Design of FIR filters using window

The Design freq response $H_d(e^{j\omega})$ of filter is periodic in frequency & can be Expanded in Fourier series. The resultant series is given by

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)\,(e^{-j\omega n}) \quad \text{—} ①$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \text{—} ②$$

$h_d(n) \rightarrow$ fourier coefficient having infinite length.
One possible way of obtaining FIR filter is to truncate (shorten) the infinite Fourier series at $n = \pm\left(\frac{N-1}{2}\right)$ where N = length of desired seq.
But abrupt truncation of fourier series results in oscillatory passband & stopband. These oscill^ns are due to slow convergence of F.S. & this Effect is known as Gibbs phenomenon.
To reduce these oscill^ns, fourier coefficients of filter are modified by multiplying infinite impulse response with a finite weighting sequence $w(n)$ called window where

$$w(n) = w(-n) \neq 0 \quad \text{for } |n| \leq \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{for } |n| > \left(\frac{N-1}{2}\right)$$

After multiplying $w(n)$ with $h_d(n)$ we get finite duration sequence $h(n)$ that satisfies desired mag. response

$$h(n) = h_d(n) \cdot w(n). \quad \text{for all } |n| \leq \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{for } |n| > \left(\frac{N-1}{2}\right)$$

The frequency response $H(e^{j\omega})$ of the filter can be obtained by convol^n of $H_d(e^{j\omega})$ & $W(e^{j\omega})$ i.e given

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$$

$$= H_d(e^{j\omega}) * W(e^{j\omega})$$

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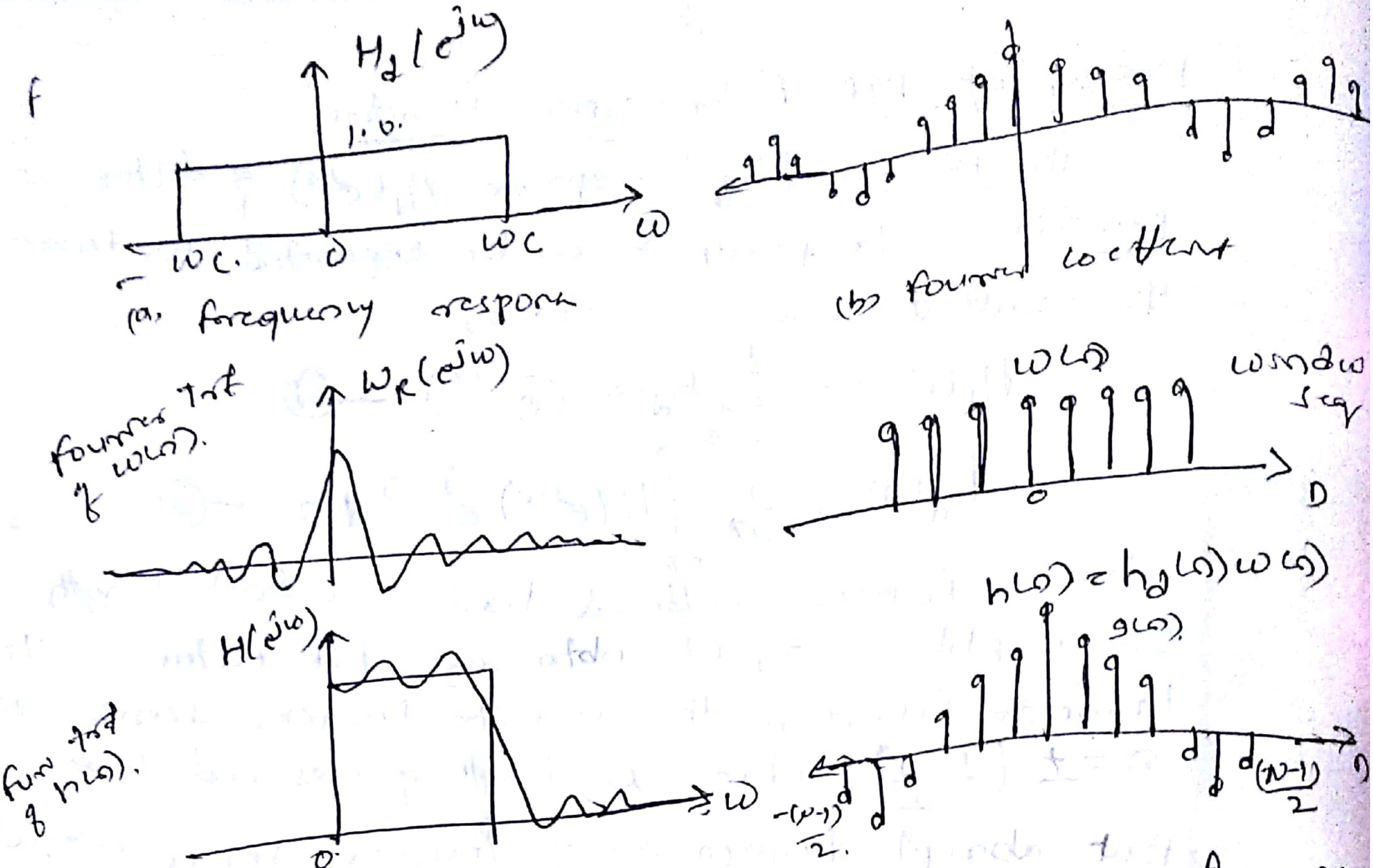


freq resp of filter $H(e^{j\omega})$ depends on freq resp of window $(W(e^{j\omega}).)$

∴ window should contain following desirable charactery of window & hold

(1) Central lobe of freq resp of window should be small contain most of Energy

(2) Highest side lobe level of freq resp should decrea n

(3) Side lobes of freq resp tends to π. Energy rapidly as ω

Rectangular window

The rectangular window seq is given by

$$w_R(n) = 1 \quad \text{for } -(N-1)/2 \le n \le (N-1)/2$$
$$= 0 \quad \text{otherwise}$$

Ex for $N = 25$

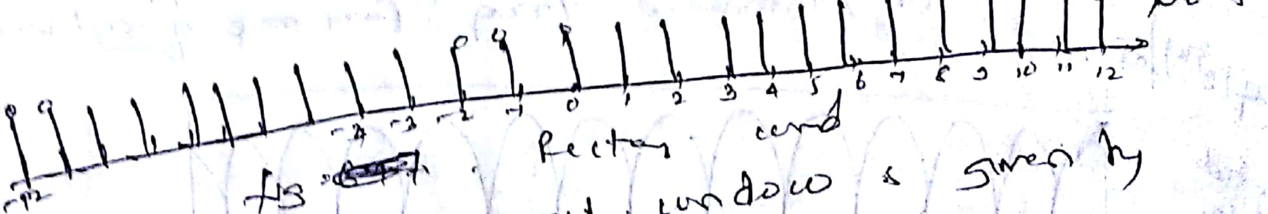


fig [redacted] Rectan. wind

The spectrum of rect window is given by

$$W_R(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\omega n}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} + \dots + e^{j\omega} + 1 + e^{-j\omega} + \dots e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[1 + e^{-j\omega} + \dots + e^{-j\omega(N-1)}\right]$$

$$\therefore 1 + a + a^2 \dots a^{N-1} = \frac{1-a^N}{1-a}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}\right]$$

$$= \frac{e^{j\omega N/2}(1 - e^{-j\omega N})}{e^{j\omega/2}(1 - e^{-j\omega})}$$

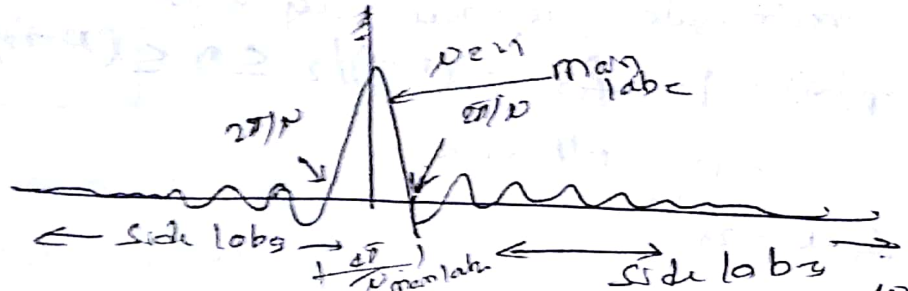
$$= \frac{e^{j\omega N/2} - e^{j\omega N/2 - j\omega N}}{e^{j\omega/2} - e^{j\omega/2 - j\omega}}$$

$$= \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$\boxed{W_R(e^{j\omega}) = \frac{\sin\frac{\omega N}{2}}{\sin\omega/2}}$$

freq Spectrum for N = 25

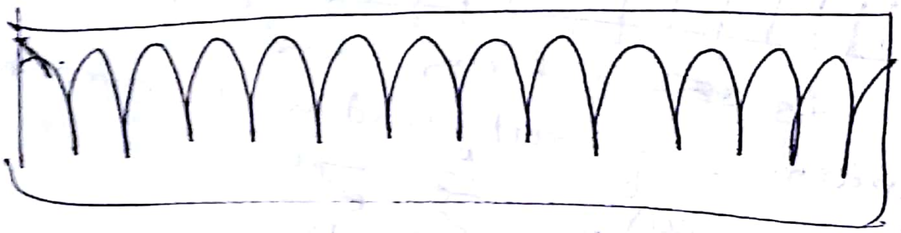
$W_R(e^{j\omega})$



(a) Normalized freq. freq resp of rect window N = ...

$|W_R(e^{j\omega})|$ dB.

Magnitude in dB.



(b) Magnitude resp of rect window for N = 25.

freq resp is real & it is zero when $\frac{N\omega}{2} = k\pi$

$\omega = \frac{2k\pi}{N}$. where k is an integer.

Main lobe is ω betw $\frac{2\pi}{N}$ to $-\frac{2\pi}{N}$ it lies betw two zero crossing.

To get finite impulse response filter we multiply $h_d(n)$ with rectan window.

$$h(n) = h_d(n)\, \omega_R(n)$$

freq response of truncated filter can be obtained by periodic convol^n

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, W_R(e^{j(\omega-\theta)})\, d\theta.$$

The desired resp of LPF changes abruptly from pass band to stopband but freq resp changes slowl...

This region of gradual change is called filter tran^n region, which is due to convol^n of desired respons with window resp main lobe.

width of tran^n region depends on width of main lobe

As N ↑ main lobe becomes narrower ↓ width of tran^n region

for rect wind ampl of side lobe is unaffected by length of window. So ↑ length N will not reduce ripple bw ...

Hanning Window

Hanning window sequence can be obtained by substituting $\alpha = 0.5$ in Equ^n as shown below

$$W_\alpha(n) = \alpha + (1-\alpha)\cos\frac{2\pi n}{N-1} \quad \text{for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2} \quad \text{—(1)}$$

$$= 0 \quad \text{other.}$$

$\alpha = 0.5$ in above Eq^n

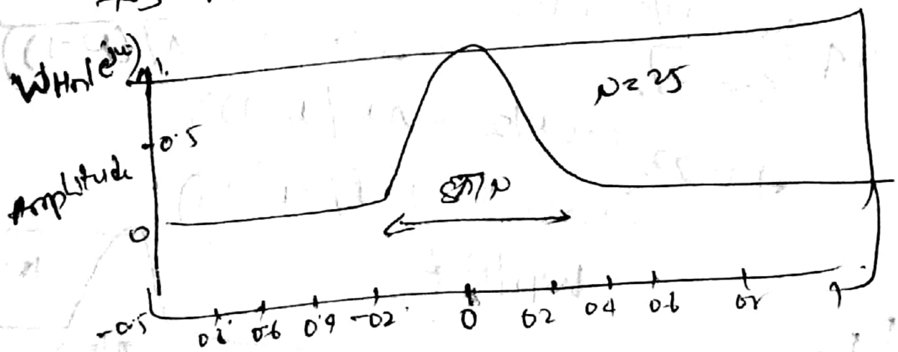
$$W_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$

$$= 0 \quad \text{otherwise}$$

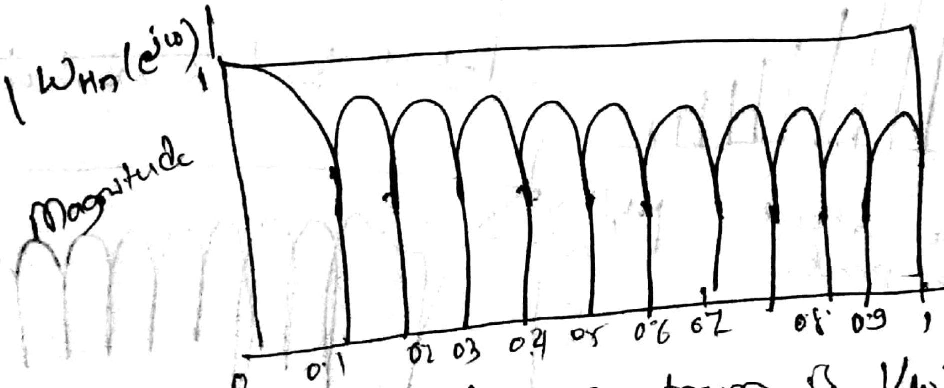
The freq. resp of Hanning window,

$$W_{Hn}(e^{j\omega}) = 0.5\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.25\frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))} + 0.25\frac{(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))}$$

Fig: Hanning window sequence.



freq. response for N=25.



Magnitude N=25

Here main lobe is twice that of rectang window side lobe of ... of rectan window. Th Hann window spectrum is 1/10th of ...

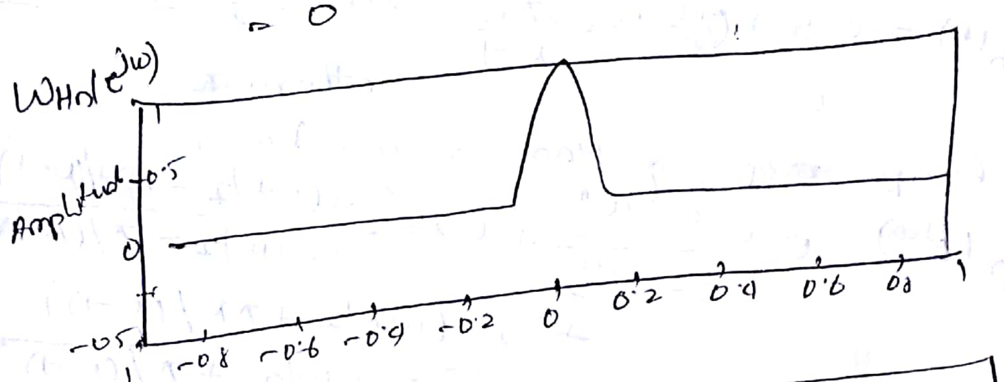
... passband & stop band of LPF design ... window

Hamming Window

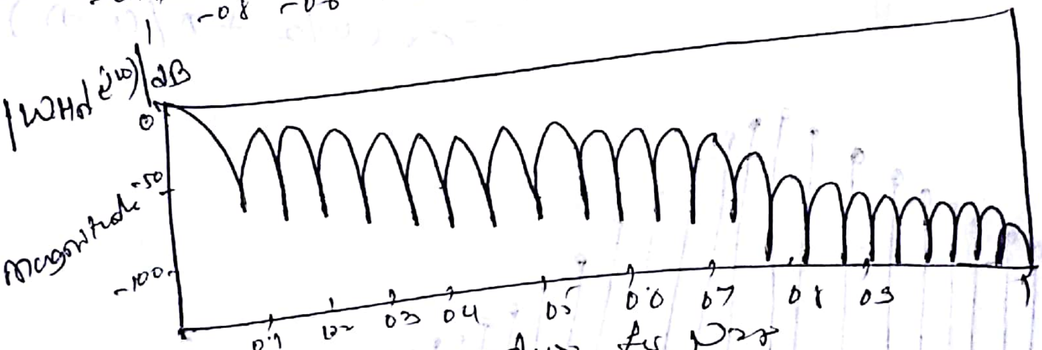
The Eqn for Hamming window can be obtained by substitute $\alpha = 0.54$ in Eq (1)

$$W_H(n) = 0.54 + 0.46\cos(2\pi n/(N-1)) \text{ for } -\frac{(N-1)}{2} \le n \le \frac{N-1}{2}$$

$$= 0 \quad \text{otherwise}$$



Fig

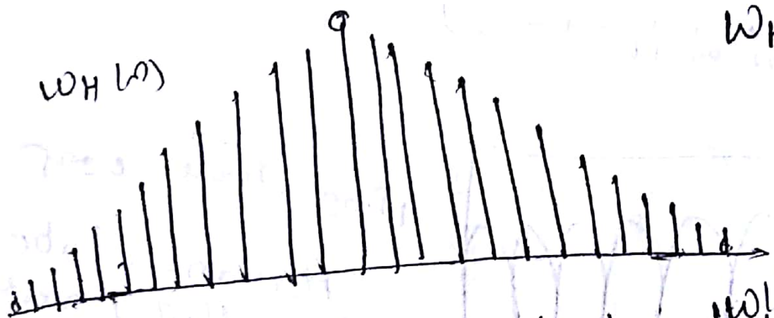


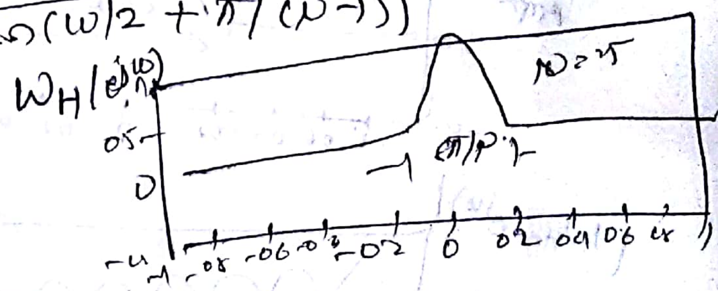
N = 51

Fig. Hamm window for N=51

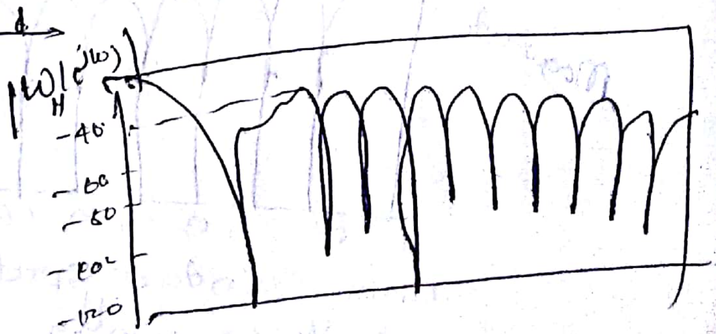
The freq response of Hamming window is

$$W_H(e^{j\omega}) = 0.54\frac{\sin \omega N/2}{\sin \omega/2} + 0.23\frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))} + 0.23\frac{(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))}$$





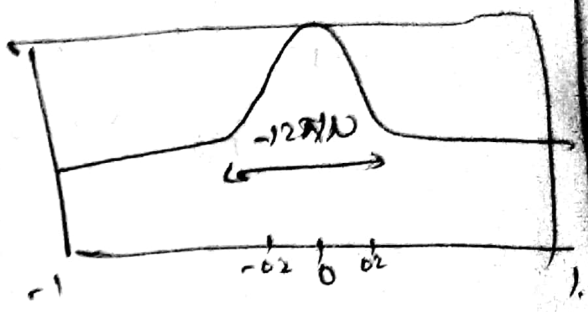
Peak side lobe is down about 41dB from main lobe peak.

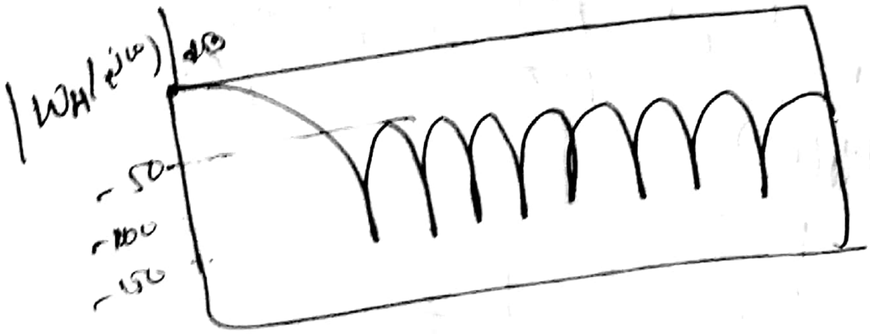


Blackman window

$$w_B(n) = 0.42 + 0.5\cos\frac{2\pi n}{N-1} + 0.08\cos\frac{4\pi n}{N-1} \quad \text{for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$

$= 0$ other





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Ex Design an ideal HP filter with a freq response

$$H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \le |\omega| \le \pi$$
$$= 0 \quad \text{for } |\omega| \le \frac{\pi}{4}$$

Find the values of $h(n)$ for $N=11$, using (a) Hanning window (b) Hamming window and shown in fig ①

The desired freq response as

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/4} e^{j\omega n}\, d\omega + \int_{\pi/4}^{\pi} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi j n}\left[e^{j\omega n}\Big|_{-\pi}^{-\pi/4} + e^{j\omega n}\Big|_{\pi/4}^{\pi}\right]$$

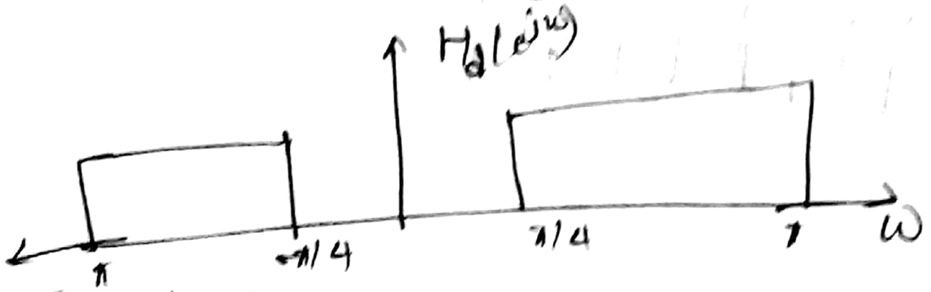

fig ①

$$= \frac{1}{\pi n (2j)}\left[e^{-j\pi n/4} - e^{-j\pi n} + e^{j\pi n} - e^{j\pi n/4}\right]$$

$$h_d(n) = \frac{1}{\pi n}\left[\sin \pi n - \sin \frac{\pi}{4} n\right] \qquad -\infty \le n \le \infty$$

Truncating $h_d(n)$ to 11 samples we have

$$h(n) = h_d(n) \quad \text{for } |n| \le 5$$
$$= 0 \quad \text{other}$$

For $n=0$

$$h(0) = \lim_{n\to 0}\frac{\sin \pi n}{\pi n} - \lim_{n\to 0}\frac{\sin \frac{\pi}{4} n}{\frac{\pi}{4} n}$$

$$= \left[\left(1 - \frac{1}{4}\right)\right]$$

$$\boxed{\because \lim_{\theta\to 0}\frac{\sin \theta}{\theta} = 1 \qquad \lim_{\theta\to 0}\frac{\sin n\theta}{\theta} = n}$$

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From the given freq response we can find that $\alpha = 0$.
∴ filter coefficients are symmetrical about $n = 0$.
satisfying the condn $h(n) = h(-n)$

for $n = 1$.

$$h(1) = h(-1) = \frac{\sin \pi - \sin \frac{\pi}{9}}{\pi}$$

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Ex 1. Design a FIR filter approximating the ideal frequency response

$$H_d(e^{j\omega}) = e^{-j\alpha\omega} \quad \text{for } |\omega| \le \pi/6$$
$$= 0 \quad \text{for } \frac{\pi}{6} \le |\omega| \le \pi$$

Determine the filter coefficients for $N = 13$

Soln:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} e^{-j\alpha\omega} \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} e^{-j(\alpha-n)\omega}\, d\omega$$

$$= \frac{1}{2\pi(-(\alpha-n)j)} \left. e^{-j(\alpha-n)\omega} \right|_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{-2\pi(\alpha-n)j} \left. e^{-j(\alpha-n)\omega} \right|_{-\pi/6}^{\pi/6}$$

$$= \frac{e^{-j(\alpha-n)\pi/6} - e^{-j(\alpha-n)(-\pi/6)}}{2\pi j(n-\alpha)}$$

$$= \frac{e^{j(n-\alpha)\pi/6} - e^{-j(n-\alpha)\pi/6}}{2\pi j(n-\alpha)}$$

$$h_d(n) = \frac{\sin\frac{\pi}{6}(n-\alpha)}{\pi(n-\alpha)}$$

From the freq resp. we can find that filter coefficients are symmetrical about $\alpha = \frac{(N-1)}{2} = 6$ satisfying $h(n) = h(N-1-n)$.

Truncating $h_d(n)$ to $N = 13$ samples we have

$$h(n) = h_d(n) \quad \text{for } 0 \le n \le 12$$
$$= 0 \quad \text{otherwise}$$

$h(n) = \frac{\sin\frac{\pi}{6}(n-\alpha)}{\pi(n-\alpha)} = \frac{\sin\frac{\pi}{6}(n-6)}{\pi(n-6)}$

$h(6) = \lim_{n\to 6} \frac{\sin\frac{\pi}{6}(n-6)}{\pi(n-6)} = 1/6 = 0.167$

$h(0) = h(13-1-0) = h(12) = 0$

$h(1) = h(11) = 0.0318$

$h(2) = h(10) = \ldots 685$

$h(3) = h(9) = 0.106$, $h(4) = h(8) = 0.1\ldots$

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Ex **Samp pbs using Hamming window**

The noncausal window seq for $N=13$ can be obtained from the eqn

$$\omega_H(n) = 0.54 + 0.46 \cos \frac{2\pi n}{N-1} \quad \text{for } |n| \le \frac{N-1}{2}$$
$$= 0 \quad \text{other}$$

$$\Rightarrow \omega_H(n) = 0.54 + 0.46 \cos \frac{n\pi}{6} \quad \text{for } |n| \le 6$$
$$= 0 \quad \text{other}$$

$\omega_H(0) = 0.54 + 0.46 = 1$

$\omega_H(1) = \omega_H(-1) = 0.54 + 0.46 \cos \frac{\pi}{6} = 0.94$

$\omega_H(2) = \omega_H(-2) = 0.54 + 0.46 \cos \frac{2\pi}{6} = 0.77$

$\omega_H(3) = \omega_H(-3) = 0.54 + 0.46 \cos \frac{3\pi}{6} = 0.54$

$\omega_H(4) = \omega_H(-4) = 0.31$

$\omega_H(5) = 0.142 = \omega_H(-5)$

$\omega_H(6) = \omega_H(-6) = 0.08$

The causal window seq obtained by shifting above seqn right by 6 samples

$\omega_{H6} = 1$

$\omega_{H5} = \omega_H(7) = 0.94$, $\omega_H(2) = \omega_H(10) = 0.31$

$\omega_H(3) = \omega_H(9) = 0.54$, $\omega_H(1) = \omega_H(11) = 0.142$

$\omega_H(4) = \omega_H(8) = 0.77$, $\omega_H(0) = \omega_H(12) = 0.08$

The filter coeffs is for $0 \le n \le N-1$

$h(n) = h_d(n)\,\omega_H(n)$

$h(0) = h_d(0)\,\omega_H(0) = 0 \times 0.08 = 0$

0 1 2 3 4 5 6

$h(1) = h(11) = h_d(1)\,\omega_H(1) = 0.0318 \times 0.142 = 0.0045$

$h(2) = h(10) = h_d(2)\,\omega_H(2) = 0.0689 \times 0.31 = 0.02136$

$h(3) = h(9) = h_d(3)\,\omega_H(3) = 0.106 \times 0.54 = 0.0572$

$h(4) = h(8) = h_d(4)\,\omega_H(4) = 0.1378 \times 0.77 = 0.1[illegible]$

$h(5) = h(7) = h_d(5)\,\omega_H(5) = 0.159 \times 0.94 = 0.14[illegible]$

$h(6) = h_d(6)\,\omega_H(6) = 0.167 \times 1 = 0.167$

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