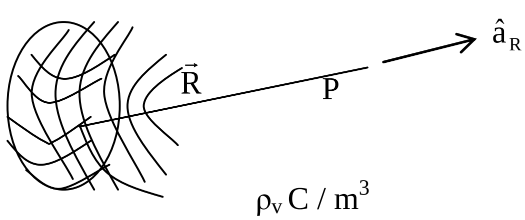
1. $\vec{E}$ **due to Array of Discrete charges**

Let Q, Q_1 , Q_2 , ……… Q_n be +ve charges at P, P_1 , P_2 , ……….. P_n . It is required to find $\vec{E}$ at P.



$$\vec{E}_r = \frac{1}{4\pi \in_0} \sum \frac{Q_m}{|r - r_m|^2} \hat{a}_m \ \ V/m$$

2. $\vec{E}$ **due to continuous volume charge distribution**



The charge is uniformly distributed within in a closed surface with a volume charge density of ρ_v C / m^3 i.e,

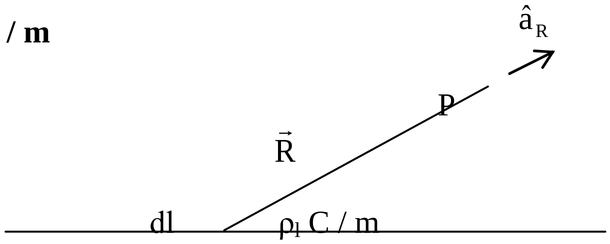
$$Q = \int_V \rho_V \, dv \quad \text{and} \quad \rho_V = \frac{dQ}{dv}$$

$$\Delta \vec{E} = \frac{\Delta Q}{4\pi \epsilon_0 R^2}\, \hat{a}_R = \frac{\rho_V \Delta V}{4\pi \epsilon_0 R^2}\, \hat{a}_R$$

$$\vec{E}_r = \int_{V^1} \frac{\rho_V (r^1)}{4\pi \epsilon_0 (r - r^1)^2}\, \hat{a}_R \ \ N / C$$

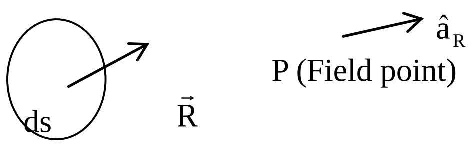
$\hat{a}_R$ is unit vector directed from ‘source’ to ‘filed point’.

3. **Electric field intensity $\vec{E}$ due to a line charge of infinite length with a line charge density of ρ_l C / m**



$$\vec{E}_p = \frac{1}{4\pi \epsilon_0} \int_L \frac{\rho_l \, dl}{R^2}\, \hat{a}_R \ \ N / C$$

4. $\vec{E}$ due to a surface charge with density of ρ_S C / m^2



(Source charge)

$$\vec{E}_p = \frac{1}{4\pi \epsilon_0} \int_S \frac{\rho_S \, ds}{R^2}\, \hat{a}_R \ \ N / C$$

Electrical Potential (V) The work done in moving a unit +ve charge from Infinity to that is called the Electric Potential at that point. Its unit is volt (V).

Electric Potential Difference (V_{12}) is the work done in moving a unit +ve charge from one point to (1) another (2) in an electric field.

Relation between $\vec{E}$ and V

If the electric potential at a point is expressed as a Scalar function of co-ordinate variables (say x,y,z) then V = V(x,y,z)

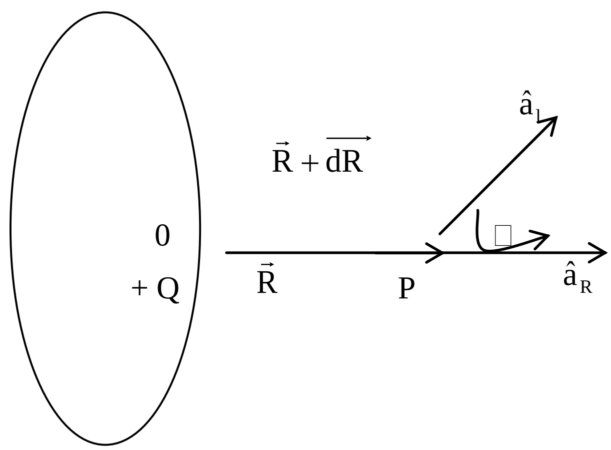
$$dV = -\frac{\vec{f}}{q}\,\overrightarrow{dl} = -\vec{E}\,.\,\overrightarrow{dl} \qquad \text{--------}(1)$$

$$\text{Also, } dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz$$

$$dV = \nabla V\,.\,\overrightarrow{dl} \qquad \text{---------}(2)$$

$$\text{From (1) and (2) } \vec{E} = -\nabla V$$

Determination of electric potential V at a point P due to a point charge of + Q C



At point P, $\vec{E} = \frac{Q}{4\pi \in_0 R^2}\,\hat{a}_R \;\; N/C$

Therefore, the force $\vec{f}$ on a unit charge at P.

$$\therefore \vec{f} = 1 \times \vec{E}_p = \frac{Q}{4\pi \in_0 R^2}\,\hat{a}_R \;\; N$$

The work done in moving a unit charge over a distance dl in the electric field is

$$dV = -\vec{f}\,.\,\overrightarrow{dl} = -\vec{E}\,.\,\overrightarrow{dl}$$

$$\therefore V_p = -\int_{\infty}^{R} \frac{Q\;dl}{4\pi \in_0 R^2}(\hat{a}_R\,.\,\hat{a}_l) = -\int_{\infty}^{R} \frac{Q}{4\pi \in_0 R^2}\,dR$$

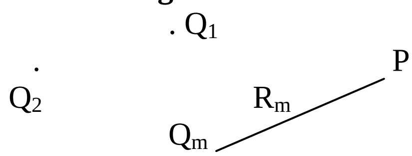
$$V_P = \frac{Q}{4\pi \in_0 R^2}\;\text{Volt} \qquad \text{(a scalar field)}$$

Electric Potential Difference between two points P & Q distant R_p and R_q from 0 is

$$V_{pq} = (V_p - V_q) = \frac{Q}{4\pi \in_0} \left[\frac{1}{R_p} - \frac{1}{R_q} \right] \text{ volt}$$

Electric Potential at a point due to different charge configurations.

1. **Discrete charges**

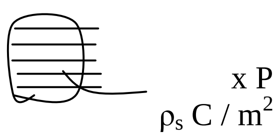


$$V_{1P} = \frac{1}{4\pi \in_0} \sum_{1}^{n} \frac{Q_m}{R_m} \text{ V}$$

2. **Line charge**

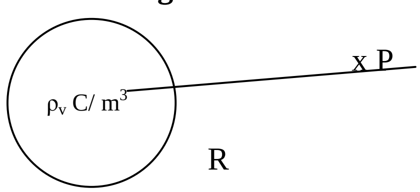
$$V_{2P} = \frac{1}{4\pi \in_0} \int_{l} \frac{l_l}{R} \, dl \text{ V}$$

3. **Surface charge**



$$V_{3P} = \frac{1}{4\pi \in_0} \int_{S} \frac{\rho_S \, ds}{R} \text{ V}$$

4. **Volume charge**



$$V_{4P} = \frac{1}{4\pi \in_0} \int_{V} \frac{\rho_V \, dv}{R} \text{ V}$$

5. **Combination of above** $V_{5P} = V_{1P} + V_{2P} + V_{3P} + V_{4P}$

Potential at every point on the spherical surface is

$$V_R = \frac{Q}{4\pi \in_0 R} \text{ volt}$$

V_{PQ} is difference of potential two equipotential surface potential

Gauss's law : The surface integral of normal component of $\vec{D}$ emerging from a closed surface is equal to the charge contained in the space bounded by the surface.

i.e., $\int_S \vec{D} \cdot \hat{n} \, ds = Q \ C$ (1)

where 'S' is called the 'Gaussian Surface'.

By Divergence Theorem,

$$\int_S \vec{D} \cdot \hat{n} \, ds = \int_V \nabla \cdot \vec{D} \, dv \quad \text{-----------} \quad (2)$$

Also, $Q = \int_V \rho_V \, dv$ ---------- (3)

From 1, 2 & 3,

$\nabla \cdot \vec{D} = \rho$ ----------- (4) is point form (or differential form) of Gauss's law while equation (1) is Integral form of Gauss law.

Poisson's equation and Laplace equation

In equation 4, $\vec{D} = \in_0 \vec{E}$

$\therefore \nabla \cdot \vec{E} = \rho / \in_0$ or $\nabla \cdot (-\nabla V) = \rho / \in_0$

$\nabla^2 V - \frac{\rho}{\in_0}$ Poisson equation

If $\rho = 0$, $\nabla^2 V = 0$ Laplace equation

Till now, we have discussed (1) Colulomb's law (2) Gauss law and (3) Laplace equation. The determination of $\vec{E}$ and V can be carried out by using any one of the above relations. However, the method of Coulomb's law is fundamental in approach while the other two use the physical concepts involved in the problem.

(1) **Coulomb's law** : Here $\vec{E}$ is found as force $\vec{f}$ per unit charge. Thus for the simple case of point charge of Q C,

$$\vec{E} = \frac{1}{4 \pi \in_0} \left(\frac{Q}{R^2} \right) \text{V/ M}$$

$$V = \int_l \vec{E} \, dl \quad \text{Volt}$$

(2) **Gauss's law** : An appropriate Gaussian surface S is chosen. The charge enclosed is determined. Then

$$\int_S \vec{D}\ \hat{n}\ ds = Q_{enc}$$

Then $\vec{D}$ and hence $\vec{E}$ are determined

Also $V = \int_l \vec{E}\ dl$ volt

The Laplace equation $\nabla^2 V = 0$ is solved subjecting to different boundary conditions to get V.

Then, $\vec{E} = -\nabla V$

Solutions to Problems on Potential :-

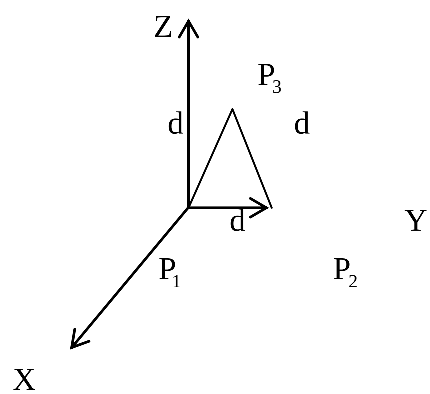
1. Data : Q_1 = 12 □C , Q_2 = 2 □C , Q_3 = 3 □C at the corners of equilateral triangle d m.

To find : $\vec{F}$ on Q_3

Solution :

Let Q_1 , Q_2 and Q_3 lie at P_1 , P_2 and P_3 the corners of equilateral triangle of side d meter.

If P_1 , P_2 and P_3 lie in YZ plane, with P_1 at origin then



$P_1 = (0,0,0)$ m

$P_2 = (0, d, 0)$ m

$P_3 = (0, 0.5\,d, 0.866\,d)$ m

$\vec{r}_1 = 0$

$\vec{r}_2 = d\,\hat{a}_y$

$\vec{r}_3 = 0.5\,d\,\hat{a}_y + 0.866\,\hat{a}_z$

The force $\vec{F}_3$ is $\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23}$

$$\vec{F}_3 = \frac{Q_3}{4\pi\epsilon_0}\left[\frac{Q_1}{d^2}\,\hat{a}_{13} + \frac{Q_2}{d^2}\,\hat{a}_{23}\right]$$

$$\hat{a}_{13} = \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|} = \frac{0.5\,d\,\hat{a}_y + 0.866\,d\,\hat{a}_z}{d} = 0.5\,\hat{a}_y + 0.866\,\hat{a}_z$$

$$\hat{a}_{23} = \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|} = -\ 0.5\,\hat{a}_y + 0.866\,\hat{a}_z$$

Substituting,

$$\vec{F}_3 = (3 \times 10^{-6}) \; 9 \times 10^{9} \left[\frac{12 \times 10^{-6}}{d^2} (0.5\,\hat{a}_y + 0.866\,\hat{a}_z) + \frac{2 \times 10^{-6}}{d^2} (-0.5\,\hat{a}_y + 0.866\,\hat{a}_z) \right]$$

$$= \frac{27 \times 10^{-3}}{d^2} \left[\frac{5\,\hat{a}_y + 12.12\,\hat{a}_z}{\sqrt{5^2 + 12.12^2}} \right] 13.11$$

$$\vec{F}_3 = 0.354\,\hat{a}_F \text{ N} \quad \text{where} \quad \hat{a}_F = (0.38\,\hat{a}_y + 0.924\,\hat{a}_z)$$

2. Data : At the point P, the potential is $V_p = (x^2 + y^2 + z^2)$ V

To find :

(1) $\vec{E}_p$ (2) V_{PQ} given P(1,0.2) and Q (1,1,2) (3) V_{PQ} by using general expression for V

Solution :

$$(1)\ \vec{E}_p = -\nabla V_p = -\left[\frac{\partial V_p}{\partial x}\,\hat{a}_x + \frac{\partial V_p}{\partial y}\,\hat{a}_y + \frac{\partial V_p}{\partial z}\,\hat{a}_z \right]$$

$$= -[\,2x\,\hat{a}_x + 2y\,\hat{a}_y + 3z^2\,\hat{a}_z\,] \text{ V/m}$$

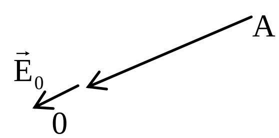
$$(2)\ V_{PQ} = -\int_Q^P \vec{E}_p \cdot \overrightarrow{dl} = \int_1^1 2x\,dx + \int_1^0 2y\,dy + \int_2^2 3z^2\,dz$$

$$= 0 + y^2\Big]^0 + 0 = -1\,\text{V}$$

$$(3)\ V_{PQ} = V_Q - V_P = -1\,\text{V}$$

3. Data : Q = 64.4 nC at A (-4, 2, -3) m

To find : $\vec{E}$ at 0 (0,0,0) m



Solution :

$$\vec{E}_0 = \frac{Q}{4\pi \epsilon_0 (AO)^2}\,\hat{a}_{AO} \text{ N/C}$$

$$= \frac{64.4 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} (AO)^2} [\hat{a}_{AO}] \text{ N/C}$$

$$\overrightarrow{AO} = (0+4)\,\hat{a}_x + (0-2)\,\hat{a}_y + (0+3)\,\hat{a}_z = 4\hat{a}_x - 2\,\hat{a}_y + 3\,\hat{a}_z$$

$$\hat{a}_{AO} = \frac{\overrightarrow{AO}}{|\overrightarrow{AO}|} = \frac{1}{\sqrt{29}} (\overrightarrow{AO}) = (0.743\,\hat{a}_x - 0.37\,\hat{a}_y + 0.56\,\hat{a}_z)$$

$$\vec{E}_0 = \frac{64.4 \times 9}{29}\,\hat{a}_{AO} = 20\,\hat{a}_{AO} \text{ N/C}$$

4. $Q_1 = 100\ \mu C$ at P_1 (0.03 , 0.08 , - 0.02) m

$Q_2 = 0.12\ \mu C$ at P_2 (- 0.03 , 0.01 , 0.04) m

F_{12} = Force on Q_2 due to Q_1 = ?

Solution :

$$\vec{F}_{12} = \frac{Q_1\ Q_2}{4\pi \in_0 R_{12}^2}\,\hat{a}_{12}$$

$$R_{12} = \vec{R}_2 - \vec{R}_1 = (-0.03\,\hat{a}_x + 0.01\,\hat{a}_y + 0.04\,\hat{a}_z) - (0.03\,\hat{a}_x + 0.08\,\hat{a}_y - 0.02\,\hat{a}_z)$$

$$= (-0.06\,\hat{a}_x - 0.07\,\hat{a}_y + 0.06\,\hat{a}_z)\ ;\ \left| R_{12} \right| = 0.11\,\text{m}$$

$$\hat{a}_{12} = (-0.545\,\hat{a}_x - 0.636\,\hat{a}_y + 0.545\,\hat{a}_z)$$

$$\vec{F}_{12} = \frac{100 \times 10^{-6} \times 0.121 \times 10^{-6}}{0.11^2} \times 9 \times 10^9\ \hat{a}_{12}$$

$$\vec{F}_{12} = 9\,\hat{a}_{12}\ \ \text{N}$$

5. $Q_1 = 2 \times 10^{-9}$ C , $Q_2 = - 0.5 \times 10^{-9}$ C C

(1) $R_{12} = 4 \times 10^{-2}$ m , $\vec{F}_{12} = ?$

(2) Q_1 & Q_2 are brought in contact and separated by $R_{12} = 4 \times 10^{-2}$ m $\vec{F}^{`}_{12} = ?$

Solution :

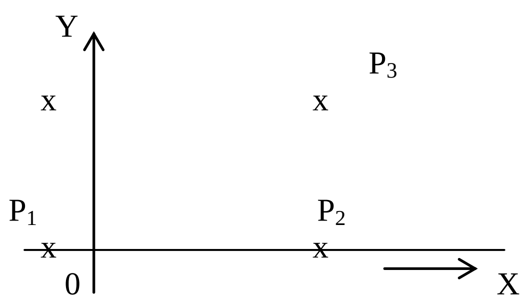
$$\vec{F}_{12} = \frac{2 \times 10^{-9} \times -0.5 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} \times (4 \times 10^{-2})^2} \; \hat{a}_{12} = \frac{-9}{16} \times 10^{-5} \; \hat{a}_{12} = +5.63\,\mu\text{N (attractive)}$$

(2) When brought into contact $Q^{`}_1 = Q^{`}_2 = \frac{1}{2}(Q_1 + Q_2) = 1.5 \times 10^{-9}$ C

$$\vec{F}^{`}_{12} = \frac{(1.5 \times 10^{-9})^2}{4\pi \times \frac{10^{-9}}{36\pi} \times (4 \times 10^{-2})^2} \; \hat{a}_{12} = \frac{1.5^2}{16} \times 9 \times 10^{-18+13} \; \hat{a}_{12} = 12.66\,\mu\text{N}\,\hat{a}_{12}$$

(1) $F^{`}_{12} = 12.66\,\mu$N (repulsive)

6.



$Q_1 = Q_2 = Q_3 = Q_4 = 20$ □ C
$Q_P = 200$ □ C at P(0,0,3) m

$P_1 = (0, 0 , 0)$ m $P_2 = (4, 0, 0)$ m
$P_3 = (4, 4, 0)$ m $P_4 = (0, 4, 0)$ m
$F_P = ?$

Solution :

$$\vec{F}_p = \vec{F}_{1p} + \vec{F}_{2p} + \vec{F}_{3p} + \vec{F}_{4p}$$

$$\vec{R}_{1p} = 3\,\hat{a}_z \quad \left| R_{1p} \right| = 3\,m \quad \hat{a}_{1p} = \hat{a}_z$$

$$\vec{R}_{2p} = -4\,\hat{a}_x + 3\,\hat{a}_z \; ; \left| R_{2p} \right| = 5\,m \quad \hat{a}_{2p} = -0.8\,\hat{a}_x + 0.6\,\hat{a}_z$$

$$\vec{R}_{3p} = -4\,\hat{a}_x - 4\,\hat{a}_y + 3\,\hat{a}_z \; ; \left| R_{3p} \right| = 6.4\,m \; ; \quad \hat{a}_{3p} = -0.625\,\hat{a}_x - 0.625\,\hat{a}_y + 0.47\,\hat{a}_z$$

$$\vec{R}_{4p} = -4\,\hat{a}_y + 3\,\hat{a}_z \; ; \left| R_{4p} \right| = 5\,m \; ; \quad \hat{a}_{4p} = -0.8\,\hat{a}_y + 0.6\,\hat{a}_z$$

$$\vec{F}_p = \frac{Q_p}{4\pi \frac{10^{-9}}{36\pi}} \left[\frac{Q_1}{R_{1p}^2}\hat{a}_{1p} + \frac{Q_2}{R_{2p}^2}\hat{a}_{2p} + \frac{Q_3}{R_{3p}^3}\hat{a}_{3p} + \frac{Q_4}{R_{4p}^2}\hat{a}_{4p} \right]$$

$$= 200 \times 10^{-6} \times 9 \times 10^{9} \left[\begin{array}{l} \frac{1}{3^2}\hat{a}_z + \frac{1}{5^2}(-0.8\,\hat{a}_x + 0.6\,\hat{a}_z) + \frac{1}{6.4^2}\;(-0.625\,\hat{a}_x - 0.625\,\hat{a}_y + 0.47\,\hat{a}_z) \\ + \frac{1}{5^2}(-0.8\,\hat{a}_y + 0.6\,\hat{a}_z) \end{array} \right] 20 \times 10^{-6}$$

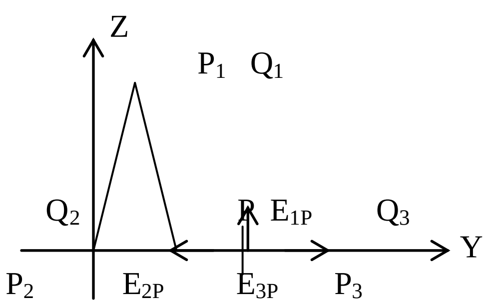
$$= 200x10^{-6} x9x10^{9} x10^{9} x10^{-6}\; x10^{-2} \left[\begin{array}{l} \frac{100}{9}\hat{a}_z + \frac{100}{25}(-0.8\,\hat{a}_x + 0.6\,\hat{a}_z) + \frac{100}{40.96}(-0.625\,\hat{a}_x - 0.625\,\hat{a}_y + 0.47\,\hat{a}_z) \\ + \frac{100}{25}(-0.8\,\hat{a}_y + 0.6\,\hat{a}_z) \end{array} \right]$$

$$= 0.36 \left[(-3.2 - 1.526)\,\hat{a}_x + \frac{1}{6.4^2}\;(-1.526 - 3.2)\,\hat{a}_y + (11.11 + 2.4 + 1.15 + 2.4)\,\hat{a}_z) \right]$$

$$= (-1.7\,\hat{a}_x - 1.7\,\hat{a}_y + 17\,\hat{a}_z)\;N \quad = 17.23\,\hat{a}_p\;N$$

7. Data : Q_1 , Q_2 & Q_3 at the corners of equilateral triangle of side 1 m.
Q_1 = - 1□C, Q_2 = -2 □C , Q_3 = - 3 □C
To find : $\vec{E}$ at the bisecting point between Q_2 & Q_3 .

Solution :



P_1 : (0, 0.5, 0.866) m
P_2 : (0, 0, 0) m
P_3 : (0, 1, 0) m
P : (0, 0.5, 0) m

$$\vec{E}_P = \vec{E}_{1P} + \vec{E}_{2P} + \vec{E}_{3P}$$

$$= \frac{1}{4\pi \epsilon_0}\left[\frac{Q_1}{R_{1P}^{\ 2}}\hat{a}_{1P} + \frac{Q_2}{R_{2P}^{\ 2}}\hat{a}_{2P} + \frac{Q_3}{R_{3P}^{\ 2}}\hat{a}_{3P}\right]$$

$$R_{1P} = -0.866\,\hat{a}_z \qquad |R_{1P}| = 0.866 \qquad \hat{a}_{1P} = -\hat{a}_z$$

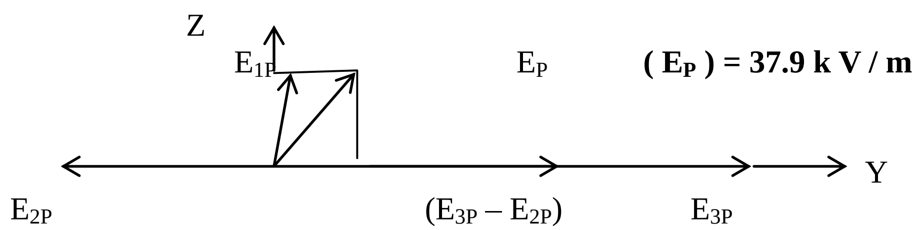
$$R_{2P} = +0.5\,\hat{a}_y \qquad |R_{2P}| = 0.5 \qquad \hat{a}_{2P} = \hat{a}_y$$

$$R_{3P} = -0.5\,\hat{a}_y \qquad |R_{3P}| = 0.5 \qquad \hat{a}_{3P} = -\hat{a}_y$$

$$\vec{E}_P = \frac{1}{4\pi\frac{10^{-9}}{36\pi}}\left[\frac{-1\,x\,10^{-6}}{0.866^2}(-\hat{a}_z) + \frac{-2\,x\,10^{-6}}{0.5^2}(-\hat{a}_y) + \frac{-3\,x\,10^{-6}}{0.5^2}(-\hat{a}_y)\right]$$

$$= 9\,x\,10^3\left[1.33\,\hat{a}_z - 8\,\hat{a}_y + 12\,\hat{a}_y\right]$$

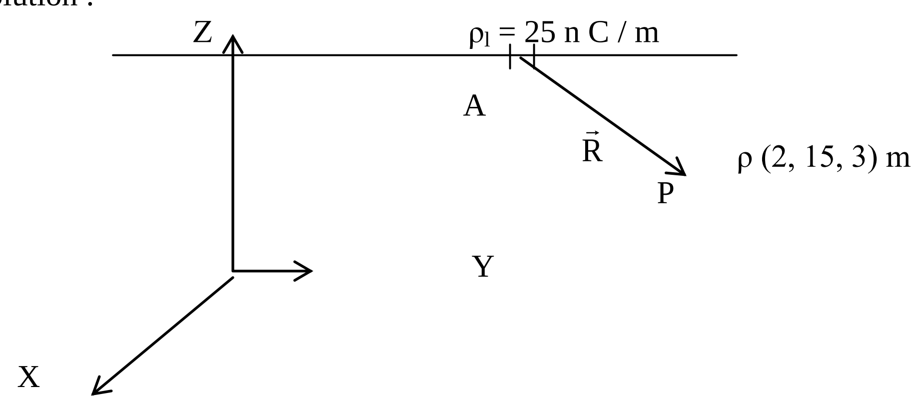
$$= 9\,x\,10^3\left[4\,\hat{a}_y + 1.33\,\hat{a}_z\right] = \left[36\,\hat{a}_y + 12\,\hat{a}_z\right] + 0^3\ V/m = 37.9\angle 18^0\ \ k\,V/m$$



8. Data P_l = 25 n C /m on (-3, y, 4) line in free space and P : (2,15,3) m

To find : E_P

Solution :



The line charge is parallel to Y axis. Therefore E_{PY} = 0

$$\vec{R} = \overrightarrow{AP} = (2-(-3))\,\hat{a}_x + (3-4)\,\hat{a}_z = (5\,\hat{a}_x - \hat{a}_z)\,; \; = \left| R \right| = 5.1\,\text{m}$$

$$\hat{a}_R = \frac{\vec{R}}{\left| R \right|} = (0.834\,\hat{a}_x - 0.167\,\hat{a}_z)$$

$$\vec{E}_P = \frac{\rho_l}{2\,\pi \in_0 \, R}\;\hat{a}_R = \frac{25\,\text{x}}{2\,\pi\,\frac{10^{-9}}{36\,\pi}\,\text{x}\,5.1}\;\hat{a}_R$$

$$\vec{E}_P = 88.23\,\hat{a}_R\,\text{V / m}$$

9. Data : P_1 (2, 2, 0) m ; P_2 (0, 1, 2) m ; P_3 (1, 0, 2) m

Q_2 = 10 □C ; Q_3 = - 10 □C

To find : E_1 , V_1

Solution :

$$\vec{E}_1 = \vec{E}_{21} + \vec{E}_{21} = \frac{1}{4\,\pi \in_0}\left[\frac{Q_2}{R_{21}^{\;2}}\,\hat{a}_{21} + \frac{Q_3}{R_{31}^{\;2}}\,\hat{a}_{31}\right]$$

$$\vec{R}_{21} = (2\,\hat{a}_x + \hat{a}_y - 2\,\hat{a}_z) \qquad \left|\vec{R}_{21}\right| = 3 \quad \hat{a}_{21} = 0.67\,\hat{a}_x + 0.33\,\hat{a}_y - 0.67\,\hat{a}_z$$

$$\vec{R}_{31} = \hat{a}_x + 2\,\hat{a}_y + 2\,\hat{a}_z \qquad \left|\vec{R}_{31}\right| = 3 \quad \hat{a}_{31} = 0.33\,\hat{a}_x + 0.67\,\hat{a}_y + 0.67\,\hat{a}_z$$

$$\vec{E}_1 = 9\,\text{x}\,10^9\left[\frac{10^{-6}}{9}(0.67\,\hat{a}_x + 0.33\,\hat{a}_y - 0.67\,\hat{a}_z) + \frac{10^{-6}}{9}(0.33\,\hat{a}_x + 0.67\,\hat{a}_y + 0.67\,\hat{a}_z)\right]$$

$$= 10^3\,[\,\hat{a}_x + \hat{a}_y\,] = 14.14\,(0.707\,\hat{a}_x + 0.707\,\hat{a}_y)\,\text{V / m}$$

$$V_1 = \frac{1}{4\,\pi \in_0}\left[\frac{Q_2}{R_{21}} + \frac{Q_3}{R_{31}}\right] = 9\,\text{x}\,10^9\left[\frac{10^{-6}}{3} + \frac{10^{-6}}{3}\right] = 3000\,\text{V}$$

$$\mathbf{\left| E_1 \right| = 14.14\,V / m \qquad V_1 = 3000\,V}$$

10. Data : Q_1 = 10 □C at P_1 (0, 1, 2) m ; Q_2 = - 5 □C at P_2 (-1, 1, 3) m
P_3 (0, 2, 0) m

To find : (1) $\vec{E}_3$ (2) Q at (0, 0, 0) for $E_{3x} = 0$

Solution :

$$\text{(1)}\quad \vec{E}_3 = \frac{1}{4\pi\in_0}\left[\frac{Q_1}{R_{13}{}^2}\hat{a}_{13} + \frac{Q_2}{R_{23}{}^2}\hat{a}_{23}\right]$$

$$\vec{R}_{13} = (2-1)\,\hat{a}_y + (0-2)\,\hat{a}_z = \hat{a}_y - 2\,\hat{a}_z \qquad |R_{13}| = \sqrt{5}$$

$$\vec{R}_{23} = (0+1)\,\hat{a}_x + (2-1)\,\hat{a}_y + (0-3)\,\hat{a}_z = \hat{a}_x + \hat{a}_y - 3\,\hat{a}_z \qquad |R_{23}| = \sqrt{11}$$

$$\hat{a}_{13} = \frac{\vec{R}_{13}}{|\vec{R}_{13}|} = (0.447\,\hat{a}_y - 0.894\,\hat{a}_z)$$

$$\hat{a}_{23} = \frac{\vec{R}_{23}}{|\vec{R}_{23}|} = 0.3\,\hat{a}_x + 0.3\,\hat{a}_y - 0.9\,\hat{a}_z$$

$$\vec{E}_3 = 9 \times 10^9\left[\frac{10 \times 10^{-6}}{(\sqrt{5})^2}(0.447\,\hat{a}_y - 0.894\,\hat{a}_z) + \frac{-5 \times 10^{-6}}{(\sqrt{11})^2}(0.3\,\hat{a}_x + 0.3\,\hat{a}_y - 0.9\,\hat{a}_z)\right]$$

$$= \left[(8\,\hat{a}_y - 16\,\hat{a}_z) + (-1.23\,\hat{a}_x - 1.23\,\hat{a}_y + 3.68\,\hat{a}_z)\right]$$

$$= \left[-1.23\,\hat{a}_x + 6.77\,\hat{a}_y - 12.32\,\hat{a}_z\right] 10^3 \;\; \text{V / m}$$

$$\text{(2)}\quad \vec{E}_3 = 9 \times 10^9\left[\frac{Q_1}{R_{13}{}^2}\hat{a}_{13} + \frac{Q_2}{R_{23}{}^2}\hat{a}_{23} + \frac{Q}{R_{03}{}^2}\hat{a}_{03}\right];\ \vec{R}_{03} = 2\,\hat{a}_y$$

$$\vec{E}_{3x} = -1.23\,\hat{a}_x$$

$\vec{E}_{3x}$ cannot be zero

11. Data : Q_2 = 121 x 10^{-9} C at P_2 (-0.02, 0.01, 0.04) m
Q_1 = 110 x 10^{-9} C at P_1 (0.03, 0.08, 0.02) m
P_3 (0, 2, 0) m

To find : $\vec{F}_{12}$

Solution :

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\in_0 R_{12}{}^2}\ \hat{a}_{12}\ \text{N} ; \quad R_{12} = -0.05\,\hat{a}_x - 0.07\,\hat{a}_y + 0.02\,\hat{a}_z$$

$$\vec{F}_{12} = \frac{121 \times 10^{-9} \times 110 \times 10^{-9}}{4\pi\dfrac{10^{-9}}{36\pi} \times 7.8 \times 10^{-3}}\ [\hat{a}_{12}] \qquad |R_{12}| = 0.088$$

$$\mathbf{\vec{F}_{12} = 0.015\,\hat{a}_{12}\ N}$$

12. Given V = (50 $x^2yz + 20y^2$) volt in free space

Find V_P , $\vec{E}_P$ and $\hat{a}_{np}$ at P (1, 2, - 3) m

Solution :

$$V_P = \left[50\,(1)^2\,(2)\,(-3) + 20\,(2)^2\right] = -\,220\text{ V}$$

$$\vec{E} = -\nabla V = -\frac{\partial}{\partial x} V\,\hat{a}_x - \frac{\partial}{\partial y} V\,\hat{a}_y - \frac{\partial}{\partial z} V\,\hat{a}_z$$

$$\vec{E} = -100\,x\,y\,z\,\hat{a}_x - 50\,x^2 z\,\hat{a}_y - 50\,x^2 y\,\hat{a}_z$$

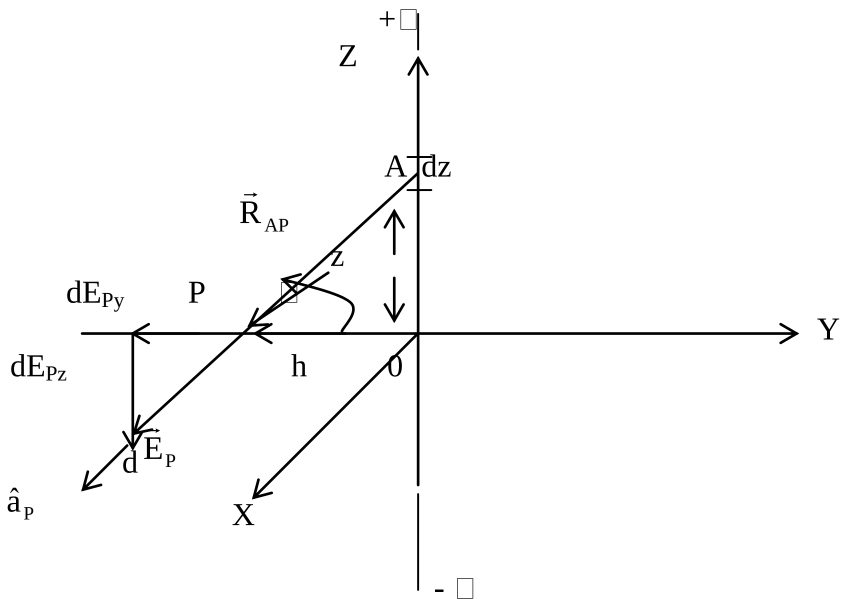
$$\vec{E}_P = -100\,(2)\,(-3)\,\hat{a}_x - 50\,(-3)\,\hat{a}_y - 50\,(2)\,\hat{a}_z$$

$$= 600\,\hat{a}_x + 150\,\hat{a}_y - 100\,\hat{a}_z$$

$$= 62\,6.5\,\hat{a}_P\ \text{V / m}\ ;\ \hat{a}_P = 0.957\,\hat{a}_x + 0.234\,\hat{a}_y - 0.16\,\hat{a}_z$$

Additional Problems

A1. Find the electric field intensity $\vec{E}$ at P (0, -h, 0) due to an infinite line charge of density ρ_l C / m along Z axis.



Solution :

Source : Line charge ρ_l C / m. Field point : P (0, -h, 0)

$$\overrightarrow{dE_P} = \frac{dQ}{4\pi \in_0 R^2}\,\hat{a}_R = \frac{\rho_l\, dz}{4\pi \in_0 R^2}\,\hat{a}_R \ \ V/m\,; \ \vec{R} = \overrightarrow{AP} = -z\,\hat{a}_z - h\,\hat{a}_y$$

$$\left|\vec{R}\right| = \left|\overrightarrow{AP}\right| = \sqrt{z^2 + h^2}$$

$$\hat{a}_R = \frac{\vec{R}}{\left|\vec{R}\right|} = \frac{1}{R}\left[-h\,\hat{a}_y - z\,\hat{a}_z\right]$$

$$\overrightarrow{dE_P} = \frac{\rho_l\, dz}{4\pi \in_0 R^2}\left[-\frac{h}{R}\,\hat{a}_y - \frac{z}{R}\,\hat{a}_z\right] = d\,E_{Py}\,\hat{a}_y + d\,E_{Pz}\,\hat{a}_z$$

$$dE_{Py} = -\frac{\rho_l\, dz}{4\pi \in_0 R^2}\,\frac{h}{R}\,\hat{a}_y \qquad dE_{Pz} = -\frac{\rho_l\, dz}{4\pi \in_0 R^2}\,\frac{z}{R}\,\hat{a}_z$$

Expressing all distances in terms of fixed distance h,
$h = R \cos\theta$ or $R = h \sec\theta$; $z = h \tan\theta$, $dz = h \sec^2\theta\, d\theta$

$$dE_{Py} = -\frac{\rho_l\, h\, Sec^2\theta\, d\theta}{4\pi \in_0 h^2\, Sec^2\theta}\ x\ Cos\,\theta = -\frac{\rho_l}{4\pi \in_0 h}\,Cos\,\theta\, d\theta$$

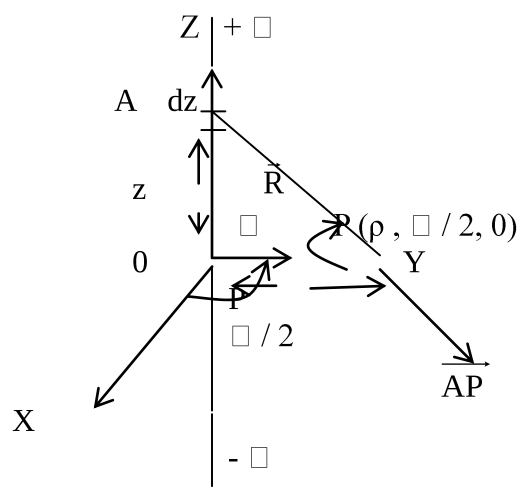
$$E_{Py} = -\frac{\rho_l}{4\pi \in_0 h}\left[+Sin\,\theta\right]_{-\pi/2}^{\pi/2} = -\frac{\rho_l}{4\pi \in_0 h}\ x\ 2 = -\frac{\rho_l}{2\pi \in_0 h}\ \hat{a}_y$$

$$dE_{Pz} = -\frac{\rho_l\, h\, Sec^2\theta\, d\theta}{4\pi \in_0 h^2\, Sec^2\theta}\ x\ \frac{\not{h}\,\tan\theta}{\not{h}\,Sec\,\theta} = -\frac{\rho_l}{4\pi \in_0 h}\,Sin\,\theta\, d\theta$$

$$\vec{E}_{Pz} = \frac{\rho_l}{4\pi \in_0 h}\left[Cos\,\theta\right]_{-\pi/2}^{\pi/2} = 0$$

$$\mathbf{E} = -\frac{\boldsymbol{\rho_l}}{\mathbf{2\,\pi \in_0 h}}\,\hat{\mathbf{a}}_y \ \ \mathbf{V/m}$$

An alternate approach uses cylindrical co-ordinate system since this yields a more general insight into the problem.



$dQ = \rho_l \overrightarrow{dz}$ is the elemental change at Z.

The field intensity $\overrightarrow{dE_P}$ due to dQ is

$$\overrightarrow{dE_P} = \frac{dQ}{4\pi \in_0 |\vec{R}|^2} \hat{a}_R \ V/m$$

where $\vec{R} = \rho \hat{a}_\rho - z \hat{a}_z$ and $\hat{a}_R = \frac{1}{R}(\rho \hat{a}_\rho - z \hat{a}_z)$

$$dQ = \rho_l \overrightarrow{dz} \ C$$

$$\overrightarrow{dE_P} = \frac{\rho_l \, dz}{4\pi \in_0 R^2}\left[\frac{\rho}{R} \hat{a}_\rho - \frac{z}{R} \hat{a}_z\right] = dE_{P\rho} \hat{a}_\rho + dE_{Pz} \hat{a}_z$$

(i) $dE_{P\rho} = \frac{\rho_l}{4\pi \in_0 R^2} \rho \, dz$; (ii) $dE_{Pz} = -\frac{\rho_l}{4\pi \in_0 R^2} z \, dz$

Taking $\theta = O\hat{P}A$ as integration variable, and expressing all distances in terms of ρ and θ

$z = \rho \tan\theta$, $dz = \rho \, Sec^2 \theta \, d\theta$ and $R = \frac{\rho}{Cos\theta} = \rho \, Sec\theta$

(i) $dE_{P\rho} = \frac{\rho_l \ x \ \rho \ x \ \rho \, Sec^2 \theta}{4\pi \in_0 \rho^3 Sec^3 \theta} d\theta = \frac{\rho_l}{4\pi \in_0 \rho} Cos\theta \, d\theta$

$$E_{P\rho} = \frac{\rho_l}{4\pi \in_0 \rho} [Sin\theta]_{-\pi/2}^{\pi/2} = \frac{\rho_l}{4\pi \in_0 \rho} \ x \ 2 = \frac{\rho_l}{2\pi \in_0 \rho}$$

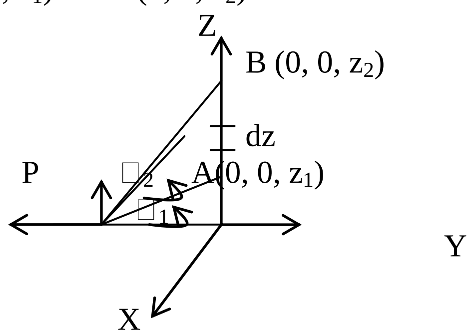
(ii) $dE_{Pz} = \frac{-\rho_l \ x \ \rho \ \tan\theta \ x \ \rho \, Sec^2 \theta}{4\pi \in_0 \rho^3 Sec^3 \theta} d\theta = \frac{\rho_l}{4\pi \in_0 \rho} (-Sin\theta) \, d\theta$

$$E_{Pz} = \frac{\rho_l}{4\pi \in_0 \rho} [Cos\theta]_{-\pi/2}^{\pi/2} = 0$$

$$\therefore \vec{E}_P = \frac{\rho_l}{2\pi \in_0 \rho} \hat{a}_\rho \ V/m$$

Thus, $\vec{E}$ is radial in direction

A2. Find the electric field intensity $\vec{E}$ at (0, -h, 0) due to a line charge of finite length along Z axis between A (0, 0, z_1) and B(0, 0, z_2)



Solution :

$$\overrightarrow{dE_P} = \frac{\rho_l\, dz}{4\pi\epsilon_0 R^2}\left[-\frac{h}{R}\,\hat{a}_y - \frac{z}{R}\,\hat{a}_z\right]$$

$$\vec{E}_P = \int_{z_1}^{z_2} d\,\vec{E}_P = -\frac{\rho_l}{4\pi\epsilon_0 h}\int_{\theta_1}^{\theta_2}\text{Cos}\,\theta\, d\theta\,\hat{a}_y - \frac{\rho_l}{4\pi\epsilon_0 h}\int_{\theta_1}^{\theta_2}\text{Sin}\,\theta\, d\theta\,\hat{a}_z$$

$$= +\frac{\rho_l}{4\pi\epsilon_0 h}(-\text{Sin}\,\theta)_{\theta_1}^{\theta_2}\,\hat{a}_y + \frac{\rho_l}{4\pi\epsilon_0 h}(\text{Cos}\,\theta)_{\theta_1}^{\theta_2}\,\hat{a}_z$$

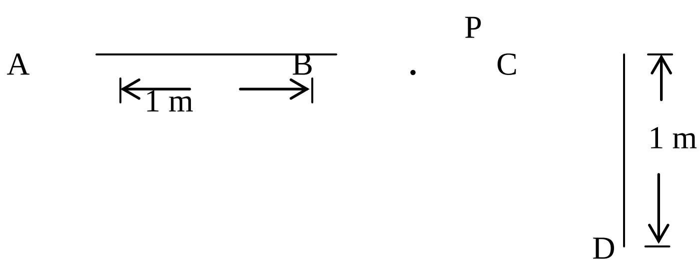
$$\vec{E}_P = +\frac{\rho_l}{4\pi\epsilon_0 h}\left[(\text{Sin}\,\theta_1 - \text{Sin}\,\theta_2)\,\hat{a}_y - (\text{Cos}\,\theta_1 - \text{Cos}\,\theta_2)\,\hat{a}_z\right] \text{V/m}$$

If the line is extending from $-\infty$ to ∞,

$$\theta_2 = \frac{\pi}{2}\ ,\ \theta_1 = -\frac{\pi}{2}$$

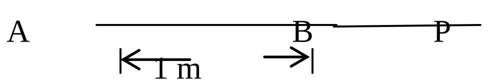
$$\mathbf{\vec{E}_P = \frac{-\rho_l}{2\pi\epsilon_0 h}\,\hat{a}_y\ V/m}$$

A3. Two wires AB and CD each 1 m length carry a total charge of 0.2 μC and are disposed as shown. Given BC = 1 m, find $\vec{E}$ at P, midpoint of BC.



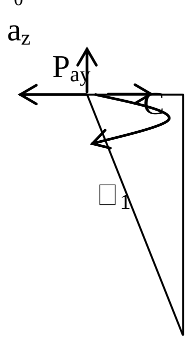
Solution :(1)

$\theta_1 = 180^0$ $\theta_2 = 180^0$



$$\vec{E}_{P_{AB}} = \frac{\rho_l}{4\pi\epsilon_0 h}\left\{\left[-(\text{Sin}\,\theta_2 - \text{Sin}\,\theta_1)\right]\hat{a}_y + \left[\text{Cos}\,\theta_2 - \text{Cos}\,\theta_1\right]\hat{a}_z\right\} = \frac{0}{0}\ \text{(Indeterminate)}$$

(2)



$$\theta_1 = -\tan^{-1}\frac{1}{0.5} = -63.43^0$$

$\theta_2 = 0$

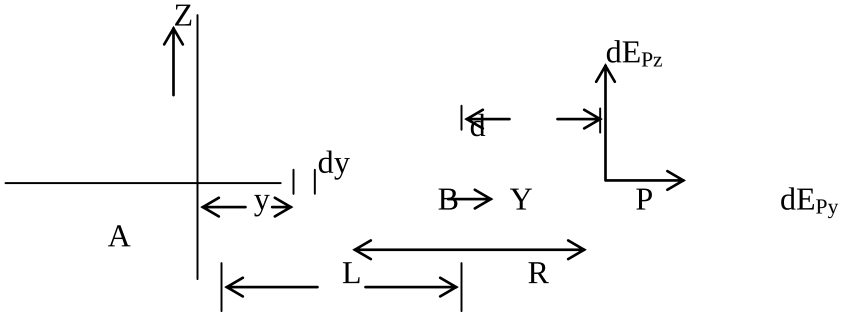
D

$$\vec{E}_{P_{CD}} = \frac{\rho_l}{4\pi\in_0 h}\left[-(\sin\theta_2 - \sin\theta_1)\,\hat{a}_y + (\cos\theta_2 - \cos\theta_1)\,\hat{a}_z\right]$$

$$= \frac{0.2 \times 10^{-6}}{4\pi\frac{10^{-9}}{36\pi}0.5}\left[-(\sin(-63.43))\,\hat{a}_y + (\cos 0 - \cos 63.43)\,\hat{a}_z\right]$$

$$\vec{E}_{P_{CD}} = 3.6 \times 10^3\left[-0.894\,\hat{a}_y + (1 - 0.447)\,\hat{a}_z\right] = (-3218\;\hat{a}_y + 1989.75\,\hat{a}_z)$$

Since $\vec{E}_{\rho_{AB}}$ is indeterminate, an alternate method is to be used as under :



$$\overrightarrow{dE_P} = \frac{\rho_l\,dy}{4\pi\in_0 R^2}\;\hat{a}_R\;\; V/m$$

$$\vec{R} = (L + d - y)\,\hat{a}_R\;;\;\;\hat{a}_R = \frac{1}{R}(-\hat{a}_y)$$

$$dE_{Py} = \frac{-\rho_l\,\hat{a}_y}{4\pi\in_0(L + d - y)^2}\;dy$$

$$\text{Let } L + d - y = -t\;\;;\; -dy = -dt\;;\;\; y = 0,\, t = \frac{1}{L + d}$$

$$y = L\,;\, t = \frac{1}{d}$$

$$dE_P = \frac{-\rho_l}{4\pi\in_0 t^2}\,dt$$

$$E_P = +\left[\frac{\rho_l}{4\pi\in_0 t}\right]_{\frac{1}{L+d}}^{\frac{1}{d}} = \frac{\rho_l}{4\pi\in_0}\left[\frac{1}{d} - \frac{1}{L + d}\right]$$

$$\therefore\;\; \mathbf{\vec{E}_P} = \frac{\boldsymbol{\rho_l}}{\mathbf{4}\pi\in_0}\left[\frac{\mathbf{1}}{\mathbf{d}} - \frac{\mathbf{1}}{\mathbf{L + d}}\right]\mathbf{V/m}$$

$$\vec{E}_{P_{AB}} = \frac{0.2 \times 10^{-6}}{4\pi \frac{10^{-9}}{36\pi}} \left[\frac{1}{0.5} - \frac{1}{1.5}\right] \hat{a}_y$$

$$\vec{E}_{P_{AB}} = 1800[\,2 - 0.67]\ \hat{a}_y = 2400\ \hat{a}_y\ \text{V/m}$$

$$\therefore\ \vec{E}_P = \vec{E}_{P_{AB}} + \vec{E}_{P_{CD}} = 2400\,\hat{a}_y - 3218\,\hat{a}_y + 1990\ \hat{a}_z$$

$$= (-820\,\hat{a}_y + 1990\,\hat{a}_z)$$

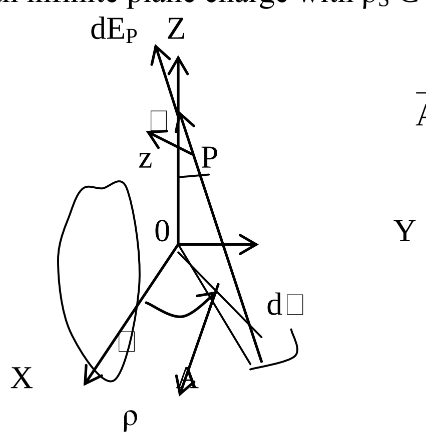
$$= 2152\,\hat{a}_P\ \ \text{V / m}$$

where $\hat{a}_P = (-0.381\,\hat{a}_y + 0.925\,\hat{a}_z)$

A4. Develop an expression for $\vec{E}$ due to a charge uniformly distributed over an infinite plane with a surface charge density of ρ_S C / m^2.

Solution :

Let the plane be perpendicular to Z axis and we shall use Cylindrical Co-ordinates. The source charge is an infinite plane charge with ρ_S C / m^2 .



$\overrightarrow{AP} = \vec{R}$

$$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\overrightarrow{OA} + \overrightarrow{OP}$$

$$\vec{R} = (-\rho\,\hat{a}_\rho + z\,\hat{a}_z)$$

$$\hat{a}_R = \frac{1}{R}(-\rho\,\hat{a}_\rho + z\,\hat{a}_z)$$

The field intensity $\overrightarrow{dE_P}$ due to dQ = ρ_S ds = ρ_S (d□ dρ) is along AP and given by

$$\overrightarrow{dE_P} = \frac{\rho_S\,\rho\,d\phi\,d\rho}{4\pi\epsilon_0 R^2}\hat{a}_R = \frac{\rho_S}{4\pi\epsilon_0 R^3}\,(-\rho\,\hat{a}_\rho + z\,\hat{a}_z)\ d\phi\,\rho\,d\rho$$

Since radial components cancel because of symmetry, only z components exist

$$\therefore\ \overrightarrow{dE}_P = \frac{\rho_S\, z}{4\pi \in_0 R^3}\, d\phi\, \rho\, d\rho$$

$$\vec{E}_P = \int_S \overrightarrow{dE}_P = \frac{\rho_S}{4\pi\in_0} \int_0^{2\pi} d\phi \int_0^{\infty} \frac{z\,\rho\, d\rho}{R^3} = \frac{\rho_S}{4\pi\in_0}\ \text{x}\ 2\pi \int_0^{\infty} \frac{z\,\rho}{R^3}\ d\rho$$

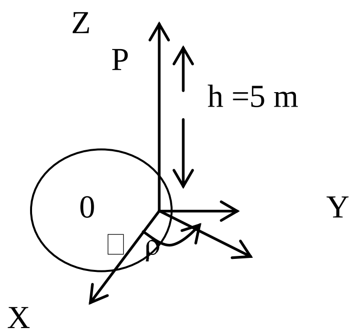
‘z’ is fixed height of ρ above plane and let $O\hat{P}A = \theta$ be integration variable. All distances are expressed in terms of z and □

ρ = z tan □ , d ρ = z Sec^2 □ d□ ; R = z Sec □ ; ρ = 0, □ = 0 ; ρ = □ , □ = □ / 2

$$\vec{E}_P = \frac{\rho_S}{2\in_0} \int_0^{\infty} \frac{z\, z \tan\theta}{z^3 \operatorname{Sec}^3\theta}\ z \operatorname{Sec}^2\theta\, d\theta = \frac{\rho_S}{2\in_0} \int_0^{\pi/2} \operatorname{Sin}\theta\, d\theta = \frac{\rho_S}{2\in_0} [-\operatorname{Cos}\theta]_0^{\pi/2}\ \hat{a}_z$$

$$= \frac{\rho_S}{2\in_0}\hat{a}_z \text{ (normal to plane)}$$

A5. Find the force on a point charge of 50 □C at P (0, 0, 5) m due to a charge of 500 □□C that is uniformly distributed over the circular disc of radius 5 m.



Solution :
Given : ρ = 5 m, h = 5 m and Q = 500 □□C
To find : f_p & q_p = 50 □C

$$\vec{f}_P = \vec{E}_P \text{ x } q_P \quad \text{where } \vec{E}_P = \frac{\rho_S}{2\in_0}\hat{a}_z$$

$$= \frac{\frac{Q}{A}}{2\in_0}\hat{a}_z = \frac{500\pi \text{ x } 10^{-6}}{2(\pi 5^2) \text{ x } \frac{10^{-9}}{36\pi}}\hat{a}_z$$

$$= \frac{500}{2 \text{ x } 25} \text{ x } 36\pi \text{ x } 10^3 \hat{a}_z$$

$$= 1131 \text{ x } 10^3 \hat{a}_z \text{ N / C}$$

$$\vec{f}_P = 1131 \text{ x } 10^3 \hat{a}_z \text{ x } 50 \text{ x } 10^{-6}$$

$$\vec{f}_P = 56.55\hat{a}_z \text{ N}$$