Infinite Impulse Response filters

Digital filter is a linear time-invariant discrete time system mainly used for filtering of sequences (arrays). The sequences are obtained by sampling the I/P analog signals. The digital filters performs the frequency related operations such as lowpass, high pass, band pass & allpass etc.

The design specifications include cut off frequency, sampling frequency of I/P signal, pass band variation, approximation, type of filter etc.

Digital filters are realized through H/W and S/W. Usually S/W digital filters require digital H/W for their operation.

Digital filters of two types

(1) finite impulse response (FIR) filters

(2) Infinite impulse response (IIR) filters

These two types of filters are best described by difference equations. IIR filters are recursive i.e they use feedback. Hence O/P of IIR filter depend upon present I/P as well as past I/P & O/P's. But FIR systems are nonrecursive. ∴ of FIR filter depends only upon present & past I/Ps. i.e it does not uses feedback.

FIR ~~filter~~ system has finite duration unit sample response i.e

$$h(n) = 0, \quad \text{for } n<0 \text{ \& } n \geq M.$$

The unit sample response exists only for the duration from 0 to M-1. Hence it is a FIR system, where IIR system has infinite duration unit sample response

$$h(n) \neq 0 \text{ for } n < 0$$

The unit sample response exists for duration 0 to ∞. Hence it is ...

04 As we know digital filters are described by difference equation of LTI system is given:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad \text{--- (1)}$$

In above Eq^n Ist term represents past & second summation indicates present & (All) i/p's. Hence above Eq (1) represents IIR system.

when first summation term^n absent in Eq (1)

then $$y(n) = \sum_{k=0}^{M} b_k x(n-k). \quad \text{--- (2)}$$

then Eq (2) represents FIR filter.

Then o/p of LTI system can be given by convolution property.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k) \quad \text{--- (3)}$$

In FIR filter unit samples response $h(k)$ exists for finite duration 0 to M.

∴ o/p FIR filter is

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k) \quad \text{--- (4)}$$

from Eq (1) & Eq (4) we can conclude that unit sample response of FIR filter is given

$$h(k) = b_k$$

The unit sample response can be directly obtained in terms of coefficients of difference equation.

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k).$$

Note: cutoff frequency is defined as freq where gain changed by some specified amount relative to the mean midband gain.

Frequency Selective Filters

Filter is the one which rejects unwanted frequencies from i/p signal & allows the desired frequencies to obtain the reqd shape of o/p signal.

The range of frequencies of signal that are passed through filter is called pass band & those frequencies that are blocked is called stopband.

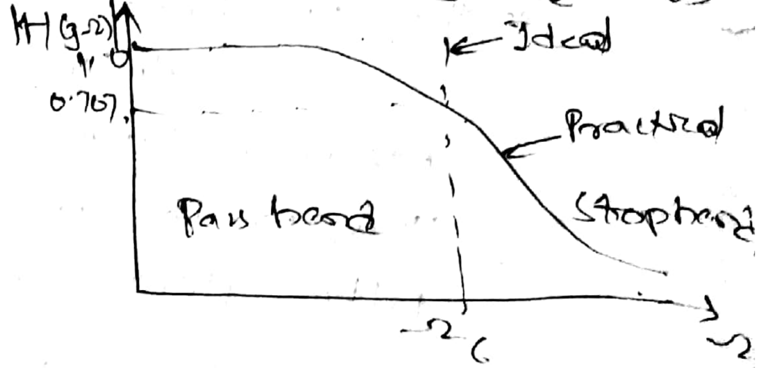
According to frequency point of view, filters are of different types.

(1) Low pass (2) High pass (3) Band pass (4) Band reject.

(1) Low pass filter

This filter allows low frequency in the passband $0 < \Omega < \Omega_c$ to pass & blocks the higher frequency in stop band $\Omega > \Omega_c$. The frequency Ω_c betn the two bands is cut off frequency, where the magnitude is given by

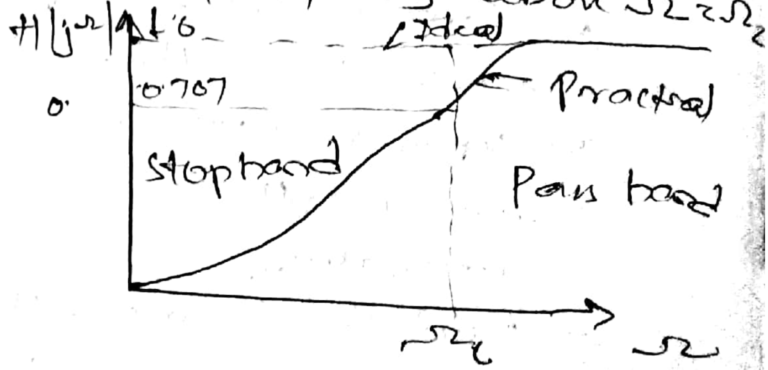
$|H(j\Omega)| = 1/\sqrt{2}$



(2) High pass filter

These filters allow high frequency above $\Omega > \Omega_c$ & rejects the frequencies between $\Omega = 0$ & $\Omega = \Omega_c$

Magnitude response of ... filters



(3) Band pass filter

It allows only a band of frequencies from Ω_1 to Ω_2 to pass & stops all other frequencies

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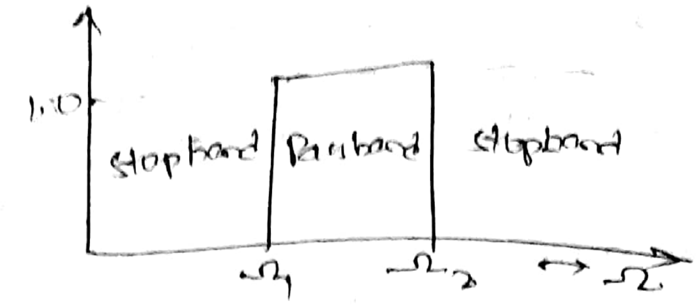
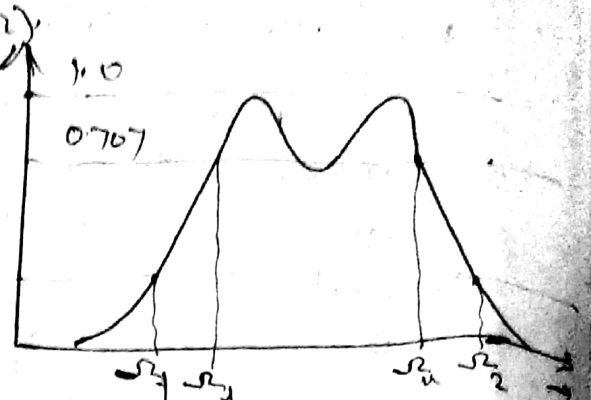
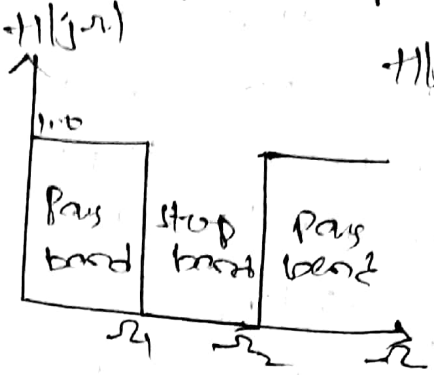
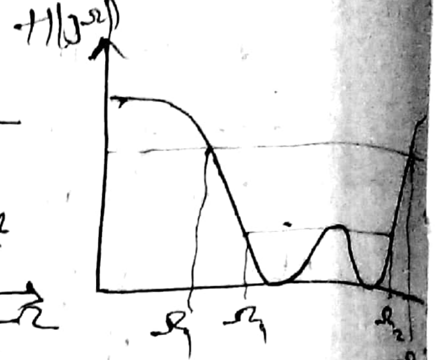




the magnitude response of Band pass filter (in ideal) (b) ...

(4) Band reject filter
It rejects all frequency between Ω_1 & Ω_2 & allows remaining frequencies





FIR filters have the following Adv over IIR

(1) They can have exact linear phase

always stable.

Design methods are generally linear

They can be realised efficiently in H/W..

Filter start up transients have finite duration

Some objectives of the filters are

(1) Eliminate noise present in the signal

(2) Remove signal distortion due to imperfect transmission channel

(3) Separate two or more distinct signals which were purposely mixed for maximising the channel utilization.

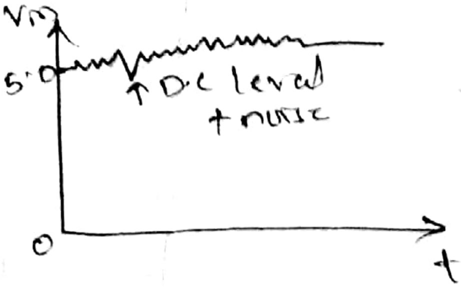
(4) Resolves signals into their frequency components

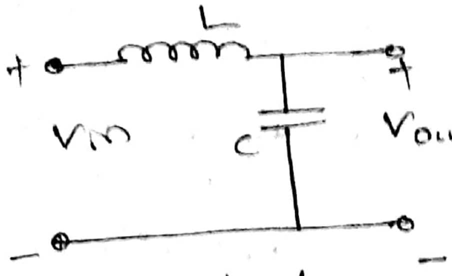
(5) ~~Re~~ Demodulates the signals which were modulated at the transmission end.

(6) Limit the bandwidth of signal

Example of Analog low pass filter $X_L > X_C$ ∴ large V

Consider LC low pass filter, used to remove ripple & noise in the input signal.





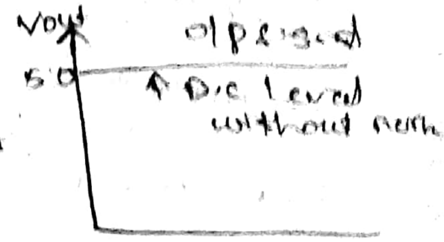


fig: Analog low pass filter

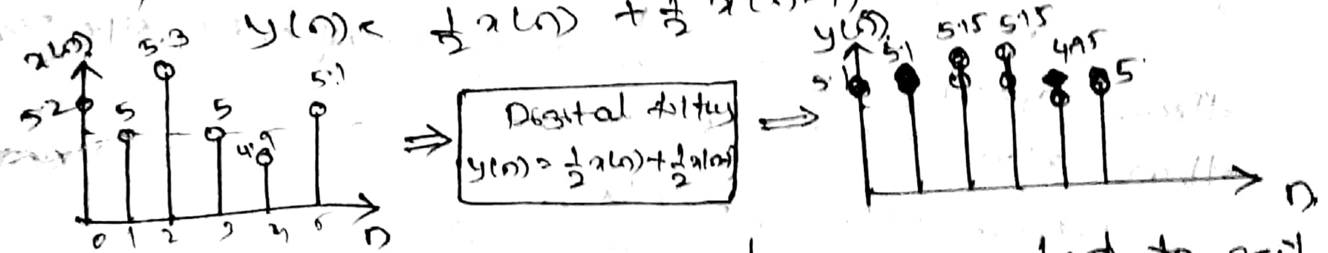
As shown in the above figure I/P is 5V noisy DC voltage. & this is removed by LC filter & o/p signal is pure DC level of 5V. ∵ noise is high frequency signal hence low pass filter removes this noise & passes only DC level.

Digital low pass filter

Only change in the digital filter is donot uses capacitor's & inductor's. Digital filter is repre^d by difference Eqn. i.e

1.2.3.6.6.9.11.13 18.19.21.22.23 ... 24 27.28 41 30 31.32 34 36 37 38 35 40 41 43 44

$y(n) = \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$



Noisy analog 5V signal is sampled to get x(n) hence samples of x(n) vary above & below 5V level. when the noisy signal given to digital low pass filter repre^d by difference Eqn y(n). & then it computes various values of y(n). according to Eqn

$y(n) = \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$, for different values of n are o/p samples y(n) for different Assume $x(-1) = 5V$

$y(n) = \frac{1}{2}x(n) + \frac{1}{2}x(n-1) = \frac{1}{2}(5.2) + \frac{1}{2}\times 5) = 5.1$

$y(0) = \frac{1}{2}x(0) + \frac{1}{2}x(-1) = 5.1$

$y(1) = \frac{1}{2}x(1) + \frac{1}{2}x(0) = \frac{1}{2}\times 5 + \frac{1}{2}5.2 = 5.1$

$y(2) = \frac{1}{2}5.3 + \frac{1}{2}\times 5 = 5.15$

$y(3) = \frac{1}{2}\times 5 + \frac{1}{2}\times 5.3 = 5.15$

$y(4) = \frac{1}{2}\times 4.9 + \frac{1}{2}\times 5 = 4.95$

$y(5) = \frac{1}{2}\times 5.1 + \frac{1}{2}\times 4.9 = 5$

Noise level in samples is reduced compared to x(n)

Comparison between Analog filters & Digital filters

Parameters	Analog filters	Digital filters
(1) I/P / O/P signals	Analog	Digital (discrete time)
2) Composition	Elements like R, L, C or analog IC's	S/W + digital H/W
(3) Filter repr^n	In terms of system components	By difference eq
(4) Flexibility	Not flexible.	Highly flexible
5) Portability	Not Easily portable	Portable
(6) Interference noise & other Effects	Maximum effect	Minimum / negligible effect
(7) storage / maintenance failure	Difficult storage & maintenance & high failure rates.	Easy storage & maintenance & reduced failure rate.

Consider simple Example of IIR filters

· IIR filters are described by difference Eq^n

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Here $a_1 = +1$, $b_0 = 1$ & rest of coefficients are zero

then above Eq^n becomes

$$y(n) = y(n-1) + x(n)$$

Take z-trf of above Eq^n

$$Y(z) = z^{-1}Y(z) + X(z)$$

$$Y(z)[1 - z^{-1}] = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1-z^{-1}} = H(z)$$

Inverse z-trf of above Eq^n gives unit sample response $h(n)$.

$h(n) = u(n)$ i.e. unit step fun^n.

$$u(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases} \quad \text{to } n = \infty$$

step fun^n has ∞ duration. $\therefore$ unit sample has ∞ duration & hence it is IIR filters.

Design of Digital filters from Analog filters

Common method of designing IIR filters is indirect method which involves designing an analog prototype filter & then transmitting prototype to a digital filter. For given specⁿs of digital filter the derivation of digital filter transfer funⁿ requires 3 steps

(1) Map the desired digital filter specⁿ into those for an equivalent analog filter

(2) convert specⁿ of digital IIR filter to eqⁿ specⁿ of analog IIR filter

(2) Derive the analog transfer funⁿ for analog prototype

(3) Transform the transfer funⁿ of the analog prototype into an equivalent digital filter transfer funⁿ.

Various parameters associated with magnitude response of a digital low pass filter.

of a digital low pass filter in radian

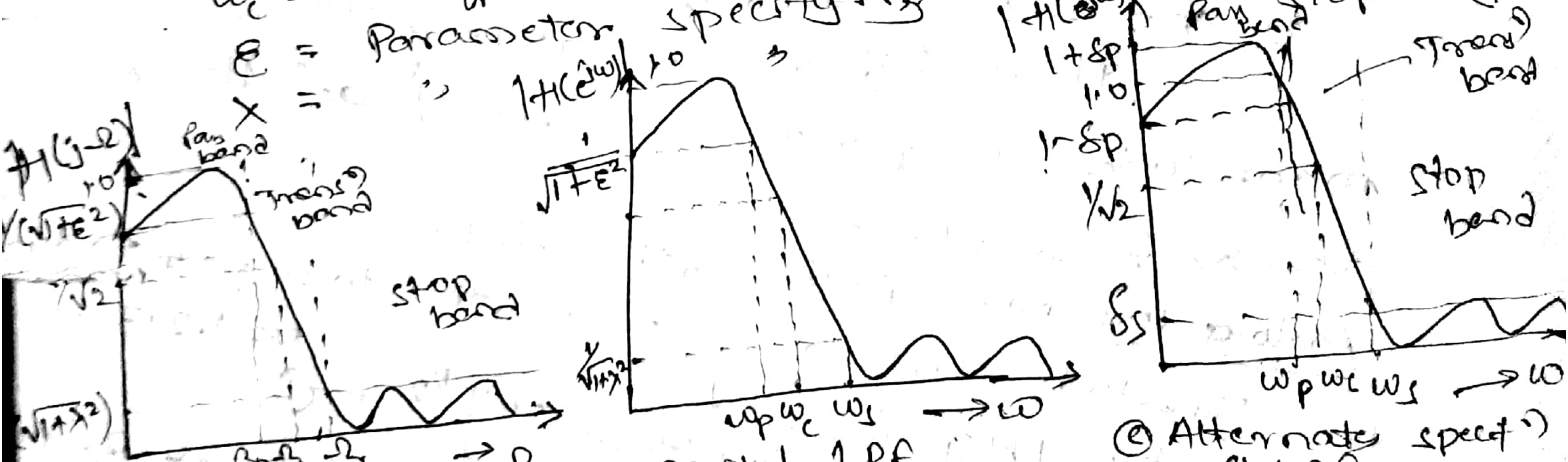
ω_s = stop band frequency in radian

ω_p = Pass band frequency in radian

ω_c = Cutoff frequency in radian

ε = Parameter specifying allowable pass band

λ = " " stopband



(a) Analog low pass filter (b) Digital LPF (c) Alternate specⁿ of LPF

Epsilon (ε), omega, Delta, lambda

Digital frequencies ω_p, ω_c & ω_s are replaced by the analog frequencies Ω_p, Ω_s & Ω_c hence digital filter can be modified as analog filter

δ_p (pass band error tolerance) & δ_s (maximum allowable magnitude in stopband) parameters also used for specifying the magnitude of digital LPF frequency response

Relation between λ & ε is $\varepsilon = \frac{2\sqrt{\delta_p}}{1-\delta_p}$ & $\lambda = \sqrt{(1+\delta_p)^2 - \delta_s^2}/\delta_s$.

Using Analog filter specⁿs the transfer funⁿ of analog LPF is designed & it is transformed to digital filter

Analog low pass filter design

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General form of analog filter transfer fun is

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{n} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i}$$

$H(s)$ is laplace transform of impulse response $h(t)$ or

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

for stable analog filter poles of $H(s)$ lies in the left half of s-plane.

1.2.5 14,17,18. 34,38.40.42.

Mainly there are two types of analog filter design

(1) Butterworth filter

(2) chebyshev filter,

They differs in nature of there magnitude responses as well as there design & implementation.

Analog low pass Butterworth filter

Magnitude fun. of Butterworth low pass filter is given by

$$|H(j\Omega)| = \frac{1}{\left[1 + (\Omega/\Omega_c)^{2N}\right]^{1/2}} \quad N = 1, 2, 3 \ldots \quad \text{—(1)}$$

where N shows the order of filter. & Ω_c is cutoff frequency. Magnitude response graph with various values of N & Ω are shown below.

Maximum response is at $\Omega = 0$. & function decreases monotonically when Ω increases.

As N increases magnitude response approach ideal characteristic's of low pass filter. when $\Omega < \Omega_c$, $H|j\Omega| \cong 1$. but $\Omega > \Omega_c$ the value $|H(j\Omega)|$ decreases rapidly

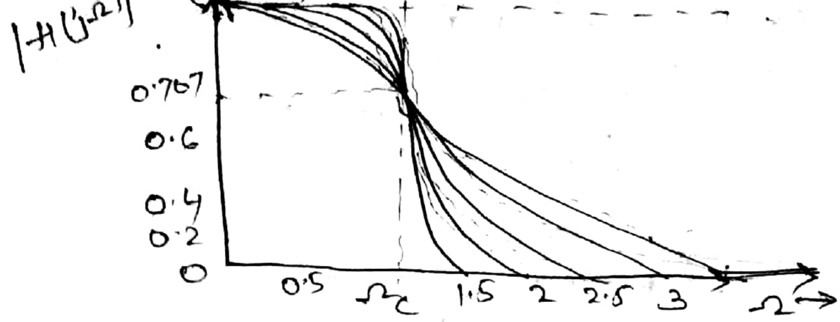

fig: lowpass Butterworth magnitude response.

At $\Omega = \Omega_c$ all curves passes through 0.707 point.

from Eq (1) Magnitude square of normalized butterworth filter is $\Omega_c = 1$ rad/sec.

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2N}}, \quad N = 1, 2, 3 \dots \quad \text{--- (2)}$$

To derive transfer fun^n of a stable filter substitute $\Omega = \frac{s}{j}$ in Eq (2) we get

$$|H(j\Omega)^2| = H(-\Omega^2) = H(s/j)^2 = H\left(\frac{s^2}{j^2}\right) = H\left(\frac{s^2}{-1}\right)$$

$$= H(-s^2)$$

$$= H(s) \cdot H(-s)$$

$$= H(j\Omega) \cdot H(-j\Omega)$$

$$H(s) \cdot H(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}} = \frac{1}{1+\frac{s^{2N}}{j^{2N}}} = \frac{1}{1+(-1)^N s^{2N}}$$

$$H(s) \cdot H(-s) = \frac{1}{1+(-s^2)^N} \quad \text{--- (3)}$$

from the above Eq^n. because of two factors $H(s)$ & $H(-s)$ poles exists both in LHP & RHP side of s-plane. $H(s)$ has roots in LHP & $H(-s)$ has corresponding roots in RHP.

W.K.T by equating denominator to zero we can obtain roots (poles). & roots depends on N value

i.e $1+(-s^2)^N = 0$ --- (4)

if N is odd $s^{2N} = 1 = e^{j2\pi k}$.

Now the roots of Eq --- (4) can be found as

$$S_k = e^{j\pi k/N} \quad k = 1, 2 \dots 2N. \quad \text{--- (5)}$$

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when N is even. $j(2k-1)\pi$

then $s^{2N} = -1 = e^{j(2k-1)\pi}$

$= e^{j2\pi k} \cdot e^{-j\pi} = e^{j2\pi k}(\cos\pi - j\sin\pi)$

$= -e^{j2\pi k}$

then roots are given by

$S_k = e^{j(2k-1)\pi/2N}$ for $k = 1, 2 \ldots 2N$ — (6)

$N=1$ $k=1,2$
$S_1 = -1$
$S_2 = 1$

$s^{2N} = 1$ $\quad s^6 = 1$ $\quad k = 1, \ldots 2N$.

For $N=3$

$S_k = e^{j\pi k/N}$ $\quad k = 1, 2, 3, 4, 5, 6$.

$S_1 = e^{j\pi/3} = \cos(\pi/3) + j\sin\pi/3 = 0.5 + j0.866$

$S_2 = e^{j2\pi/3} = \cos(2\pi/3) + j\sin 2\pi/3 = -0.5 + 0.866j$

$S_3 = e^{j3\pi/3} = e^{j\pi} = \cos\pi + j\sin\pi = -1$.

$S_4 = e^{j4\pi/3} = \cos(4\pi/3) + j\sin(4\pi/3) = -0.5 - j0.866j$

$S_5 = e^{j5\pi/3} = \cos(300) + j\sin(300) = 0.5 - j0.866j$

$S_6 = e^{j2\pi} = \cos(360) + j\sin(360) = 1$

Representation of all poles in s-plane is shown

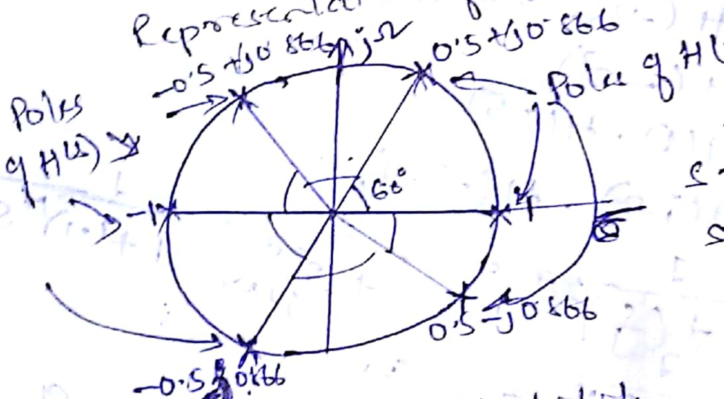


Fig: pole locations in the s-plane for magnitude square fun of Butterworth filters,

To Ensure stability conditions, only poles that lies in left half of s-plane then transfer of denominator $H(s)$ is

$(s+1)(s+0.5-j0.866)(s+0.5+j0.866)$

$(s+1)(s^2+0.5s+j0.866s+0.5s+0.5^2+j0.5\times0.866-j0.866s$
$-j0.5\times0.866+0.866^2)$

$(s+1)(s^2+s+\underbrace{0.5^2+0.866^2}_{0.9996})$

$(s+1)(s^2+s+1)$

Thus transfer fun of 3rd order Butterworth filter at $\Omega_c = 1$ rad/sec is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)} \quad (7)$$

Roots can be calculated by using the formula

$s_k = e^{j\phi_k}$ where —— (8)

$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$, $k = 1, 2, 3 \ldots N$. —— (9) 13

for $N=4$ from Eqn – (9) poles can be calculated a

$s_1 = e^{j\phi_1} = s_1$

$\phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{2 \times 4} = \frac{\pi}{2} + \frac{\pi}{8} = \frac{4\pi + \pi}{8} = 5\pi/8$

$s_1 = e^{j5\pi/8} = \cos 112.5 + j\sin 112.5 = -0.3827 + j0.9239$

$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{8} = 157.5$

$s_2 = e^{j157.5} = \cos 157.5 + j\sin 157.5 = -0.9239 + j0.3826$

$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{8} = 202.5$

$s_3 = e^{j202.5} = \cos 202.5 + j\sin 202.5 = -0.9239 - j0.3827$.

$\phi_4 = \frac{\pi}{2} + \frac{7\pi}{8}$ & $s_4 = \cos 247.5 + j\sin 247.5 = -0.3827 - j0.9239$

transfer funⁿ $H(s)$ is

$(s+0.3827-j0.9239)(s+0.9239-j0.3826)(s+0.9239+j0.3827)$

$(s+0.3827+j0.9239)$

$\{(s+0.3827)^2 + 0.9239^2)\}\{(s+0.9239)^2 + 0.3827)^2\}$

$(a+b)^2 = a^2+b^2+2ab$

$a^2+b^2 = (a+b)^2 - 2ab$

$\{ s^2 + 0.1464 + 0.7654s) + 0.8536\}\{s^2 + 0.8536 + 1.8478s + 0.1464\}$

$\{(s^2 + 0.7654s + 1)\}\{s^2 + 1.8478s + 1\}$ transfer funⁿ for fourth order Butterworth filter

$\Omega_c = 1$ rad/sec. is given by —— (10)

$H(s) = \frac{1}{(s^2+0.7654s+1)(s^2+1.84776s+1)}$

Eq – (9) is called Eqⁿ for normalized poles.

The unnormalized poles are given by

$s_k' = \Omega_c s_k$. —— (11)

Transfer funⁿ of such type of butterworth filter can be obtained by substituting $s \to s/\Omega_c$ in table

g
oloe√2
3dB

14 WKT filter was restricted to −3dB attenuation at Ω_c. Max pass band attenuation in +dB = α_p (< ... at pass band freq Ω_p & α_s = stopband attenuation in +dB at stop band freq Ω_s.

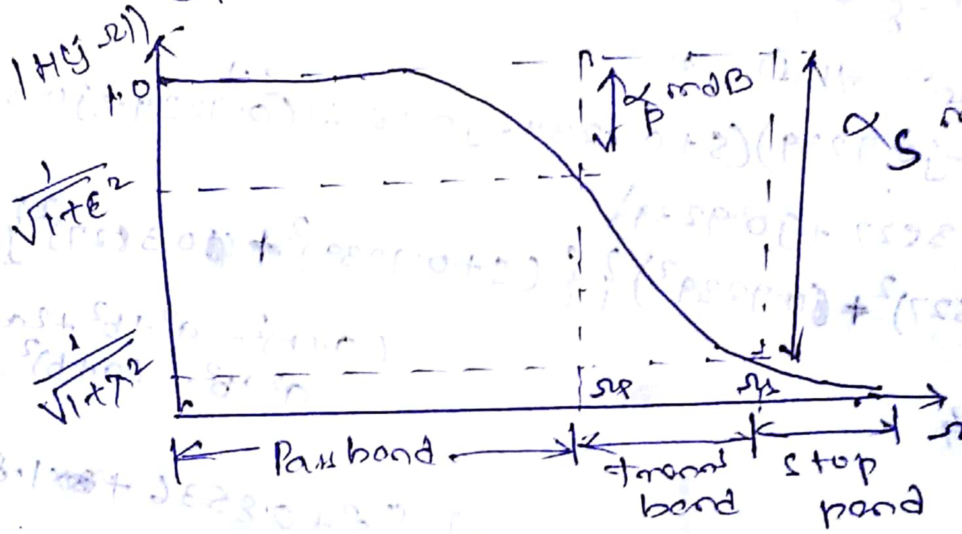
Then magnitude function can be written as

$$|H(j\Omega)| = \frac{1}{\left[1+\varepsilon^2(\Omega/\Omega_p)^{2N}\right]^{1/2}} \quad — (12)$$

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}} \quad — (13)$$

Taking logarithm on both sides

$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right] \quad — (14)$$



For Butterworth approximation of magnitude response at $\Omega = \Omega_p$ attenuation is equal to α_p

from eq (14)

$$20\log|H(j\Omega_p)| = -10\log(1+\varepsilon^2) = -\alpha_p \quad \text{at } \Omega_c$$

$$\alpha_p = 10\log(1+\varepsilon^2)$$

$$0.1\alpha_p = \log(1+\varepsilon^2)$$

Taking antilog on both sides we get

$$1+\varepsilon^2 = 10^{0.1\alpha_p}$$

$$\varepsilon^2 = 10^{0.1\alpha_p} - 1$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} \quad — (15)$$

At $\Omega = \Omega_s$ minimum stopband attenuation is Ω_s Eq – (14) becomes

$$20\log|H(j\Omega_s)| = 10\log 1 - 10\log\left[1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right] = -\alpha_s \quad 15$$

$$-\alpha_s = -10\log\left[1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

$$0.1\alpha_s = \log\left[1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

Taking antilog on both sides we get

$$\left[1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right] = 10^{0.1\alpha_s}$$

$$\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s}-1. \quad \text{——— (16)}$$

Substituting Eq – (15) in Eq – (16) we get

$$\left(10^{0.1\alpha_p}-1\right)\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s}-1$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}. \quad \text{——— (17)}$$

1.2.3. 4.10.12.13. 15.18 31.3 4.35.39 40 42 43 44.

To find out Expression for the order of filter then take Logarithm on Eq – (17) we get

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{N} = \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}.$$

$$N\log\left(\frac{\Omega_s}{\Omega_p}\right) = \log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}. \quad \text{——— (18)}$$

$$N = \frac{\log\sqrt{\left(10^{0.1\alpha_s}-1\right)/\left(10^{0.1\alpha_p}-1\right)}}{\log(\Omega_s/\Omega_p)}$$

Result of this Expn is not an integer value N is round off to the next higher hence always integer.

$$N \geq \frac{\log\sqrt{10^{0.1\alpha_s}-1/10^{0.1\alpha_p}-1}}{\log(\Omega_s/\Omega_p)}. \quad \text{——— (19)}$$

or

$$\geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)} \quad \text{——— (20)}$$

where $\varepsilon = \left(10^{0.1\alpha_p}-1\right)^{1/2}$

$\lambda = \left(10^{0.1\alpha_s}-1\right)^{1/2}$

As k notations are used in the above eqn

$$A = \frac{\lambda}{\varepsilon} \left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right)^{1/2} \; \& \; k = \Omega_p/\Omega_s = \text{transition ratio}$$

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Final order eqn for low pass butterworth analog filters given by

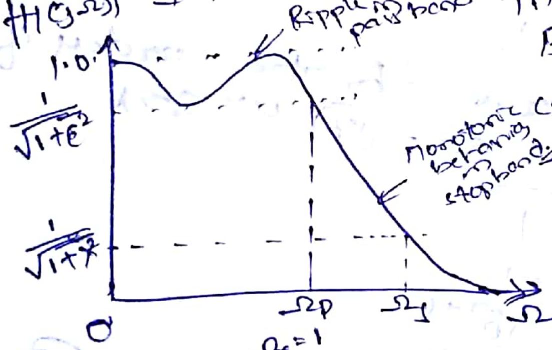
$$\boxed{N \geq \frac{\log A}{\log(1/k)}} \quad \text{— (21)}$$

Analog low pass chebyshev filters

In this filter there is ripple in the passband or stopband. Based on the characteristic there are two types of chebyshev filters.

(1) Type-I chebyshev filters: a ripple in the passband. In this type there is a ripple in the passband. These filters are all pole filters. Behaviour in the passband is also called as equiripple characteristic. These type of filters have monotonic characteristic in the stopband. The poles of Type-I chebyshev filter lie on the ellipse in s-plane.

2,3,4,5,8
10/12/18
25,29,36,38



(31)

The magnitude square response of Nth order type-I filter can be expressed as

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)} \qquad N = 1, 2 \ldots \quad \text{— (1)}$$

where $C_N(x)$ is Nth order chebyshev polynomial

$C_N(x) = \cos(N \cos^{-1} x)$ — (2), $|x| \leq 1$, passband

$C_N(x) = \cosh(N \cosh^{-1} x)$ — (3), $|x| > 1$ stopband

The higher order chebyshev polynomials are obtained by following recursive formula.

$C_N(x) = 2x\,C_{N-1}(x) - C_{N-2}(x), \quad N > 1$ — (4)

where $C_0(x) = 1 \quad x = 0 \qquad \cos(0) = 1 = \cos(0 \times \cos^{-1} 1)$

$C_1(x) = x, \quad x = 1 \qquad \cos(1) = \cos(1 \times \cos^{-1} 1)$

Properties of chebyshev polynomials

(1) $C_N(x) = -C_N(-x)$ for N is odd

$C_N(x) = C_N(-x)$ for N is even

$C_N(0) = (-1)^{N/2}$ for N is even

$C_N(1) = 1$ for all N

$C_N(-1) = 1$ for N even

$C_N(-1) = -1$ for N odd

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(2) $C_N(\Omega)$ oscillates with equal ripple betn ± 1 for $|\Omega| \leq 1$

(3) For all N $0 \leq |C_N(\Omega)| \leq 1$ for $0 \leq |\Omega| \leq 1$

$|C_N(\Omega)| > 1$ for $|\Omega| \geq 1$

(4) $C_N(\Omega)$ is monotonically increasing for $|\Omega| > 1$ for all N.

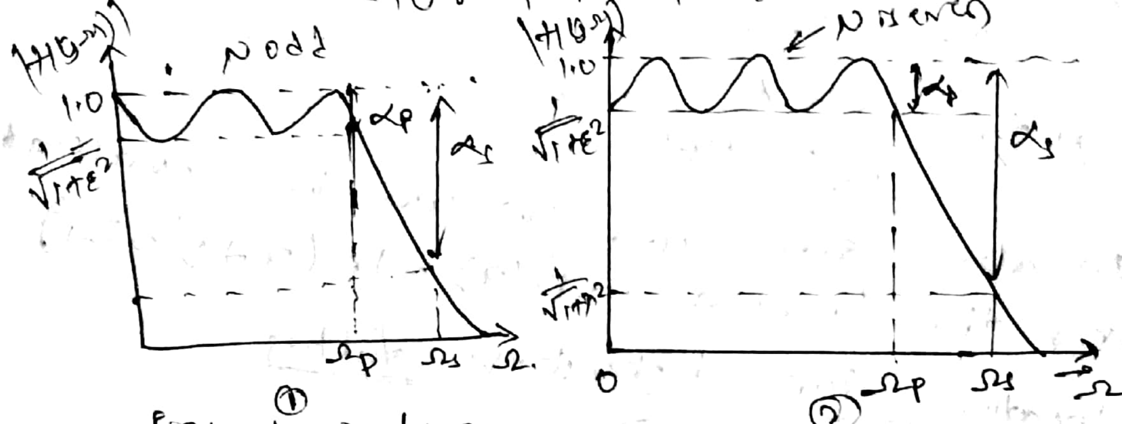

Fig: shows lowpass chebyshev filter magnitude response for odd values of N oscillatory curve starts from unity & for even values of N oscillatory curve starts from $\left(1/\sqrt{1+\epsilon^2}\right)$.

Taking logarithm for Eq -① we get

$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)\right] \quad \text{—⑤}$$

Here α_p is attenuation in the dB at passband freq Ω_p

α_s " " " " " stop band " Ω_s

&

At $\Omega = \Omega_p$ Eq-⑤ can be written as

$$-10\log\left[1+\epsilon^2 C_N^2\left(\frac{\Omega_p}{\Omega_p}\right)\right] = -\alpha_p$$

$\because C_N(1) = 1$

$$\alpha_p = 10\log(1+\epsilon^2)$$

Taking antilog on both sides

$$0.1\alpha_p = \log(1+\epsilon^2)$$

$$1+\epsilon^2 = 10^{0.1\alpha_p}$$

$$\epsilon = (10^{0.1\alpha_p} - 1)^{1/2} \quad \text{—⑥}$$

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at $\Omega = \Omega_s$ in Eq-(5) we get
$-10\log(1+\epsilon^2 C_N^2(\frac{\Omega_s}{\Omega_p}) = -\alpha_s$
$\alpha_s = 10\log(1+\epsilon^2(\cosh(N\cosh^{-1}(\Omega_s/\Omega_p)))^2)$ $\because \frac{\Omega_s}{\Omega_p} > 1$

Substituting ϵ in above Eqn & solving N & rounding it to next higher integer.

$$\therefore N \geq \frac{\cosh^{-1}\sqrt{(10^{0.1\alpha_s}-1/10^{0.1\alpha_p}-1)}}{\cosh^{-1}\Omega_s/\Omega_p} \quad \text{—— (7)}$$

let $A = \frac{\lambda}{\epsilon} = \sqrt{(10^{0.1\alpha_s}-1/10^{0.1\alpha_p}-1)}$.

$k = \Omega_p/\Omega_s$

$$N \geq \frac{\cosh^{-1} A}{\cosh^{-1}(1/k)} \quad \text{—— (8)}$$

In above Eqn $\cosh^{-1}(x)$ can be evaluated using the formula $\cosh^{-1}(x) = \ln(x+\sqrt{x^2-1})$

(*) **Pole locations for chebyshev filter**

considering Eq-(1)

$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2(\frac{\Omega}{\Omega_p})}$ by equating denominator equal to zero we obtain poles for type-I chebyshev filter.

$$1+\epsilon^2 C_N^2\left(\frac{-js}{\Omega_p}\right) = 0.$$

Important points of chebyshev filter

1. The transfer funcn of chebyshev filter like butterworth is an all pole funcn, & 4 because it is constant.
2. The numerator is constant. The finite zeros
3. Poles on transfer funcn lie on ellipse. The ellipse lies along imaginary major axis & of ellipse lies along real axis of s-plane & minor axis lies will be poles of.
4. Narrower the ellipse closer will be poles & has strong ripples.
5. Ripple magnitude will have a strong effect on location of poles & tranf func.

Simplify above Eqn. we get

$$C_N\left(\frac{-js}{\Omega_p}\right) = \pm j/\varepsilon = \cos\left[N\cos^{-1}\left(\frac{js}{\Omega_p}\right)\right] \quad \text{— (9)}$$

According to def^n $\cos^{-1}\left(\frac{js}{\Omega_p}\right) = \phi - j\theta$ — (10) 33

Substituting Eq-(10) in Eq-(9) we get

$$\frac{\pm j}{\varepsilon} = \cos\left[N(\phi - j\theta)\right] = \cos\left[N\phi - jN\theta\right]$$

$$= \cos(N\phi)\cos(jN\theta) + \sin(N\phi)\sin(jN\theta)$$

$$\pm j/\varepsilon = \cos(N\phi)\cosh(N\theta) + j\sin(N\phi)\sinh(N\theta) \quad \text{— (11)}$$

$$\because \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} \; \& \; \cos j\theta = \frac{e^{j^2\theta}+e^{-j^2\theta}}{2} = \frac{e^{-\theta}+e^{\theta}}{2} = \frac{e^{\theta}+e^{-\theta}}{2} = \cosh(\theta)$$

Equating real & imaginary parts of both side of eq-(11) gives

$$\cos(N\phi)\cdot\cosh(N\theta) = 0$$

$$\sin(N\phi)\cdot\sinh(N\theta) = \pm\frac{1}{\varepsilon}$$

when θ is real then $\cosh(N\theta) > 0$ then to satisfy real eq^n $\phi = \frac{(2k-1)\pi}{2N}$, $k = 1, 2 \ldots N$. — (12)

Using the value of ϕ & above imaginary eq^n value of θ becomes where $\sin N\phi = \pm 1$

$$\theta = \pm\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)$$

By putting the value of ϕ & θ in Eq-(10) we get the left half plane locations.

$$S_k = j\Omega_p\cos(\phi - j\theta),$$

$$= j\Omega_p(\cos\phi\cdot\cosh\theta + j\sin\phi\cdot\sinh\theta) \quad \text{— (13)}$$

$$S_k = \Omega_p(-\sin\phi\sinh\theta + j\cos\phi\cdot\cosh\theta)$$

then in Eq-(13) can be calculated as $\sinh\theta$ & $\cosh\theta$

$$\sinh^{-1}(x) = \ln(x + \sqrt{1+x^2})$$

$$\sinh^{-1}(1/\varepsilon) = \ln\left(\varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}\right)$$

$$\text{or} \quad \mu = e^{\sinh^{-1}(\varepsilon^{-1})} = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$$

$$\because \sinh x = \frac{e^{x}-e^{-x}}{2}$$

then $\sinh\theta = \sinh\left(\frac{1}{N}\sinh^{-1} 1/\varepsilon\right)$

$$= \frac{e^{(1/N)\sinh^{-1}(1/\varepsilon)} - e^{-(1/N)\sinh^{-1}(1/\varepsilon)}}{2}$$

1.2.3 5.6.7.10
11.13.14.15 17
... 1.3 31 ... 39.40

$$\sinh\theta = \frac{\left[e^{\sinh^{-1}\cdot\epsilon^{-1}}\right]^{1/N} - \left[e^{-\sinh^{-1}\epsilon^{-1}}\right]^{1/N}}{2}$$

$$= \frac{\mu^{1/N} - \mu^{-1/N}}{2}$$

Illy $\cosh\theta = \dfrac{\mu^{1/N} + \mu^{-1/N}}{2}$

then Eq – (13) becomes

$$S_k = \Omega_p\left[-\sin\phi\cdot\sinh\theta + j\cos\phi\cdot\cosh\theta\right]$$

$$= \Omega_p\left[-\sin\phi\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] + j\cos\phi\cdot\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]\right]$$

$$= -a\sin\phi + jb\cos\phi$$

$$= -a\sin\left(\frac{(2k-1)\pi}{2N}\right) + jb\cos\frac{(2k-1)\pi}{2N}$$

$\cos(\theta \pm 90) = \mp\sin\theta$

$\sin(\theta \pm 90) = \pm\cos\theta$

$$= a\cos\left(90 + \frac{(2k-1)\pi}{2N}\right) + jb\sin\left[90 + \frac{(2k-1)\pi}{2N}\right]$$

$$S_k = a\cos\phi_k + jb\sin\phi_k \quad \text{—— (14)}$$

$$= \sigma_k + j\Omega_k \qquad k = 1, 2 \ldots N$$

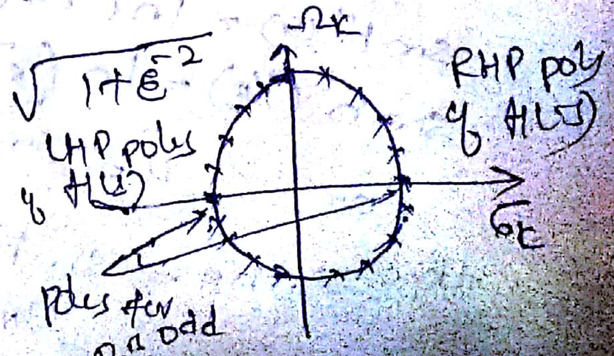
The poles of chebyshev filter can be determined by using Eq – (14)

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] \qquad k = 1, 2 \ldots N$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

$$\& \ \mu = e^{\sinh^{-1}(\epsilon^{-1})} = \epsilon^{-1} + \sqrt{1 + \epsilon^{-2}}$$



The poles of chebyshev transfer fun are located on ellipse in s-plane. & Eqn for ellipse

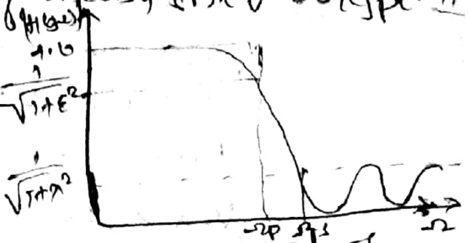
$$\frac{\sigma_k^2}{a^2} + \frac{\Omega_k^2}{b^2} = 1$$

a & b minor & major axis of ellipse

Chebyshev Type - 2 Filter & Exhibits monotonic behaviour in passband & equiripple in stopband.

This type of filter has both poles and zeros.

The magnitude square response of Chebyshev Type-II filter is given by



$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2\left[\frac{C_N^2(\Omega_s/\Omega_p)}{C_N^2(\Omega_s/\Omega)}\right]}$$

Here $C_N(x)$ is N^{th} order Chebyshev polynomial, & Ω_p & Ω_s are passband & stopband frequency parameter. These zeros are located on imaginary axis at the points

$$S_k = j\frac{\Omega_s}{\sin\phi_k} \quad k = 1, 2 \dots N.$$

Poles are located at the points (x_k, y_k) where

$$x_k = \frac{\Omega_s \sigma_k}{\sigma_k^2 + \Omega_k^2} \quad k = 1, 2 \dots N$$

$$y_k = \frac{\Omega_s \Omega_k}{\sigma_k^2 + \Omega_k^2} \quad k = 1, 2 \dots N.$$

where $\sigma_k = a\cos\phi_k \quad k = 1, 2 \dots N$

$\Omega_k = b\sin\phi_k \quad k = 1, 2 \dots N.$

& $\mu = \lambda + \sqrt{1+\lambda^2}$

For the given specifications $\epsilon, \lambda, \Omega_s$ & Ω_p the order of filter

$$N = \frac{\cosh^{-1}(\lambda/\epsilon)}{\cosh^{-1}\Omega_s/\Omega_p} = \frac{\cosh^{-1}A}{\cosh^{-1}1/k}.$$

where $A = \lambda/\epsilon$

$k = \Omega_p/\Omega_s$

$\epsilon = (10^{0.1\alpha_p} - 1)^{0.5}$

$\lambda = (10^{0.1\alpha_s} - 1)^{0.5}$

Comparison

(i) Butterworth filter

(a) Poles lies on a circle

(b) Magnitude response ↓ monotonically as frequency ↑ from 0 to ∞.

(c) Transition Band is more for more specifications

(d) no of poles are more i.e. order of filter more

(ii) Chebyshev filter

(a) Poles lies on ellipse

(b) Magnitude response Exhibits ripples in passband or stopband according to type.

(c) Transition band is less

(d) no of poles less i.e. order of filter is less.

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Frequency transformation in Analog domain

30 To design lowpass filters with different passband frequency, highpass filters, band pass filters & bandstop filters from normalized Lowpass Analog filter ($\Omega_c = 1$ rad/sec) frequency transformation can be ~~done by using~~ used.

(1) Lowpass to Lowpass filter

For a given normalized low pass filter, when it is desired to have a ~~lo~~ LPF with different cutoff frequency (Ω_c) (or pass band freq Ω_p), then transformation is given by $s \to \frac{s}{\Omega_c}$.

(2) Lowpass to High pass.

When it is desired to have a high pass filter with cutoff frequency Ω_c then transformation can be done by $s \to \frac{\Omega_c}{s}$

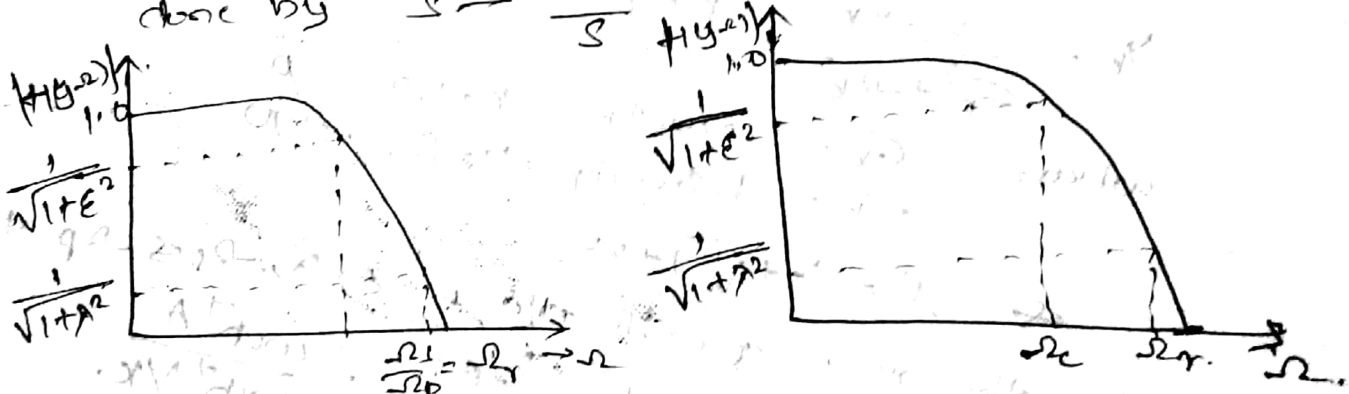

fig. Lowpass to Lowpass transf.

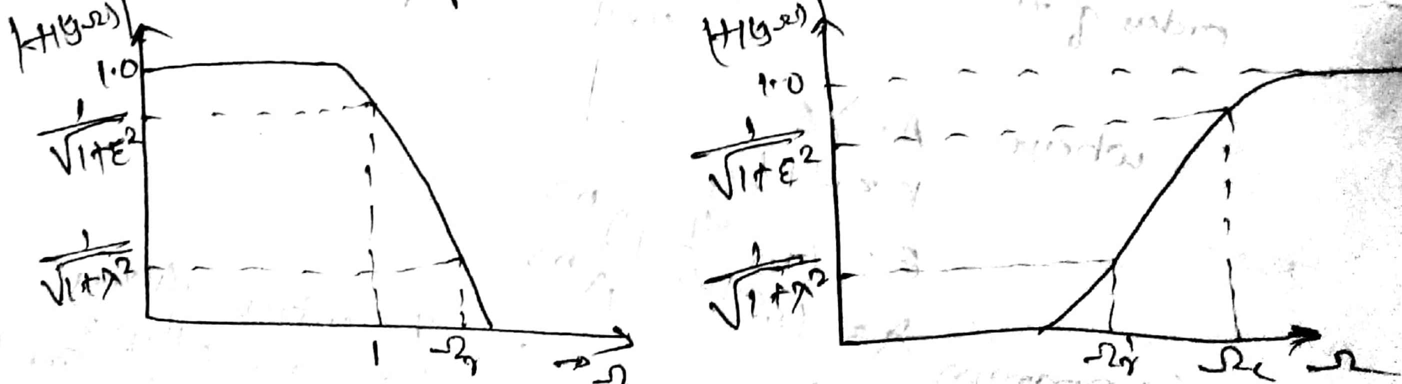

fig. Lowpass to Highpass transformation.

(3) Lowpass to Bandpass

For converting normalized LPF to band pass filter with cutoff frequencies Ω_l, Ω_u can be accomplished by $s \to \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}, \quad B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$

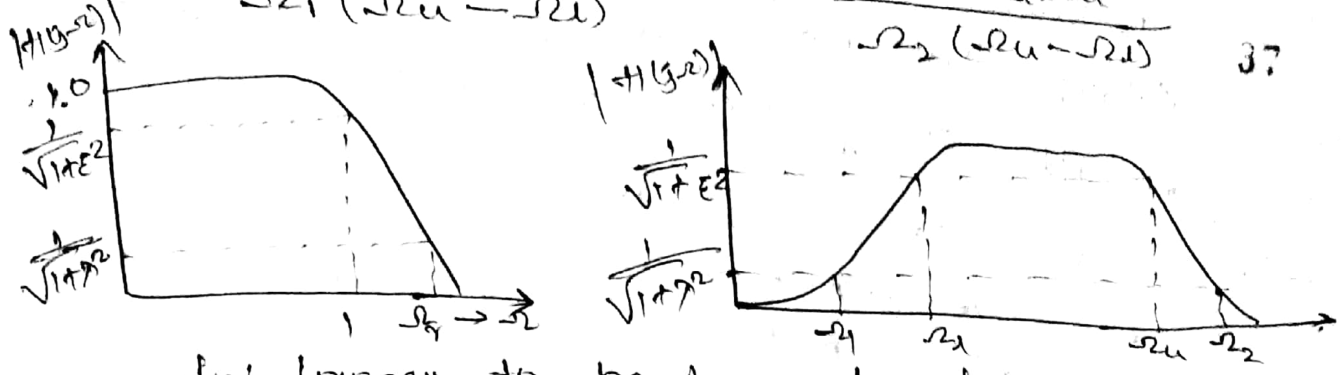

fig: Lowpass to bandpass transformation.

(4) Lowpass to Band stop

Transformation to convert a normalized LPF to bandstop filter is

$$s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l \Omega_u}$$

$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l \Omega_u}$$

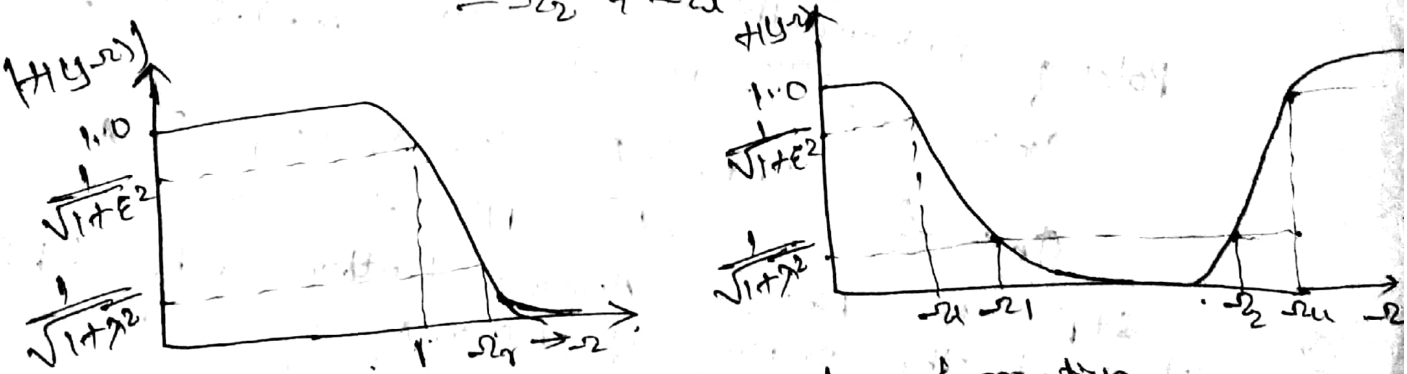

fig: Lowpass to Bandstop transformation.

Design steps for Analog Butt

(1) Find the order of filter N from the given specification
(2) Round off it to next higher integer.
(3) Find the transfer funⁿ H(s) for $\Omega_c = 1$ rad/sec for the value of N.
(4) Calculate value of cutoff frequency Ω_c.
(5) Find the transfer funⁿ $H_a(s)$ for above value of Ω_c by substituting $s \to \frac{s}{\Omega_c}$ in H(s).

Ex Determine the order & poles of LP Butterworth filter that has 3dB attenuation at 500Hz & an attenuation of 40dB at 1000 Hz.

Sol? $\alpha_p = 3dB$ $\alpha_s = 40dB$.

$\Omega_p = 2\pi f = 2\times\pi\times 500Hz = 1000\pi$ rad/sec

$\Omega_s = 2\pi f = 2\times\pi\times 1000Hz = 2000\pi$ rad/sec.

The order of the filter is given by

$$N \geq \frac{\log\sqrt{(10^{0.1\alpha_s}-1)/(10^{0.1\alpha_p}-1)}}{\log(\Omega_s/\Omega_p)}.$$

$$\geq \log\sqrt{(10^{4}-1)/(10^{0.3}-1)} \Big/ \log\left(\frac{2000\pi}{1000\pi}\right)$$

$$\geq \log\sqrt{}$$

$N \geq 6.6 \cong 7$.

Poles of Butterworth filter is given by

$S_k = \Omega_c e^{j\phi_k} = 1000\pi e^{j\phi_k}$ $k = 1, 2 \cdots 7$

$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$, $k = 1, 2, \ldots 7$.

$H(s) = (s+1)(s^2+1.801s+1)(s^2+1.247s+1)(s^2+0.445s+1)$

Ex Steps to Design Analog Butterworth LPF.

(1) Find the order of filter N from given specs

(2) Round it to next higher integer

(3) Find the Tr.f fun $H(s)$ for $\Omega_c = 1$ rad/sec for the value of N.

(4) Calculate the value of Ω_c (cut off freq)

(5) Find the transfer fun $H_a(s)$ for the above value of Ω_c by substituting $s \to s/\Omega_c$ in $H(s)$

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Design An Analog Butterworth filter that has -2dB passband attenuation at freq of 20 rad/sec & atleast -10dB stopband attn at 30 rad/sec.

Soln: $\alpha_p = 2dB$, $\Omega_p = 20 rad/sec$, $\alpha_s = 10dB$, $\Omega_s = 30 rad/sec$

$$N \geq \frac{\log \sqrt{10^{0.1\alpha_s}-1 / 10^{0.1\alpha_p}-1}}{\log \Omega_s/\Omega_p}$$

$$N \geq \log\sqrt{10^{0.1\times10}-1 / 10^{0.1\times2}-1} / \log 30/20 \geq 3.37 \approx 4$$

Normalized LP Butterworth filter for $N=4$.

$$H(s) = \frac{1}{(s^2+0.7653s+1)(s^2+1.8477s+1)}$$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{20}{(10^{0.1\times2}-1)^{1/8}} = 21.3868$$

Trf fun for $\Omega_c = 21.3868$ can be obtained by substituting $s \rightarrow s/21.3868$ in $H(s)$

$$H(s) = \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 0.7653\left(\frac{s}{21.3868}\right)+1\right]\left[\left(\frac{s}{21.3868}\right)^2 + 1.8477\frac{s}{21.3868}+1\right]}$$

$$= \frac{0.20921\times10^6}{(s^2+16.3868+457.39)(s^2+39.5176s+457.3)}$$

P.T

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$$

Magnitude square fun of Butterworth Analog LPF

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \quad \text{--- (1)}$$

from Eq (1) $$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}} \quad \text{--- (2)}$$

compare Eq (1) & (2)

$$1+(\Omega/\Omega_c)^{2N} = 1+\varepsilon^2(\Omega/\Omega_p)^{2N}$$

$$\varepsilon^2(\Omega/\Omega_p)^{2N} = (\Omega/\Omega_c)^{2N} \qquad \varepsilon = (10^{0.1\alpha_p}-1)^{1/2}$$

$$(10^{0.1\alpha_p}-1)\left(\frac{\Omega}{\Omega}\right) = (\Omega_p/\Omega_c)^{2N}$$

$$\Omega_c = \Omega_p/(10^{0.1\alpha_p}-1)^{1/2N} = \frac{\Omega_p}{\varepsilon^{1/N}}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}$$

$$\Omega_s = \Omega_p\left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]^{1/2N}$$

$$= \Omega_c(10^{0.1\alpha_p}-1)^{1/2N} \cdot \left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)$$

$$\Omega_c = \frac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$$

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Steps to design analog Chebyshev lowpass filter.

(1) From the given specification find the order of filter N.

(2) Round off it to next higher integer.

(3) Find the values of a & b, which are minor & major axis of Ellipse respectively

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] ; \quad b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$\mu = \epsilon^{-1} + \sqrt{\epsilon^{-2}+1}$$

$$\epsilon = \sqrt{10^{0.1\alpha_p} - 1}$$

(4) Calculate the poles of chebyshev filter which lies on Ellipse by using the formula

$$s_k = a\cos\phi_k + jb\sin\phi_k , \quad k=1,2 \cdots N.$$

where $\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi , \quad k=1,2 \cdots N.$

(5) Find denominator polynomial of transfer function using poles

(6) The numerator of transfer fun^n depends on N

(a) For N=odd substitute $s=0$ in denominator polynomial & find the value. This value is equal to numerator of transfer function. ($\because$ For N=odd magnitude response $|H(j\Omega)|$ starts at 1).

(b) For N=Even put $s=0$ in denominator polynomial & divide result by $\sqrt{1+\epsilon^2}$. this value is Equal to numerator

To find the transfer function of highpass, band pass etc any type find the transfer fun^n of normalized lowpass filter & use suitable transformation.

Ex

Design a chebyshev filter with a maximum passband attenuation of 2.5dB at $\Omega_p = 20$ rad/sec & stopband " of 30dB at $\Omega_s = 50$ rad/sec.

$$N = \frac{\cosh^{-1} \lambda/\epsilon}{\cosh^{-1} 1/k}$$

$$= \frac{\cosh^{-1} \sqrt{10^{0.1\alpha_s}-1} \Big/ \left(10^{0.1\alpha_p}-1\right)}{\cosh^{-1} \Omega_s/\Omega_p}$$

$\lambda = 31.607$

$\epsilon = 0.882$

$\lambda = \sqrt{10^{0.1\alpha_s}-1} = 31.607$

$\epsilon = \sqrt{10^{0.1\alpha_p}-1} = 0.882$

$k = \frac{\Omega_p}{\Omega_s}$.

$k = \frac{\Omega_p}{\Omega_s} = 0.4$

$N \geq 2.726 = 3$

$\mu = \epsilon^{-1} + \sqrt{1 + \epsilon^{-2}} = 2.65$

$a = 6.6$, $b = 21.06$

$s_k = a\cos\phi_k + jb\sin\phi_k$, $k = 1, 2, 3$

$\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi$, $k = 1, 2, 3$

$\phi_1 = 120^\circ$, $\phi_2 = 180^\circ$, $\phi_3 = 240$

$s_1 = -3.3 + j18.23$

$s_2 = -6.6$

$s_3 = -3.3 - j18.23$

Denominator of $H(s) = (s+6.6)(s^2+6.6s+343.2)$

Put $s=0$ Numerator of $H(s) = (6.6)(343.2) = 2265.27$

Transfer function of $H(s) = \frac{2265.27}{(s+6.6)(s^2+6.6s+343.2)}$