Fast Fourier Transform

Mahesh Yanagasmath

Fast Fourier transform is highly efficient procedure for computing DFT of a finite series and requires less number of computations than that of direct evaluation of DFT ∵ Direct computation of DFT involves large number of computations. So that fast Fourier transform algorithms are developed to compute DFT quickly. As value of N increases the computational efficiency of FFT algorithms increases. ∴ FFT technique is used in digital spectral analysis, filter simulation, autocorrelation & pattern recognition.

FFT is based on decomposition and breaking the transform in to smaller transforms and combining them to get total transform. It reduces the computation time reqd to compute

DFT - Direct computation :-

Consider DFT Eqn

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \qquad k = 0, 1, \ldots N-1$$

$$X(k) = x(0) \times W_N^{0} + x(1) \times W_N^{k} + x(2) W_N^{2k} + \ldots + x(N-1) \cdot W_N^{(N-1)k}$$

Here $x(n)$ may be real or complex.

Then no of complex multiplications required for calculating $X(k)$ for $k = 0, 1, \ldots N-1$ is $N \times N = N^2$

$\|$ly no of complex additions reqd for are

$$(N-1) \times N = N^2 - N$$

consider 1024 point DFT then $N = 1024$

complex multiplications $= N^2 = (1024)^2 = 1 \times 10^6$

complex additions $= N^2 - N = (1024)^2 - 1024 \approx 1 \times 10^6$

Time reqd for computation $=$ (complex multipln) (time for 1 multipln) + (complex addn) × (time for 1 addn)

$= (1 \times 10^6 \times 1 \times 10^{-6}) + (1 \times 10^6 \times 1 \times 10^{-6})$

$= 1 + 1 = 2$. Here processor execute one complex multipln or addition is 1 μsec. Again time is reqd for storing data & display etc. Hence by

Here N^2 complex multiplications & (N^2-N) complex additions are required in the direct computation of N-point DFT.

WKT DFT Eq^n: $X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$ — ① $k=0,1,\dots N-1$

Here both $x(n)$ & W_N^{kn} are complex functions.

then $x(n) = x_R(n) + j x_I(n)$

$W_N^{kn} = W_{RN}^{kn} + j W_{IN}^{kn}$

Then Eq^n ① becomes

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + j x_I(n)] [W_{RN}^{kn} + j W_{IN}^{kn}]$$

$$= \sum_{n=0}^{N-1} [x_R(n) \cdot W_{RN}^{kn} + x_R(n) j W_{IN}^{kn} + j x_I(n) W_{RN}^{kn} - x_I(n) W_{IN}^{kn}]$$

$$= \sum_{n=0}^{N-1} [x_R(n) W_{RN}^{kn} - x_I(n) W_{IN}^{kn}] + j [x_R(n) W_{IN}^{kn} + x_I(n) W_{RN}^{kn}]$$ — ②

Hence by comparing Eq ① with Eq ② One complex multiplication is converted into 4 Real multiplications and 2 Real additions.

These subtractions is also counted as addition in signal processing. & Real part and imaginary part of complex number are basically real no.

Hence for Each value of k there are N complex multiplications then 4N Real multiplications and R 2N Real additions we get.

If k varies from (0 - N-1) Then complete DFT of $(N \times N)$ ~~matrix~~ complex multiplications converted into $4N \times N = 4N^2$ Real multipl^ns & $2N \times N = 2N^2$ Real ad^ns.

Consider two complex nos $(a+jb)$ & $(c+jd)$.

Then addition of two nos is

$$(a+jb) + (c+jd) = (a+c) + j(b+d)$$

∴ Addition of 2 complex nos is converted to two real additions. Here for Each value of k there are $(N-1)$ complex addition then $2(N-1) = 2(N-1)$ real addition

Complete DFT $(N-1)$ complex addⁿ are converted to $= 2(N-1) N = 2N^2 - 2N$ real

∴ Total no of real addⁿ are $= 2N^2 + 2N^2 - 2N$
$= 4N^2 - 2N$ //

where $W_N^{kn} = e^{-j2\pi kn/N}$

$W_N^{kn} = \cos\frac{2\pi kn}{N} - j\sin\frac{2\pi kn}{N}$

If k varies from 0 to N-1
then no of trigonometric values evaluated
for computⁿ of DFT $= 2 \times N \times N = 2N^2$

Two properties of twiddle factor (W_N)

Symmetry property : $W_N^{k+N/2} = -W_N^k$

Periodicity property : $W_N^{k+N} = W_N^k$

Symmetry property

$W_N^{k+N/2} = -W_N^k$

$W_N^{nk} = e^{\frac{-j2\pi nk}{N}}$

$W_N^{(n+N/2)k} = e^{\frac{-j2\pi(n+N/2)k}{N}} = e^{\frac{-j2\pi nk}{N}} \cdot e^{\frac{-j2\pi N/2\, k}{N}}$

$= e^{\frac{-j2\pi nk}{N}} \cdot e^{-j\pi k} = -e^{\frac{-j2\pi nk}{N}} = -W_N^k$

$e^{-j\pi k} = \cos\pi k - j\sin\pi k = -1$

∴ $W_N^{k+N/2} = -W_N^k$.

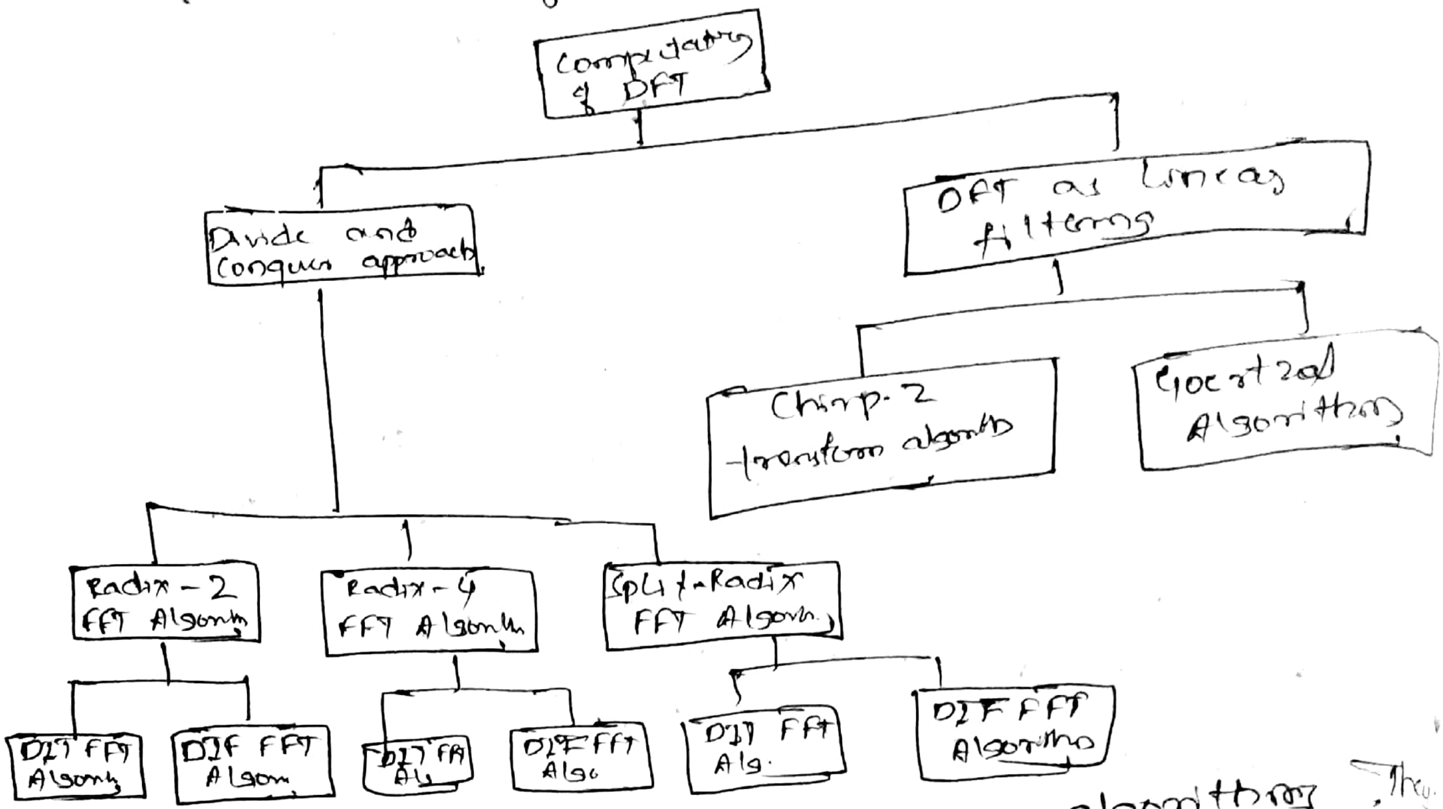
Periodicity property

$W_N^{k+N} = W_N^k$. //

Fast fourier transform

FFT refers to algorithms that compute the DFT in a numerically efficient manner. These algorithms exploit the two basic properties of W_N^{kn} (twiddle factor) to reduce no of complex multiplications reqd to perform DFT. These are based on principle of decomposing the computⁿ of DFT of a sequence of length N into successively smaller DFT's.

Classification of FFT algorithms



The Basically two classes of FFT algorithms

(1) Decimation in time algorithm
(2) Decimation in frequency algorithm.

(1) ~~In DFT the frequency samples of DFT are decomposed into smaller & smaller subsequ~~ In DIT sequence is successively divided into smaller sequences & DFT's of these subsequences are combined in a certain pattern to obtain the reqd DFT of Entire sequence.

In DFT frequency samples of the DFT are decomposed into smaller & smaller subsequences in similar manner.

Decimation in time Algorithm (DIT)
(Radix - 2 FFT Algorithm)

These are based on Divide and Conquer approach & here N-point DFT is successively decomposed into smaller DFTs. Hence no. of computations are reduced. Here no of o/p points can be expressed as a power of 2.
i.e $N = 2^M$. where M is an integer shows the no of stages

1.5.18.

$x(n) \rightarrow$ $x_e(n)$ — Even Indexed sequence of $x(n)$
 $x_o(n)$ — Odd indexed sequence of $x(n)$

Consider sequence $x(n)$ of N point and divided into two sequences of length N/2 as even indexed seq $x_e(n)$ & odd ~~seq~~ indexed sequence $x_o(n)$ of $x(n)$.

$x_e(n) = x(2n)$ $\quad n = 0.1 \ldots N/2 - 1$ —— ①

$x_o(n) = x(2n+1)$ $\quad n = 0.1 \ldots N/2 - 1$. —— ②

Then N point DFT of $x(n)$ can be written as

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \qquad K = 0.1 \ldots N-1$$

Divide $x(n)$ into $x_e(n)$ & $x_o(n)$. we get

$$X(K) = \sum_{\substack{n=0 \\ (\text{even})}}^{N-1} x_e(n) W_N^{nk} + \sum_{\substack{n=0 \\ (\text{odd})}}^{N-1} x_o(n) W_N^{nk}$$

$$= \sum_{n=0}^{N/2-1} x(2n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{(2n+1)k.}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk} \cdot W_N^{k}$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk} \qquad \text{—— ③}$$

Substituting Eq-① & ② in Eq-③ we get

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x_o(n) . W_N^{2nk}$$

But $W_N^{2nk} = e^{-j\frac{2\pi}{N} \times 2nk} = e^{-j\frac{2\pi nk}{N/2}} = W_{N/2}^{nk}$

$$\therefore X(k) = \sum_{n=0}^{N/2-1} x_e(n) . W_{N/2}^{nk} + W_N^{k} \sum_{n=0}^{N/2-1} x_o(n) . W_{N/2}^{nk} = X_e(k) + W_N^{k} X_o(k). \qquad \text{—— ④}$$

Above eqn shows the sum of $N/2$ point of the even indexed sequence with $N/2$ po DFT of the odd indexed sequence where k ranges from 0 to N-1. Here $X_e(k)$ & are periodic in nature.

If $k \geq N/2$ According to symmetry prop

$$W_N^{k+N/2} = -W_N^k$$

Now $X(k)$ for $k \geq N/2$ then k is replaced by (k

$$X(k) = X_e(k-N/2) - W_N^{k-N/2} X_o(k-\tfrac{N}{2}) \quad \text{for } k = \tfrac{N}{2}, \tfrac{N}{2}+1 \ldots \quad \text{— (5)}$$

Eq — (4)

N=8 from Eq (1) & (2)

or use below

$X(k) = X_e(k) + W_N^k X_o(k)$ &

& $X_e(k-N/2) - W_N^{k-N/2} X_o(k-N/2)$

$x_e(0) = x(0)$	$x_o(0) = x(1)$
$x_e(1) = x(2)$	$x_o(1) = x(3)$
$x_e(2) = x(4)$	$x_o(2) = x(5)$
$x_e(3) = x(6)$	$x_o(3) = x(7)$

Now $X(k) = X_e(k) + W_N^k X_o(k)$

k=0

$X(0) = X_e(0) + W_8^0 X_o(0)$

$X(1) = X_e(1) + W_8^1 X_o(1)$

$X(2) = X_e(2) + W_8^2 X_o(2)$

$X(3) = X_e(3) + W_8^3 X_o(3)$

$X(4) = X_e(4) + W_8^4 X_o(4)$

$= X_e(0) - W_8^0 X_o(0)$

$\because X_e(4) = X_e(0)$ & $W_8^4 = -W_8^0 \quad \because W_N^{k+N/2} = -W_N^k$

$X(5) = X_e(5) + W_8^5 X_o(5)$

$= X_e(1) - W_8^1 X_o(1)$

$X(6) = X_e(6) + W_8^6 X_o(6)$

$= X_e(2) - W_8^2 X_o(2)$

$X(7) = X_e(7) + W_8^7 X_o(7)$

$= X_e(3) - W_8^3 X_o(3)$

Here Eqn for $X_e(0)$ & $X_e(4)$ both are same only changes addn or substraction of $X_o(0)$ & W_8^0

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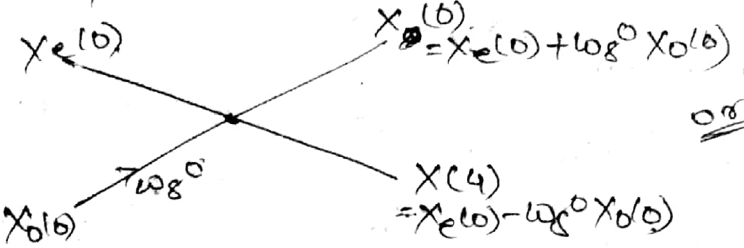
Above all Eqⁿ's from X(0) to X(7) are represented by using butterfly diagram.



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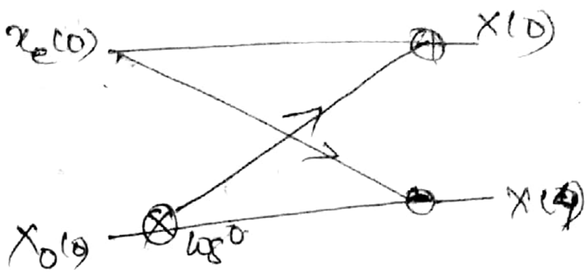
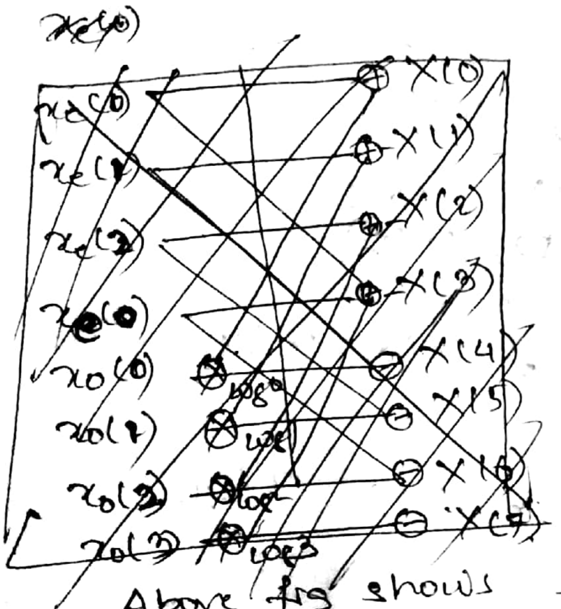
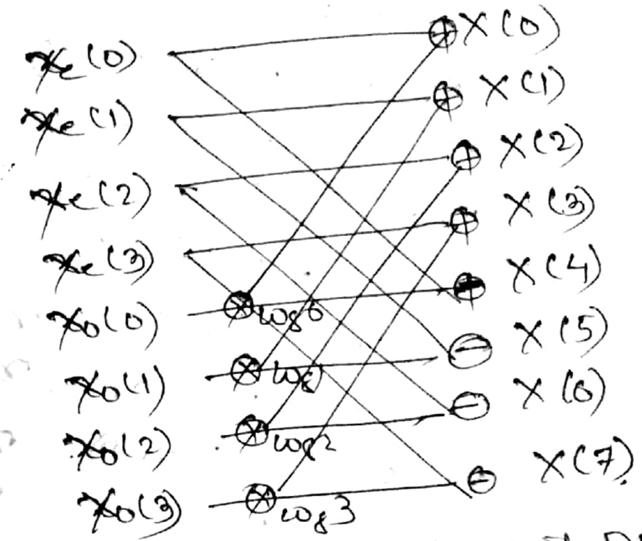
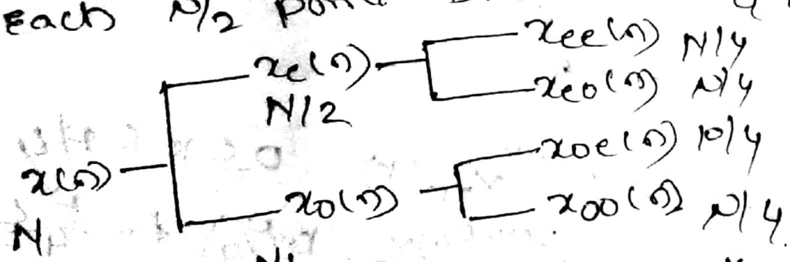






Above fig shows stage ③ or constⁿ of 8 point DFT from two 4 point DFTs.

Same approach is applied to decompose each N/2 point DFTs to $\frac{N}{4}$ point DFTs.



12. 24, 25. 40.

$$\text{Now } X_e(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n) \cdot W_N^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x_{ee}(n) + x_{eo}(n)\right] W_N^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x_{ee}(n) W_N^{2nk} + x_{eo}(n) W_N^{2nk}\right]$$

$$= \sum_{n=0}^{N/4-1} x_e(2n) W_{N/4}^{4nk} + \sum_{n=0}^{\frac{N}{4}-1} x_e(2n+1) W_{N/4}^{2(2n+1)k}$$

$$= \sum_{n=0}^{N/4-1} x_e(2n) W_{N/4}^{4nk} + \sum_{n=0}^{N/4-1} x_e(2n+1) W_{N/4}^{4nk} \cdot W_{N/4}^{2k}$$

$$= X_{ee}(k) + W_N^{2k} X_{eo}(k) \quad \text{for } 0 \le k \le \frac{N}{4}-1$$

$$= \ldots \, 2(k - N/4) X_{eo}(k - \tfrac{N}{4}) \quad \text{for } \frac{N}{4} \le k \le \frac{N}{2}-1$$

IIly

$$X_o(k) = \sum_{n=0}^{N/2-1} x_o(n) W_N^{2nk}$$

$$= \sum_{n=0}^{N/2-1} (x_{oe}(n) + x_{oo}(n)) \cdot W_N^{2nk}$$

$$X_o(k) = X_{oe}(k) + W_N^{2k} X_{oo}(k) \quad \text{for } 0 \le k \le N/4$$

$$= X_{oe}(k - \tfrac{N}{4}) - W_N^{2(k-N/4)} X_{oo}(k - \tfrac{N}{4}) \quad \text{for } \tfrac{N}{4} \le k \le$$

For N=8

$x(n) = x(0).\ x(1),\ x(2).\ x(3).\ x(4).\ x(5)\ x(6)$

$x_e(n) = x_e(0).\ x_e(1).\ x_e(2).\ x_e(3)$

$= x(0),\ x(2),\ x(4).\ x(6)$

$x_o(n) = x_o(0).\ x_o(1).\ x_o(2).\ x_o(3)$

$= x(1).\ x(3).\ x(5),\ x(7)$

$x_{ee}(n) = x_{ee}(0).\ x_{ee}(1)$

$= x_e(0),\ x_e(2) = x(0).\ x(4)$

$x_{eo}(n) = x_{eo}(0);\ x_{eo}(1)$

$= x_e(1);\ x_e(3) = x(2).\ x(6)$

$x_{oo}(n) = x_{oo}(0),\ x_{oo}(1)$

$= x_o(1).\ x_o(3) = x(3).\ x(7)$

$x_{oe}(n) = x_{oe}(0),\ x_{oe}(1)$

$= x_o(0).\ x_o(2)$

$= x(1).\ x(5)$

Then from Eqn

$$X_e(k) = X_{ee}(k) + W_N^{2k} X_{eo}(k) \quad 0 \le k \le N/4$$

$$= X_{ee}(k - N/4) - W_N^{2(k-N/4)} X_{eo}(k - N/4) \quad \text{for } \tfrac{N}{4} \le k \le \tfrac{N}{2} - 1$$

$\therefore X_e(0) = X_{ee}(0) + W_8^0 X_{eo}(0)$

$X_e(1) = X_{ee}(1) + W_8^2 X_{eo}(1)$

$X_e(2) = X_{ee}(2) + W_8^4 X_{eo}(2)$

$= X_{ee}(0) - W_8^0 X_{eo}(0)$

$X_e(3) = X_{ee}(3) + W_8^6 X_{eo}(3)$

$= X_{ee}(1) - W_8^2 X_{eo}(1)$

Here $X_{ee}(k)$ is the 2point DFT of Even members of $x_e(n)$ & $X_{eo}(k)$ 2 point DFT of odd members of $x_e(n)$

III^ly

$$X_0(k) = X_{oe}(k) + W_N^{2k} X_{oo}(k)$$

$$X_0(0) = X_{oe}(0) + W_8^0 X_{oo}(0)$$

$$X_0(1) = X_{oe}(1) + W_8^2 X_{oo}(1)$$

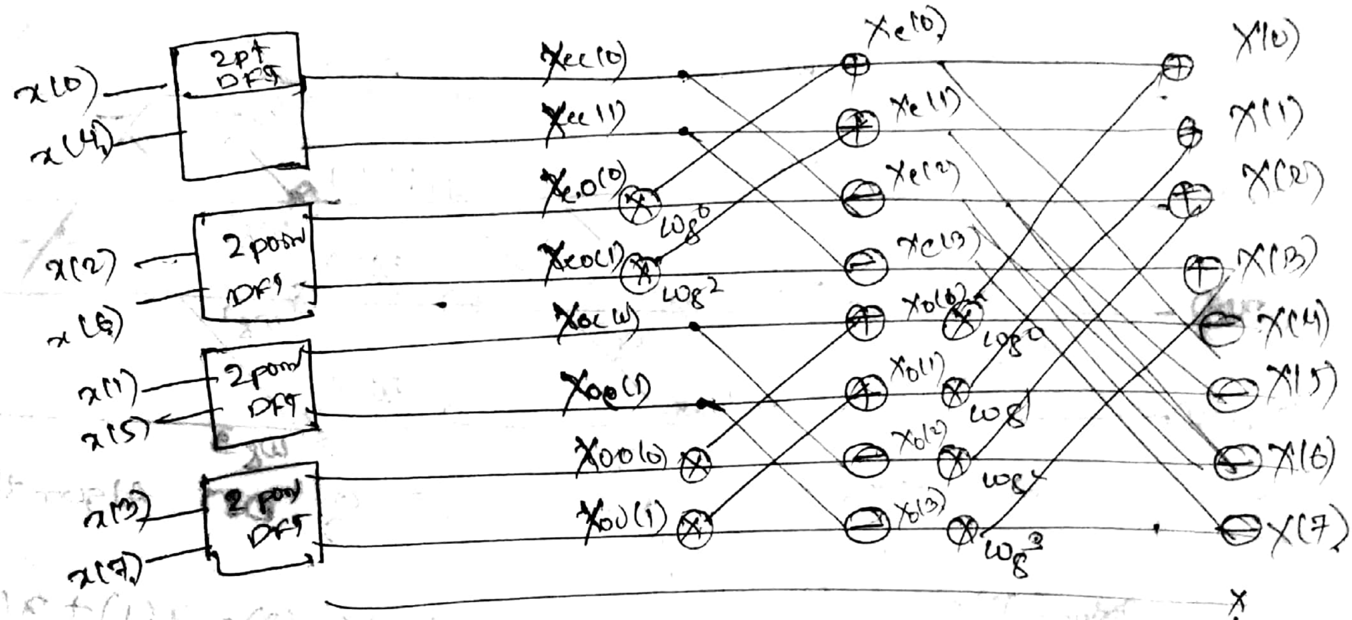
$$X_0(2) = X_{oe}(2) + W_8^4 X_{oo}(2)$$

$$X_0(2) = X_{oe}(0) - W_8^0 X_{oo}(0)$$

$$X_0(3) = X_{oe}(3) + W_8^3 X_{oo}(3)$$

$$= X_{oe}(1) - W_8^2 X_{oo}(1)$$

$W_4 \cdot e^{-j\frac{2\pi}{N}}$



Next stage of two point DFT can be found by adding & substracting the i/p sequences as twiddle factor associated with 1st stage is $W_8^0 = 1$. This stage involves no multipⁿ but addⁿ & subsⁿ.

$$x_{ee}(0) + x_{ee}(1) = x_e(0) + x_e(2) = x(0) + x(4)$$

$$x_{ee}(1) = x_{ee}(0) - x_{ee}(1)$$

$$X(k) = \sum_{n=0}^{1} x(n) \cdot W_N^{nk} \quad k = 0, 1.$$

$$X_{ee}(k) = \sum_{n=0}^{1} x_{ee}(n) \cdot W_N^{nk}$$

1, 12, 18, 23, 25, 26, 40

$k=0 \quad X_{ee}(0) = \sum x_{ee}(0) + x_{ee}(1) = x_e(0) + x_e(2) = x(0) + x(4)$

$W^{0+N/2} = -W_2^0$

$k=1 \quad X_{ee}(1) = x_{ee}(0) W_N^0 + x_{ee}(1) W_2 = x_e(0) - x_e(2) = x(0) - x(4)$

$$= X_{ee}(0) - x_{ee}(1)$$

$k=2 \quad X_{eo}(0) = \sum_0^1 x_{eo}(n) \cdot W_2^{kn} = x_e(1) + x_e(3) = x(2) + x(6)$

$$= x_{eo}(0) + x_{eo}(1)$$

$$x_{eo}(1) = x_{eo}(0) + x_{eo}(1) W_2^1 = x(2) - x(6)$$

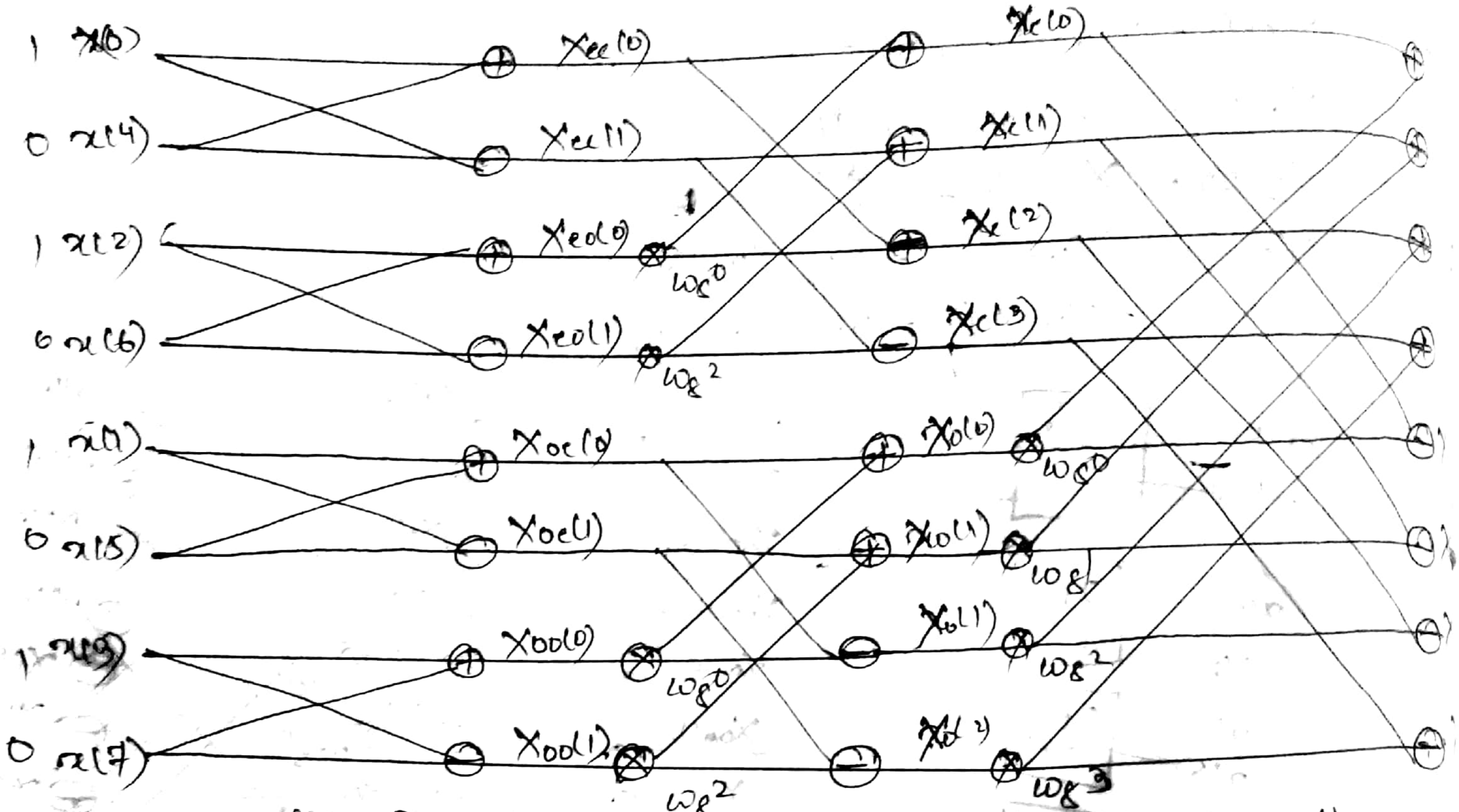


fig Decimation flow graph of DIT Algorithm.

$X_{oe}(K) = \sum_{0}^{1} x_{oe}(n) W_2^{kn}$

$X_{oe}(0) = x_{oe}(0) + x_{oe}(1) = x_o(0) + x_o(2) = x(1) + x(5)$

$X_{oe}(1) = x_{oe}(0) + x_{oe}(1) W_2^1 = x_o(0) - x_o(2) = x(1) - x(5)$

$X_{oo}(0) = x_{oo}(0) + x_{oo}(1) = x_o(1) + x_o(3) = x(3) + x(7)$

$X_{oo}(1) = x_{oo}(0) + X_{oo}(1) W_2^1 = X_o(1) - X_o(3) = x(3) - x(7)$

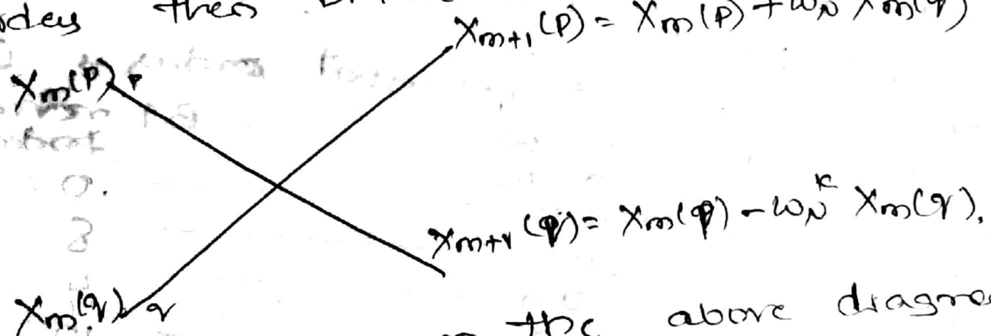
In N/8 point transforms only 2 point DFT we get. Totally each decomposition is called stage. Total no of stages m given by $m = \log_2(N)$. Each value computed from butterfly Eqn requires one complex multipn. ∴ Each stage requires total of N/2 complex multipn. ∴ Entire FFT requires total $\frac{N}{2}\log_2 N$ complex multipns & $N \log_2 N$ complex additions. In this algorithm at each stage i/p seqn is divided into smaller seqn ie decimation at each stage. ∴ It is called as Decimation in time FFT algorithm..

Bit Reversal

Considering signal flow graph we observed that i/p data sequence x shuffled as x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7), to get proper order X(k).

M/m add of x(n) in decimal	m/m add x(n) in binary			m/m add x(n) in reverse order			New m/m add of x(n) in reverse order (same are stored & applied as I/P)
n	n_2	n_1	n_0	n_0	n_1	n_2	n
0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	4
2	0	1	0	0	1	0	2
3	0	1	1	1	1	0	6
4	1	0	0	0	0	1	1
5	1	0	1	1	0	1	5
6	1	1	0	0	1	1	3
7	1	1	1	1	1	1	7

Thus in i/p data is stored in bit reversed order then DFT will be obtained in natural sequence



As shown in the above diagram $X_m(p)$ & $X_m(q)$ are the i/p value at i/p nodes p & q respectively. o/p also stored in the same m/m location as that of i/p. An algorithm that uses the same m/m locations to store both i/p & $X_{m+1}(p)$ & $X_{m+1}(q)$ are o/p sequences is called as in-place algorithm.

FFT algorithm reduce no of computations. It is greater than 100 times faster than direct computation.

No of stage	No. of points N	No of multiplications Direct eval N^2	FFT Alg $(\frac{N}{2})\log_2 N$	Speed improvement factor $= N^2/(N/2)\log_2 N$
2	4	16	4	4
3	8	64	12	5.333
5	32	1024	80	12.8
8	256	65536	1024	64
10	1024	1048576	5120	204.8

12.15.18

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Steps Of Radix-2 DIT-FFT Algorithm

(1) No of i/p samples $N = 2^M$ where M is an integer
(2) The i/p sequence is shuffled through bit reversal
(3) The no of stages in signal flow graph is $M = \log_2 N$.
(4) Each stage consists of $N/2$ butterflies
(5) m represents the stage index (m=1 for 1st stage) and no of i/p / o/p's for each Butterfly are separated by 2^{m-1}
(6) No of complex multiplications are $\frac{N}{2}\log_2 N$ & complex additions are $N\log_2 N$.
(7) twiddle factor (W_N^k) are fun of stage index m, $t = 0, 1, 2 \ldots 2^{m-1} - 1$.
i.e $k = \frac{Nt}{2^m}$
(8) No of sets of butterflies in each stage is given by 2^{M-m}.
(9) Exponent repeat factor (ERF) is no of times the exponent sequence associated with m is repeated is given by 2^{N-m}.

Ex: Draw the flow graph of 16-point DIT-FFT.

Sol^n: (1) No of i/p samples $N = 2^M = 16$

(2) I/P sequence with bit reversal order is shown below

Index	Binary repr^n	Bit reversal order	Bit reversed Index
0	0000	0000	0
1	0001	1000	8
2	0010	0100	4
3	0011	1100	12
4	0100	0010	2
5	0101	1010	10
6	0110	0110	6
7	0111	1110	14
8	1000	0001	1
9	1001	1001	9
10	1010	0101	5
11	1011	1101	13
12	1100	0011	3
13	1101	1011	11
14	1110	0111	7
15	1111	1111	15
~~16~~			

(3) No of stage $m = \log_2 N = \frac{\log 16}{\log 2} = \log_2 16 = 4$

(4) No of butterflies at each stage $16/2 = 8$ per

(5) I/p's / o/p's for each butterfly in stage m is separated by 2^{m-1} samples

stage (1) I/P / O/P for each butterfly are separated by $2^0 = 1$ samp

stage (2) " " " " " " $2^1 = 2$ samp

stage (3) " " " " " " $2^2 = 4$ samp

stage (4) " " " " " " $2^3 = 8$ samp

(6) No of complex mul's $\frac{N}{2}\log_2 N = \frac{16}{2} \times 4 = 32$

(7) No of complex add's $N \log_2 N = 16 \times 4 = 64$

No of butterflys in each stage $= 2^{m-m}$

(8) No of sets of butterfly $2^{M-m} = 2^{4-1} = 2^3 = 8$

stage (1) No of sets of butterfly " $= 2^{4-2} = 2^2 = 4$

stage (2) " " " " $= 2^{4-3} = 2 =$

stage (3) " " " " $= 2^{4-4} = 2^0 = 1$

stage (4) "

(9) Twiddle factor exponents at each stage

$k = \frac{Nt}{2^m} \quad t = 0, 1, 2 \cdots 2^{m-1} - 1$

stage 1 exponent is 0 $\quad W_N^{k=0}$ is repeated 8 times

$m = 1 \quad 2^{1-1} - 1. \quad W_{16}^0 =$ repeated 8 times

$k = \frac{Nt}{2^m} \quad t = 0, \ldots$

$t = 0$

$k = \frac{N \times 0}{2^m} = \frac{16 \times 0}{2^1} \; 0$

stage 2 exponent's are $k = \frac{Nt}{2^m} \quad t = 0 \ldots 2^{2-1} - 1$

$t = 0, 1$

0, 4,

$k = \frac{16 \times 0}{2^2} = 0. \quad k = \frac{16 \times 1}{2^2} = \frac{16}{4} = 4$

stage 3 exponent are $k = \frac{Nt}{2^m} \quad t = 0 \ldots 2^{3-1} - 1$

$t = 0, 1, 2, 3$

0, 2, 4, 6.

$k = \frac{16 \times 1}{2^3} = \frac{16}{8} = 2, \quad \frac{16 \times 2}{2^3} = \frac{32}{8} = 4, \quad \frac{16 \times 3}{8}$

stage 4 exponent are $k = \frac{Nt}{2^m} \quad t = 0 \ldots 2^{4-1} - 1$

$t = 0, 1, 2, 3, 4, 5, 6, 7.$

are 0, 1, 2, 3, 4, 5, 6, 7.

(4) ERF = 2^{m-m}.

stage (1) $2^{4-1} = 2^3 = 8$

stage (2) $2^{4-2} = 2^2 = 4$

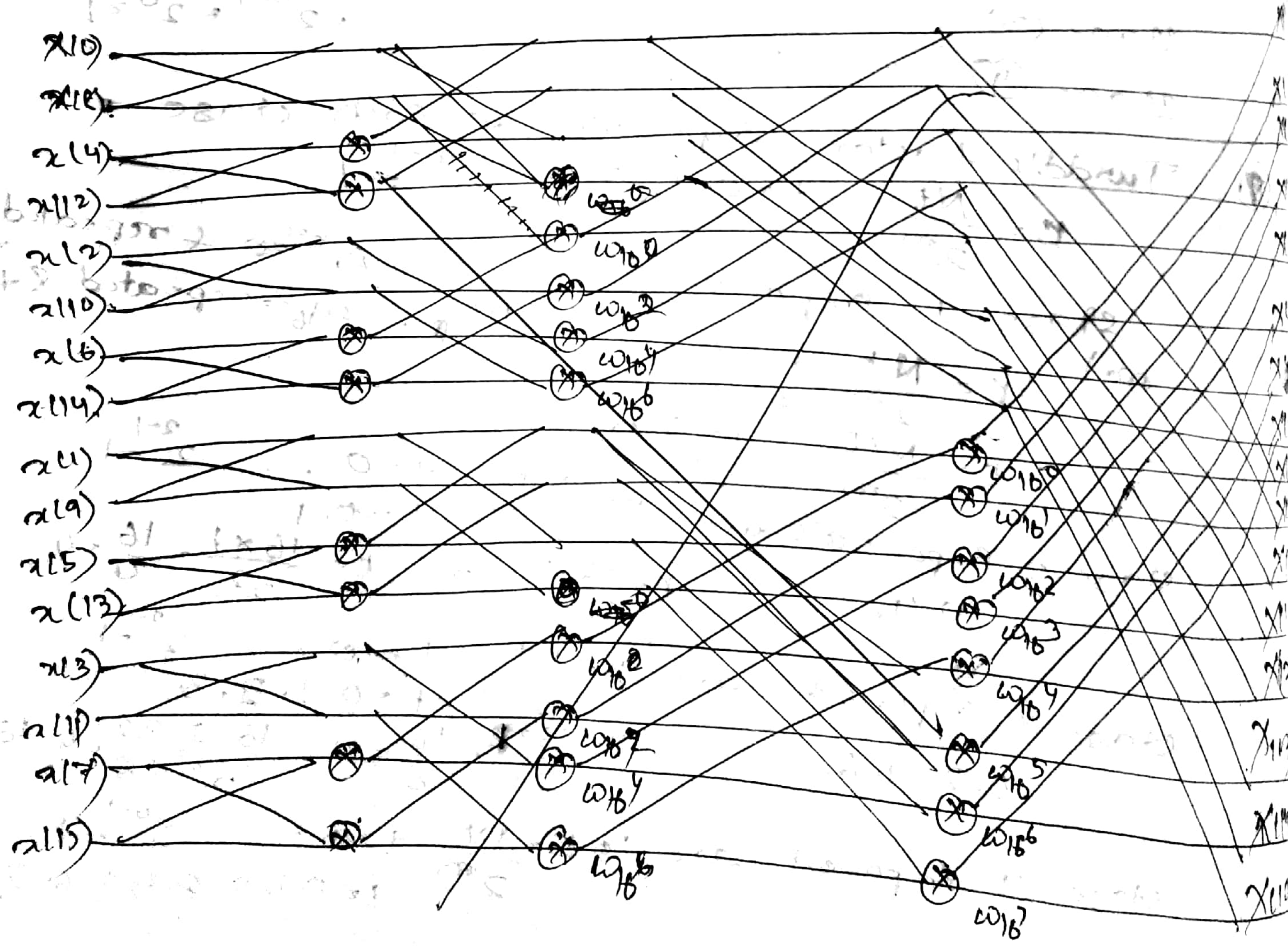
stage (3) $2^{4-3} = 2^1 = 2$

stage (4) $2^{4-4} = 2^0 = 1$.

At stage 2 twiddle factor exponents are 0, 4. & ERF = 4. All 4 sets of butterflies. Each set consists of two butterflies having twiddle factor W_{16}^0, W_{16}^4.

stage (3) Exponents are 0, 2, 4, 6. ERF = 2. Each set of 4 butterfly sequence repeats 2 times. having twiddle factors $W_{16}^0, W_{16}^2, W_{16}^4, W_{16}^6$.

Stage (4) ERF = 1 & one set of butterfly consists twiddle factors $W_{16}^0, W_{16}^1, W_{16}^2, W_{16}^3, W_{16}^4, W_{16}^5, W_{16}^6, W_{16}^7$.



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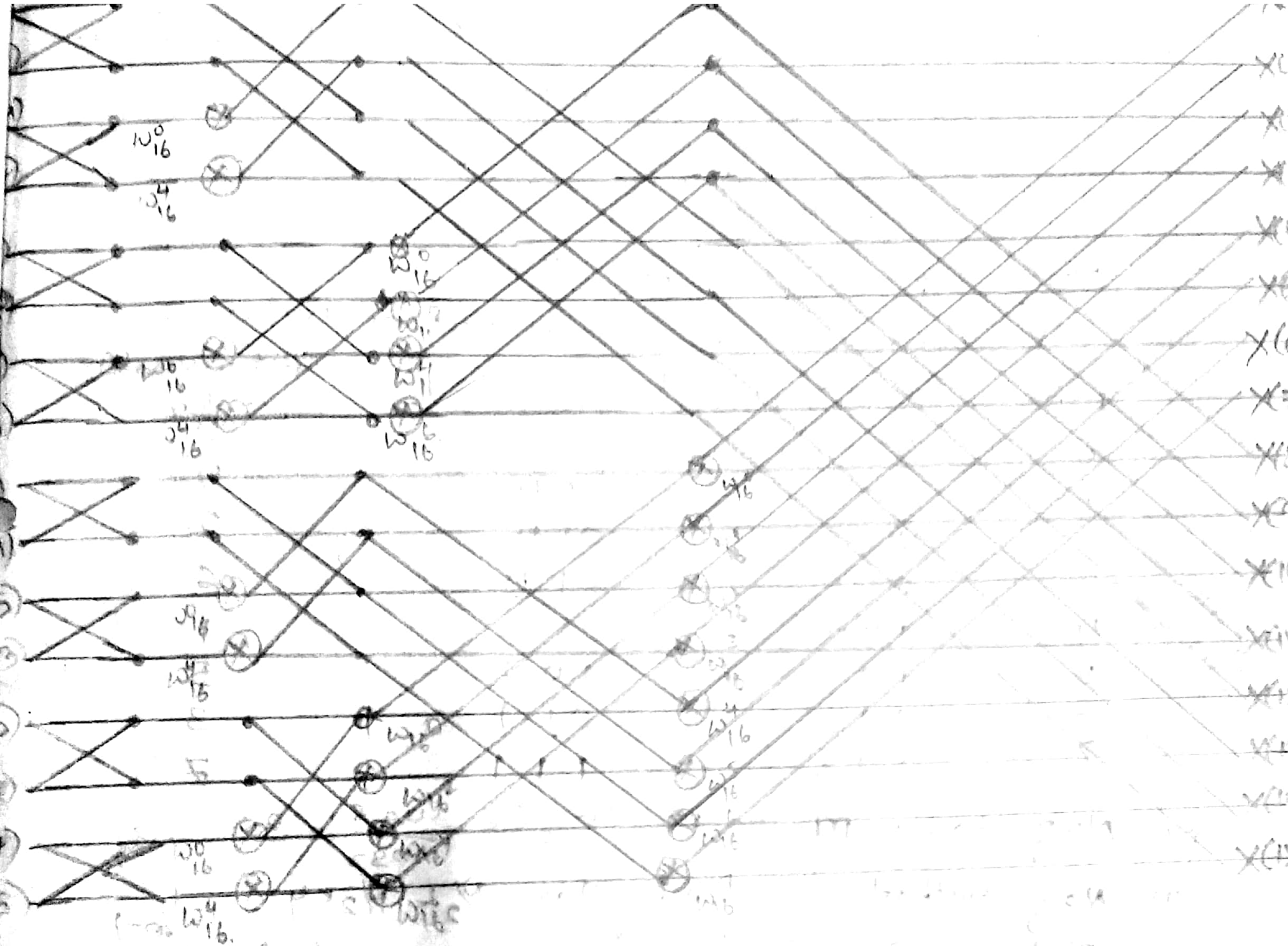
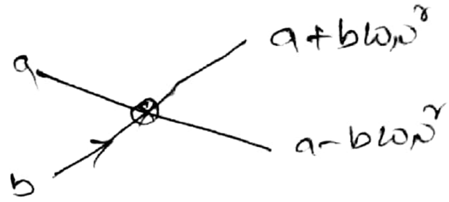

W_{16}^{0}
W_{16}^{4}
W_{16}^{6}

Ex: Find 8 point DFT of the sequence $x(n) = (1,2,3,4,4,3,2,1)$ using DIT-FFT radix-2 algorithm. The basic computation block known as the butterfly should be used as shown below:



(1) No of I/P sample $N = 2^m = 8$

(2) I/P sequence with bit reversed order

Index	Binary repr^n	Bit reversed order	Bit reversed order
0 = 1	000	000	0
1 = 2	001	100	4
2 = 3	010	010	2
3 = 4	011	110	6
4 = 4	100	001	1
5 = 3	101	101	5
6	110	011	3
7	111	111	7

1.2.3.12.14.15
19.21.25.31
32.36.0

(3) No of stages $M = \log_2 N = \log_2 8 = \frac{\log 8}{\log 2} = 3$

(4) No of butterflies at each stage $N/2 = 8/2 = 4$

(5) I/P/O/P for each butterfly are separated by 2^{m-1}

stage 1 I/P/O/P for each butterfly are separated by $2^{1-1} = 2^0 = 1$

stage 2 " " " " " " " $= 2^{2-1} = 2^1 = 2$

stage 3 " " " " " " " $2^{3-1} = 2^2 = 4$

(6) No of complex mul^n are $\frac{N}{2}\log_2 N = \frac{8}{2}\log_2 8 = 12$

(7) No of complex add^ns are $N\log_2 N = 8\log_2 8 = 24$

(8) No of sets of butterflies in each stage 2^{M-m}

stage (1) no of sets of butterflies are $2^{M-m} = 2^{3-1} = 4$

stage (2) " " " " " $2^{M-m} = 2^{3-2} = 2$

stage (3) " " " " " $= 2^{3-3} = 1$

(9) Twiddle factor exponents are in each stage & $K = \frac{Nt}{2^m}$ ... $t = 0, 1 \ldots 2^{m}$

stage (1) exponents are 0 ∴ $t = 0, 1 \ldots 2^{1-1}-1 = 0$

$K = \frac{Nt}{2^m} = \frac{0}{2^1} = 0$

stage (2) $k = \frac{Nt}{2^m} = 0$ $\quad t = 0, 1, \ldots, 2^{m-1}-1$

$t=0$ $\quad t=1,\ k = \frac{8\times 1}{2^2} = \frac{8}{4} = 2$ $\quad = 0, 1.$

$t = 0, 1, \ldots 2^{3-1}-1$

stage (3) $k = 0$ $\quad$ ~~t=1~~ $t=1,\ k = \frac{8\times1}{2^3} = 1$ $\quad = 0, 1, 2, 3$

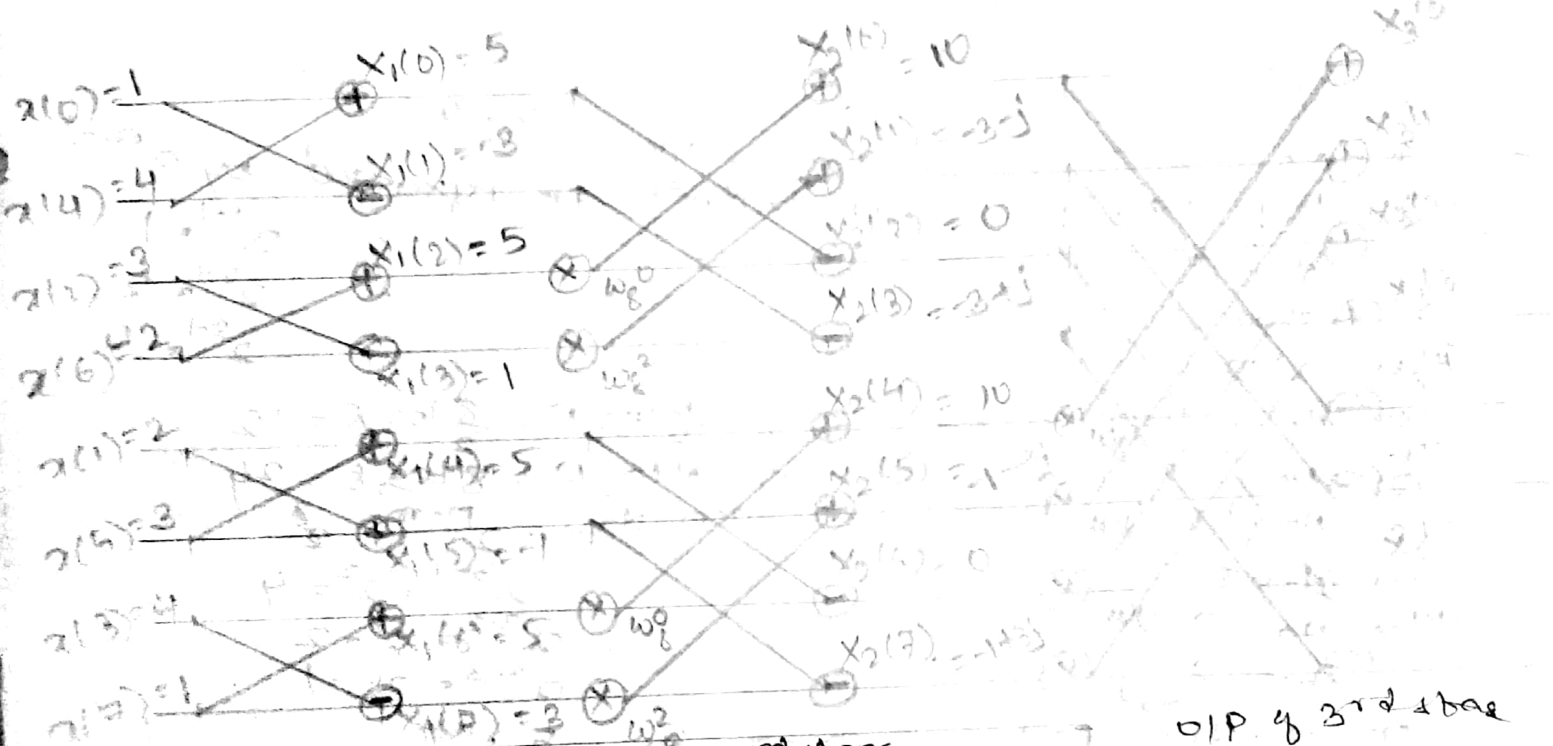
$t=0$

$t=2,\ k = \frac{8\times2}{2^3} = \frac{16}{8} = 2$ $\quad t=3,\ k = \frac{8\times3}{2^3} = \frac{24}{8} = 3$

(10) ERF stage (1) $2^{m-m} = 2^{3-1} = 4$

stage (2) $2^{3-2} = 2$

stage (3) $2^{3-3} = 2^0 = 1.$



o/p of 1st stage

$X_1(0) = 1 + 4\times w_8^0 = 5$

$X_1(1) = 1 - 4\times w_8^0 = -3$

$X_1(2) = 3 + 2\times w_8^0 = 5$

$X_1(3) = 3 - 2\times w_8^0 = 1$

$X_1(4) = 2 + 3\times w_8^0 = 5$

$X_1(5) = 2 - 3\times w_8^0 = -1$

$X_1(6) = 4 + 1\times w_8^0 = 5$

$X_1(7) = 4 - 1\times w_8^0 = 3$

o/p of 2nd stage

$X_2(0) = 5 + 5\, w_8^0 = 10$

$X_2(1) = -3 + 1\, w_8^2 = -3-j$

$X_2(2) = 5 - 5 w_8^0 = 0$

$X_2(3) = -3 - 1 w_8^2 = -3+j$

$X_2(4) = 5 + 5 w_8^0 = 10$

$X_2(5) = -1 + 3 w_8^2 = -1-3j$

$X_2(6) = 5 - 5 w_8^0 = 0$

$X_2(7) = -1 - 3w_8^2 = -1+3j$

o/p of 3rd stage

$X_3(0) = 10 + 10 w_8^0 = 20$

$X_3(1) = (-3-j) + (-1-3j) w_8^1 =$

$= (-3-j) + (-1-3j)(0.707 - j0.707)$

$= -3 - j - 0.707 + j0.707 - 2.121j + 2.121j$

$= \cancel{-3.707 - 0.293j}$

$= -5.828 - 2.414j$

$X_3(2) = 0 +$

$X_3(3) = (-3+j) + (-1+3j)\times(-0.707 - j0.707)$

$= -3 + j + 0.707 + 0.707j + 2.121j + 2.121$

$= -0.172 - 0.414j$

$w_8^5 = e^{-j5\pi/4} = -0.707 + j\,0.707$

$w_8^6 = e^{-j6\pi/4} = e^{-j3\pi/2} = +j$

$w_8^7 = e^{-j7\pi/4} = 0.707 + j\,0.707$

$w_8^0 = e^{\frac{-j2\pi t}{N}} = 1$

$w_8^1 = e^{\frac{-j2\pi}{8}} = e^{-j\pi/4} = 0.707 - j0.707$

$w_8^2 = e^{\frac{-j4\pi}{8}} = e^{-j\pi/2} = -j$

$w_8^3 = e^{\frac{-j6\pi}{8}} = e^{-j3\pi/4} = -0.707 - j0.707$

$w_8^4 = e^{-j\pi} = -1$

$X_3(4) = 10 - 10 w_8^0 = 0$

$X_3(5) = (-3-j) - (-1-3j)(0.707 - j0.707) = 0.172 + j0.414j$

$X_3(6) = 0 - 0 = 0$

$X_3(7) = (-3+j) - (-1+3j)(-0.707 - j0.707) = -5.828 + j2.414j$

DEC 08/09

12 M

Determine 8 point DFT of $x(n) = \{1, 0, -1, 2, 1, 1, 0, 2\}$ using radix-2 DIT FFT algorithm. Show clearly all the intermediate results.

(1) No of I/P samples N = 8

(2) I/P sequence with bit reversal order

0	000	000	0 = 1
1	001	100	4 = 1
2	010	010	2 = -1
3	011	110	6 = 0
4	100	001	1 = 0
5	101	101	5 = 1
6	110	011	3 = 2
7	111	111	7 = 2

(3) No of stage $m = \log_2 N = 3$ $8/2 = 4$

(4) No of butterflies at Each stage

(5) ~~No of~~ I/P/O/P for Each butterfly are separated by 2^{m-1}

stage ① " " " " " " $2^{1-1} = 1$

stage ② " " " " " " $2^{2-1} = 2$

stage ③ " " " " " " $2^{3-1} = 4$

(6) No of complex multⁿ's are $\frac{N}{2}\log_2 N = \frac{8}{2}\log_2 8 = 12$

(7) No of complex addⁿs are $N\log_2 N = 8\log_2 8 = 24$

(8) No of set of butterflies in Each stage $= 2^{M-m} = 2^{3}$

stage ① " " " " " $2^{3-1} = 2^2 = 4$

stage ② " " " " " $2^{3-2} = 2 = 2$

stage ③ " " " " " $2^{3-3} = 2^0 = 1$

(9) twiddle factor Exponents are $k = \frac{Nt}{2^m}$

stage ① $k = 0$ $t = 0$

stage ② $k = 0, 2$ $t = 0, 1$

stage ③ $k = 0, 1, 2, 3$ $t = 0, 1, 2, 3$

(10) ERF Exponent repeat factor

2^{M-m}

stage ① $= 2^{3-1} = 2^2 = 4$

stage ② $= 2^{3-2} = 2 =$

stage ③ $= 2^{3-3} = 2^0 = 1$

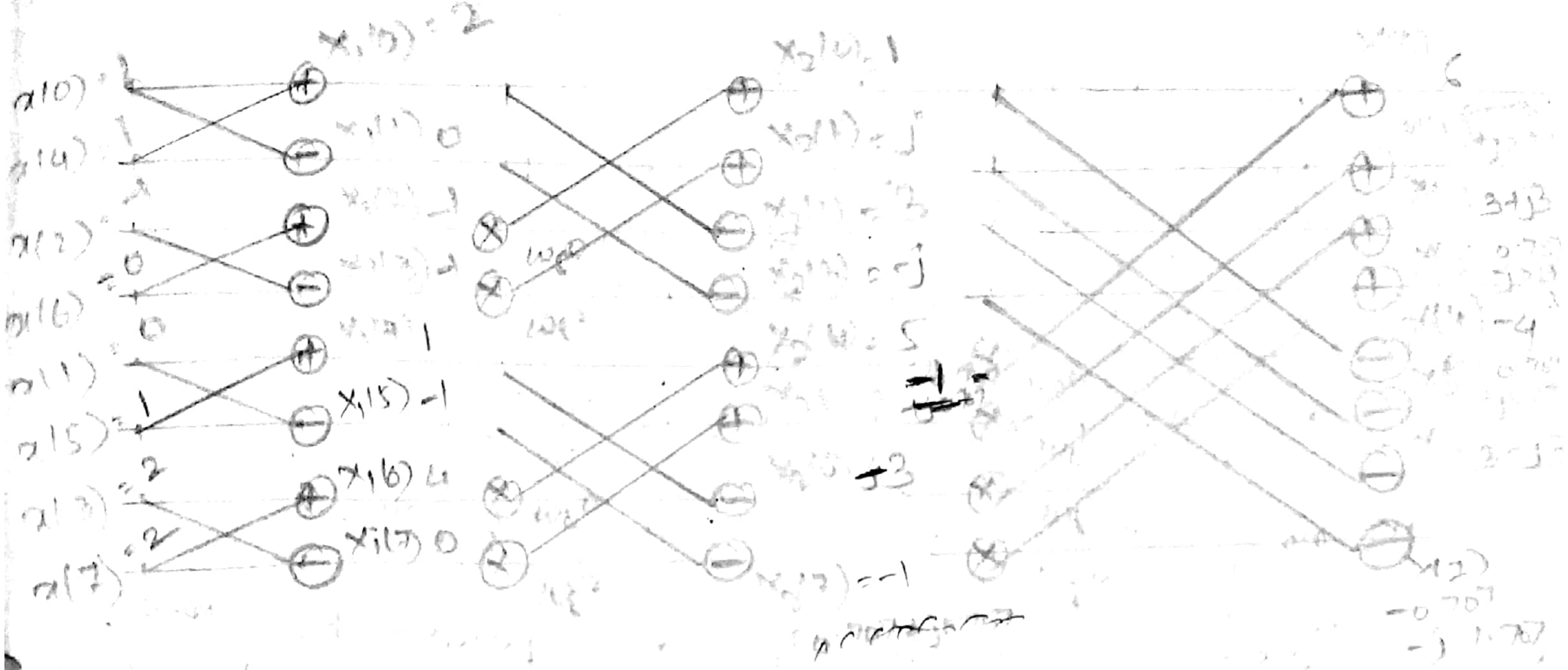
Twiddle factors

$W_8^0 = e^{\frac{-j2\pi r}{N}} = e^{-j\frac{\pi}{4}r} = 1$

$W_8^1 = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4 = 0.707 - j0.707$

$W_8^2 = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$

$W_8^3 = e^{-j3\pi/4} = \cos 3\pi/4 - j\sin 3\pi/4 = -0.707 - j0.707$



$X_2(0) = 2 + -1\,W_8^0 = 1$

$X_2(1) = 0 + -1\,W_8^2 = -1(-j) = j$

$X_2(2) = 2 - (-1)\,W_8^0 = 3$

$X_2(3) = 0 - (-1)\,W_8^2 = +1(-j) = -j$

$X_2(4) = 1 + 4\,W_8^0 = 5$

$X_2(5) = -1 + 0\,W_8^2 = -1$

$X_2(6) = +1 - (4 \times W_8^0) = -3$

$X_2(7) = -1 - (0 \times j) = -1$

$X_1(0) = 1 + 1 = 2$

$X_1(1) = 1 - 1 = 0$

$X_1(2) = -1 + 0 = -1$

$X_1(3) = -1$

$X_1(4) = 0 + 1 = 1$

$X_1(5) = 0 - 1 = -1$

$X_1(6) = 2 + 2 = 4$

$X_1(7) = 2 - 2 = 0$

$X_3(0) = 1 + (+5)\,W_8^0 = 6$

$X_3(1) = j - 1(W_8^1) = j - 1(0.707 - j0.707) = j - 0.707 + j0.707 = -0.707 + j1.707$

$X_3(2) = 3 - 3\,W_8^2 = 3 - 3(-j) = 3 + 3j$

$X_3(3) = -j - 1\,W_8^3 = -j - 1(-0.707 - j0.707) = -j + 0.707 + j0.707$

$= 0.707 - 0.293j$

$X_3(4) = 1 - 5\,W_8^0 = -4$

$X_3(5) = j - (-1)\,W_8^1 = j - (-1(0.707 - j0.707)) = j + 0.707 - j0.707 = 0.707 + j0.293$

$= j - (-0.707 + j0.707)$

$X_3(6) = 3 - (-3 \times W_8^2) = 3 - (-3 \times -j) = 3 - 3j$

$X_3(7) = -j - (-1 \times W_8^3) = -j - (-1 \times (-0.707 - j0.707)) = -j - 0.707 - j0.707$

$= -j - (+0.707 + j0.707)$

$= -0.707 - j1.707$

Compute 4 point DFT of the sequence $x(n) = (1, 0, 1, 0)$ using DIT-FFT radix-2 algorithm

(1) No of I/P samples $N = 4$

(2) I/P sequence with bit reversed order

0	00	00	0	$x(0) = 1$
1	01	10	2	$x(2) = 1$
2	10	01	1	$x(1) = 0$
3	11	11	3	$x(3) = 0$

3) No of stages $M = \log_2 N = \log_2 4 = \frac{\log 4}{\log 2} = 2$

(4) No of butterflies at each stage $N/2 = 4/2 = 2$

(5) I/p o/p for each butterfly are separated by 2^{m-1}

Stage ① " " " " $2^{1-1} = 2^0 = 1$

Stage ② " " " " $2^{2-1} = 2^1 = 2$

(6) No of complex mult^n are $\frac{N}{2}\log_2 N = \frac{4}{2}\log_2 4 = 2 \times 2 = 4$

(7) No of complex add^n are $N\log_2 N = 4 \times 2 = 8$

(8) No of sets of butterflies in each stage $2^{M-m} = 2^2$

Stage ① " " " " $2^{2-1} = 2^1 = 2$

Stage ② " " " $2^{2-2} = 2^0 = 1$

(9) Twiddle factor exponents are $k = \frac{Nt}{2^m}$ $t = 0, 1, \ldots 2^{m-1}$

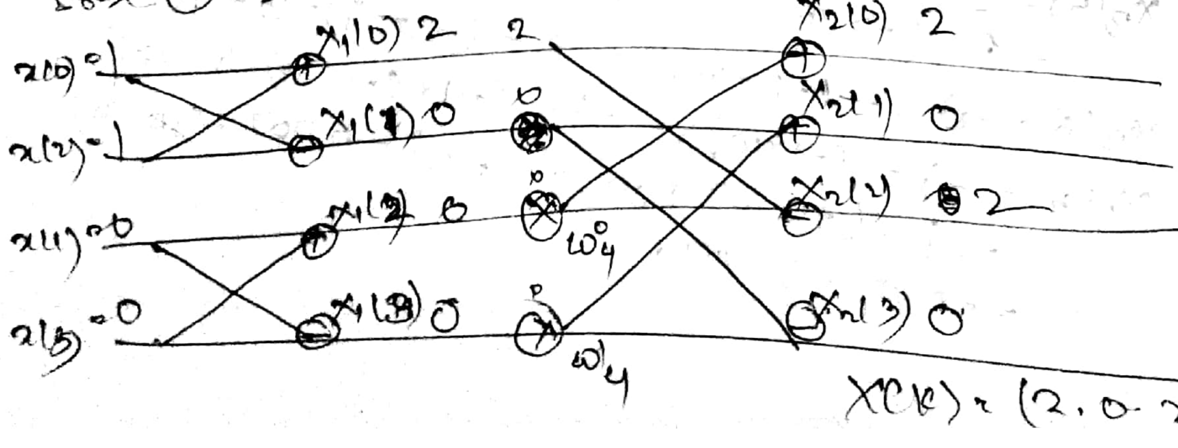
Stage ① $t = 0, 1 \ldots 2^{1-1}$ $t = 0$ $k = \frac{4 \times 0}{2^1} = 0$

stage ② $t = 0, 1 \ldots 2^{2-1} = t = 0, 1.$ $k = \frac{4 \times 0}{2^2} = 0$. $k = \frac{4 \times 1}{2^2} = 1$

(10) BPF 2^{M-m} Twiddle factors

stage ① $2^{2-1} = 2$ $W_4^0 = 1$ $W_4^1 = e^{-j2\pi \times 1/4} = e^{-j\pi/2} = -j$

stage ② $2^{2-2} = 1$



$X(k) = (2, 0, 2, 0)$

Decimation in frequency (DIF) Algorithm

This algorithm is formed by the decomposition of DFT computation by forming smaller & smaller sequences of $x(n)$. Here o/p sequence $X(k)$ is divided into smaller & smaller subsequences. In this algorithm i/p sequence $x(n)$ is partitioned into two sequences each of length $N/2$ samples.

$x_1(n)$ consists of first $N/2$ samples of $x(n)$
$x_2(n)$ " " last $N/2$ " of $x(n)$

$x_1(n) = x(n)$, $n = 0, 1, 2 \ldots N/2 - 1$
$x_2(n) = x(n + N/2)$, $n = 0, 1, 2 \ldots N/2 - 1$

consider $N = 8$ & $x_1(n)$ has values $0 \leq n \leq 3$
& $x_2(n)$ has $4 \leq n \leq 7$

$W_N^{nk} = e^{-j2\pi nk/N}$
$= e^{-j2\pi k}$

∴ N point DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=N/2}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{(n+N/2)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + W_N^{\frac{Nk}{2}} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) \cdot W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + e^{-j\pi k} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{nk}$$

$$\boxed{W_N^{\frac{Nk}{2}} = W_N^{nk} = e^{-j2\pi \frac{N}{2} k / N} \qquad W_N^{\frac{N}{2}k} = e^{-j\pi k}}$$

for even values of k, $e^{-j\pi k} = 1$ always, $k = 0, 2, 4, 6 \ldots$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)] W_N^{2kn}$$

$$\boxed{W_N^2 = W_{N/2}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)] W_{N/2}^{nk} \quad \text{--- (1)}$$

$$\boxed{\because W_N^2 = W_{N/2}.}$$

Eq (1) is $\frac{N}{2}$ point DFT of $\frac{N}{2}$ point sequence obtained by adding 1st & last $\frac{N}{2}$ of i/p sequence

2, 2, 20, 36, 43, 44, 54, 63, 67

when k is odd $e^{-j\pi k} = -1$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} (x_1(n) - x_2(n))\, W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} (x_1(n) - x_2(n)) \cdot W_N^{2kn} \cdot W_N^{n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} (x_1(n) - x_2(n))\, W_N^{n} \cdot W_{N/2}^{nk} \quad \text{—(2)}$$

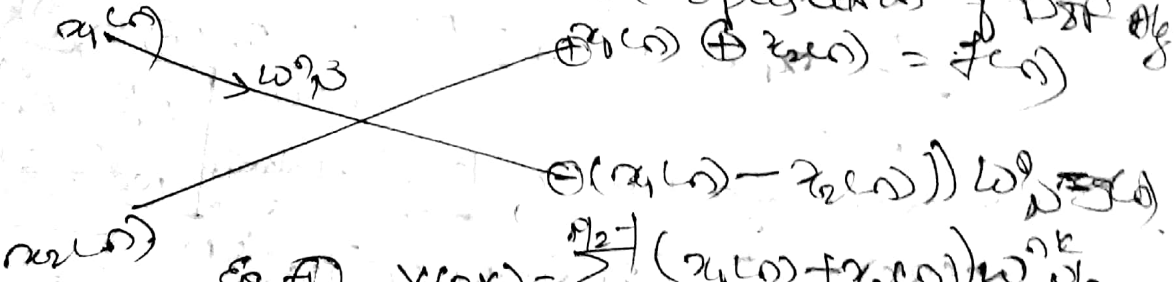
Eq —(2) is $\frac{N}{2}$ pt DFT of sequence obtained by subtracting second half of the O/P sequence from 1st half & multiplying the resulting sequence by W_N^n

Eq (1) & (2) shows the even & odd samples of DFT is obtained from $\frac{N}{2}$ pt DFTs of $f(n)$ & $g(n)$ resp.

$f(n) = x_1(n) + x_2(n)$, $n = 0, 1, 2 \ldots \frac{N}{2}-1$

$g(n) = (x_1(n) - x_2(n))\, W_N^n$ $n = 0, 1, \ldots \frac{N}{2}-1$

Fig shows the basic operation of DIF alg



from Eq (1) $X(2k) = \sum_{n=0}^{N/2-1} (x_1(n) + x_2(n))\, W_{N/2}^{nk}$ or W_N^{2nk}

$$X(0) = \sum_{n=0}^{3} (x_1(n) + x_2(n))\, W_{N/2}^{0} = \sum_{n=0}^{3} f(n)$$

$$= f(0) + f(1) + f(2) + f(3)$$

$$X(2) = \sum_{n=0}^{3} (x_1(n) + x_2(n))\, W_{N/2}^{2n} = \sum_{n=0}^{3} f(n)\, W_{N/2}^{2n}$$

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$$X(2) = \sum_{n=0}^{3} f(0).W_8^0 + f(1)W_8^2 + f(2)W_8^4 + f(3)W_8^6$$

$$= f(0) + f(1)W_8^2 - f(2) - f(3)W_8^2$$

$W_8^4 = e^{-j\frac{2\pi \times 4}{8}} = -1$

$W_8^6 = e^{-j\frac{2\pi \times 3}{8/2}} = e^{-j\frac{3\pi}{2}}$

$W_N^{k+N/2} = -W_N^k$ $\quad e^{-j\frac{2\pi\times 2}{8}}$

$W_8^{2+4} = W_8^6 = -W_8^2$ $\quad W_8^{12} = -W_8^8 = -e^{-j\frac{2\pi\times 8}{8}}$

$$X(4) = \sum_{n=0}^{3} f(n).W_8^{4n} = f(0)W_8^0 + f(1)W_8^4 + f(2)W_8^8 + f(3)W_8^{12}$$

$$= f(0) + f(1) + f(2) - f(3) = \sum_0^3 f(n)(-1)^n$$

$$X(6) = \sum_{n=0}^{3} f(n)W_8^{6n} = f(0) + f(1)W_8^6 + f(2)W_8^{12} + f(3)W_8^{18}$$

$$= \sum_0^3 f(n)(-W_8^2)^n$$

$$= f(0) - f(1)W_8^2 - f(2) + f(3)W_8^2$$

$W_8^{12} = W_8^4 = e^{-j\frac{2\pi\times 4}{8}} = e^{-j\pi} = -1$

from Eqn (2)

$g(n) = [x_1(n) - x_2(n)]W_N^n$

$$X(2k+1) = \sum_{n=0}^{N/2-1} g(n)W_N^{2kn}$$

$$X(1) = \sum_{n=0}^{3}(x_1(n) - x_2(n))W_8^n = \sum_0^3 g(n) = g(0)+g(1)+g(2)+g(3)$$

$$X(3) = \sum_{n=0}^{3}(x_1(n) - x_2(n))W_N^{2n}.W_N^n = \sum_{n=0}^{3}(x_1(n)-x_2(n))W_N^{3n}$$

$$= \sum_{n=0}^{3} g(n) W_8^{2n} = g(0) + g(1)W_8^2 - g(2) - g(3)W_8^2$$

$$X(5) = \sum_{n=0}^{3}(x_1(n) - x_2(n))W_8^{5n} = \sum_{n=0}^{3} g(n)W_8^{4n} = \sum_{n=0}^{3} g(n)(-1)^n$$

$$= g(0) - g(1) + g(2) - g(3)$$

$$X(7) = \sum_{n=0}^{3}(x_1(n) - x_2(n))W_8^{7n} = \sum_{n=0}^{3} g(n).(-W_8^3)^n$$

$$= g(0) - g(1)W_8^2 - g(2) + g(3)W_8^2$$

Even and Odd indexed sequence samples of x(n) can be obtained by two 4-point DFT of the sequence

Even Indexed seq. — $n = 0, 1, \ldots \frac{N}{2}-1$

$\therefore f(n) = x_1(n) + x_2(n)$

$f(0) = x_1(0) + x_2(0)$

$f(1) = x_1(1) + x_2(1)$

$f(2) = x_1(2) + x_2(2)$

$f(3) = x_1(3) + x_2(3)$

Odd Indexed sequence

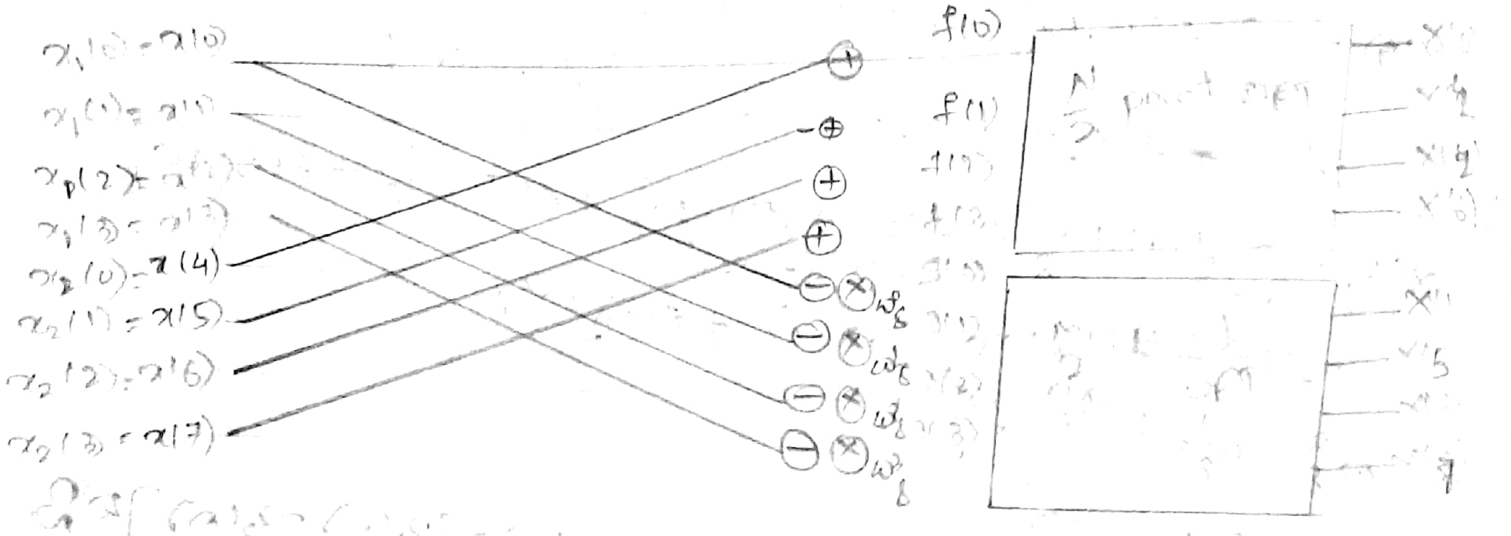
$g(n) = (x_1(n) - x_2(n))\, W_8^n$

$g(0) = (x_1(0) - x_2(0))\, W_8^0$

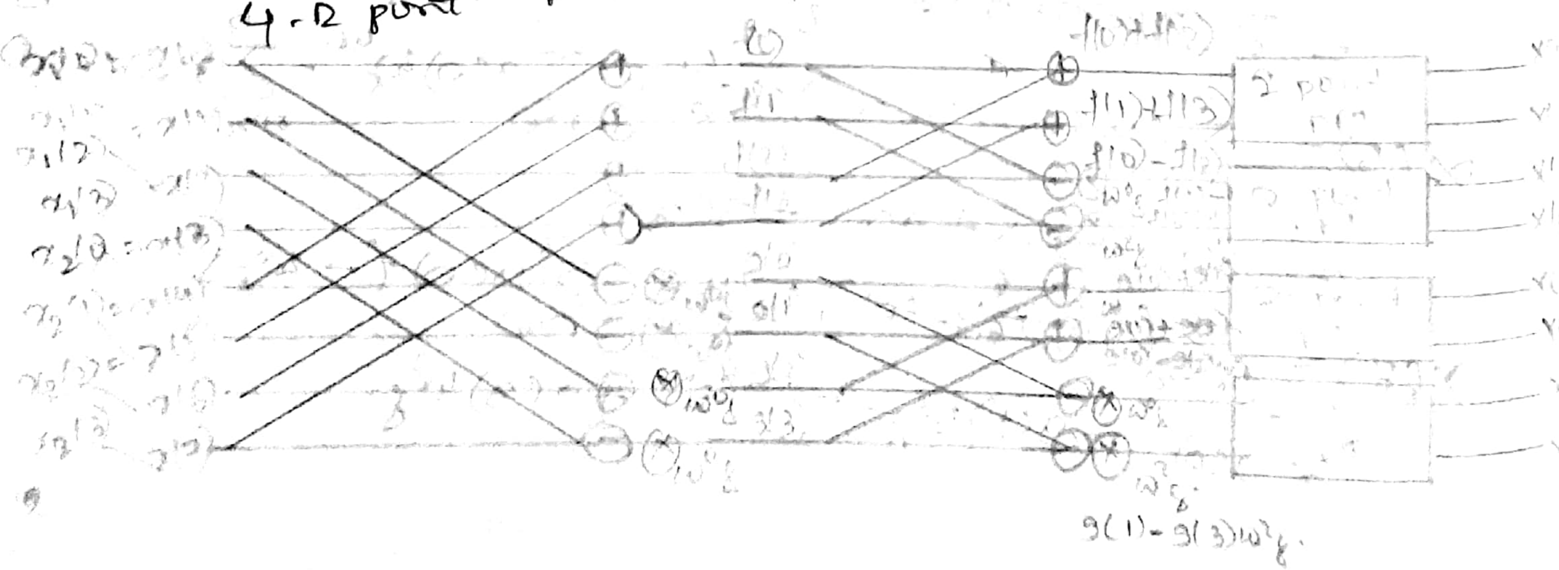
$g(1) = (x_1(1) - x_2(1))\, W_8^1$

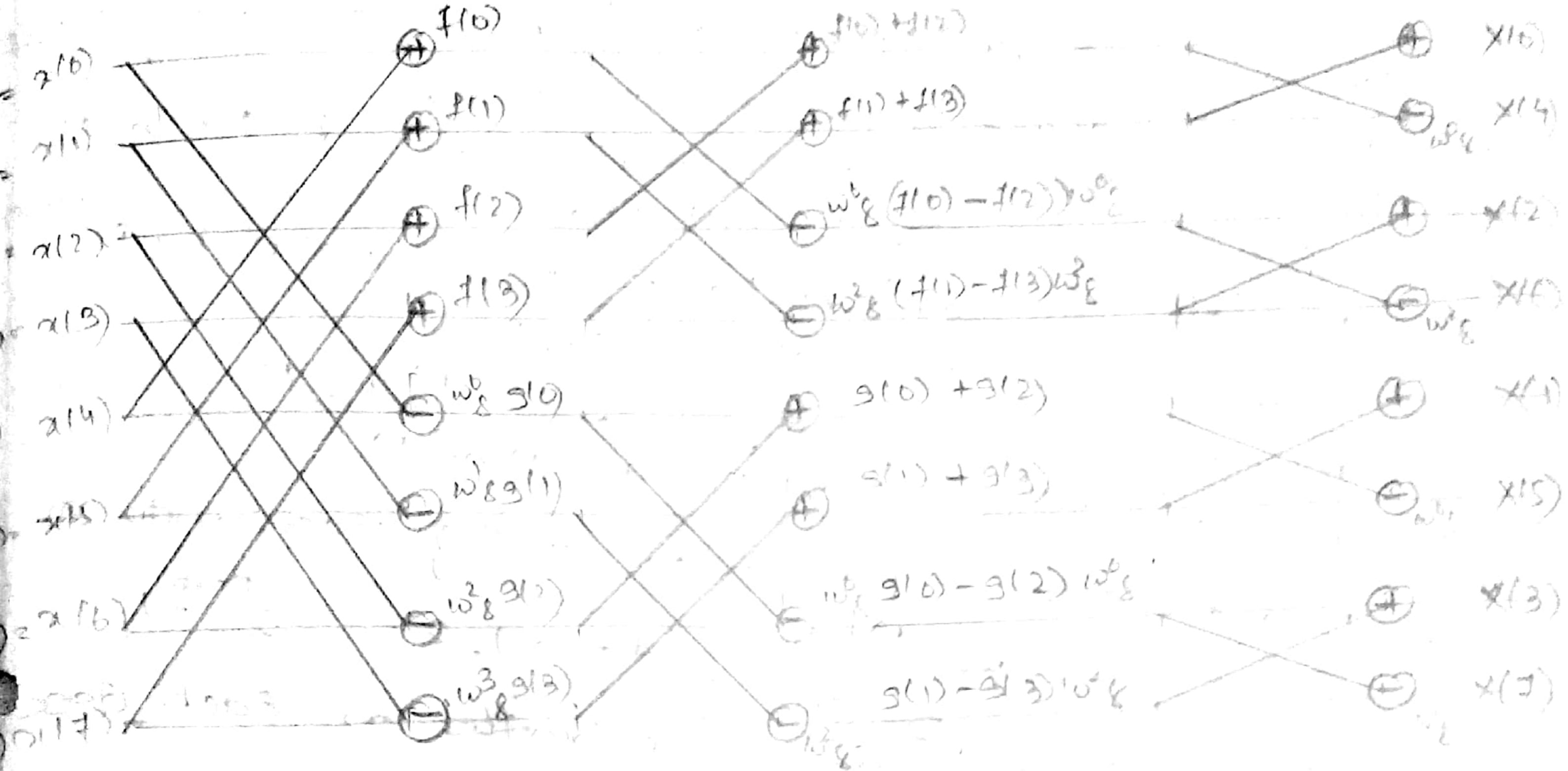
$g(2) = (x_1(2) - x_2(2))\, W_8^2$

$g(3) = (x_1(3) - x_2(3))\, W_8^3$



$\frac{N}{2}$ point DFT can be computed by combining first half & last half of the i/p points for each of the $\frac{N}{2}$ point DFT's. Same procedure for $\frac{N}{4}$ point DFT. Thus 2-4 point DFTs can be converted into 4-2 point DFTs by adding & subtracting i/p's





$X(0) = f(0)+f(2)+f(1)+f(3) = X(0)$

$f(0)+f(2) + f(1)W^0_8 - f(3)W^0_8 = X(4)$.

$f(0)W^0_8 - f(2)W^0_8 + f(1)W^2_8 - f(3)W^2_8 = X(2)$

$f(0) + f(1)W^2_8 - f(2) - f(3)W^2_8$

$f(0) - f(2) - f(1)W^2_8 + f(3)W^2_8$

$f(0) - f(1)W^2_8 - f(2) + f(3)W^2_8 = X(6)$

$g(0) + g(2) + g(1) + g(3) = X(1)$

$g(0) + g(2) - g(1) - g(3) = X(5)$

$g(0) - g(1) + g(2) - g(3)$

$g(0)W^0_8 - g(2)W^0_8 + g(1)W^2_8 - g(3)W^2_8 = X(3)$

$g(0) + g(1)W^2_8 - g(2) + g(3)W^2_8$

$g(0) - g(2) - g(1)W^2_8 + g(3)W^2_8 = X(7)$

$g(0) - g(1)W^2_8 - g(2) + g(3)W^2_8$

from above graph we can observe that D.I.F reversal i/p sequence & natural but o/p freq are in bit reversal order, thus no of complex mul^ns are $\frac{N}{2}\log_2 N$ & no of complex addition are $N\log_2 N$ to Evaluate $X(k)$. ∴ No of computations & these also same are same for DIT & DIF Algorithms & need to store i/p & o/p sequence.

Steps for Radix - 2 DIF - FFT Algorithm

(1) The number of I/P samples $N = 2^m$, where M is no of stages given by $M = \log_2 N$.

(2) The I/P sequence is in natural order.

(3) The no of stages in flow graph is given by $M = \log_2 N$.

(4) Each stage consists of $\frac{N}{2}$ butterflies.

(5) I/P's / O/P's for each butterfly are separated by 2^{M-m} samples, where m is stage index.

(6) The no of complex multipl's are $\frac{N}{2}\log_2 N$.

(7) The no of complex add's are $N\log_2 N$.

(8) Twiddle factor exponents are func of stage index m & is given by $k = \frac{Nt}{2^{M-m+1}}$, $t = 0, 1, 2 \ldots 2^{M-m}-1$.

(9) No of sets or sections of butterflies in each stage is given by 2^{m-1}.

(10) ERF (Exponent repeat factor) is no of times the exponent sequence associated with m is repeated by 2^{m-1}.

Ex: Find DFT of a sequence $x(n) = \{1, 2, 3, 4, 4, 3, \ldots\}$ using DIF Algorithm.

$8 = 2^m$. M

$M = \log_2 N = \log_2 8 = 3$

Each consists of $N/2$ butterflies

I/P & O/P are separated by 2^{M-m}

stage ① $m=1$ $\quad 2^{3-1} = 2^2 = 4$

② $m=2$ $\quad 2^{3-2} = 2^1 = 2$

③ $m=3$ $\quad 2^{3-3} = 2^0 = 1$

No of complex multipl's are $\frac{N}{2}\log_2 N$

$\frac{8}{2}\log_2 8 = 4 \times 3 = 12$

complex additions $N\log_2 N = 8 \times 3 = 24$

Twid

Twiddle factor exponents are func of stage index m

$k = \frac{Nt}{2^{n-m+1}}$; $t = 0, 1, 2 \ldots 2^{n-m} - 1$

stage ① $m = 1$ $t = 0, 1, 2 \ldots 2^{3-1} - 1$

$t = 0, 1, 2, 3$

$t = 0$ $k = \frac{8 \times 0}{2^{3-1+1}} = 0$

$t = 1$, $k = \frac{8 \times 1}{2^{3-1+1}} = 1$ $t = 2$, $k = \frac{8 \times 2}{8} = 2$, $t = 3$, $k = \frac{8 \times 3}{8} = 3$

stage ② $m = 2$ $t = 0, 1, 2 \ldots 2^{3-2} - 1$, $t = 0, 1$

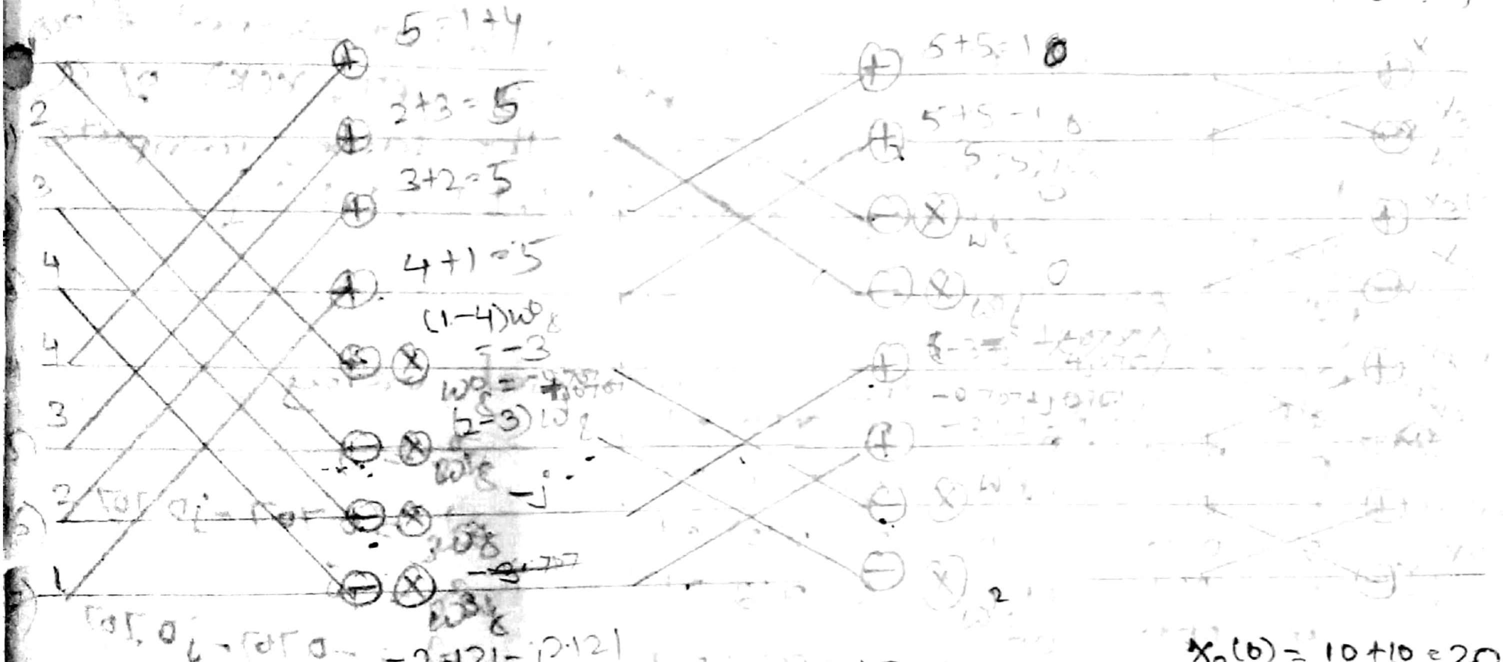
$t = 0$ $k = \frac{8 \times 0}{2^{3-2+1}} = 0$ $t = 1$. $k = \frac{8 \times 1}{2^2} = 2$

stage ③ $t = 0, 1, 2 \ldots 2^{3-3} - 1$ $t = 0$

$k = 0$ at $t = 0$ $W^0_8 = 1$, $W^1_8 = e^{\frac{-j2\pi \cdot 1}{8}} = e^{-j\pi/4} = 0.707 - j0.707$

$W^2_8 = e^{-j4\pi/8} = e^{-j\pi/2} = -j$, $W^3_8 = e^{-j6\pi/8} = e^{-j3\pi/4}$

$= -0.707 - j0.707$



$X_1(0) = 1 + 4 = 5$

$X_1(1) = 2 + 3 = 5$

$X_1(2) = 3 + 2 = 5$

$X_1(3) = 4 + 1 = 5$

$X_1(4) = (1-4)W^0_8 = -3$

$X_1(5) = (2-3)W^1_8 = -$

$= -1(0.707 - j0.707)$

$= -0.707 + j0.707$

$X_1(6) = (3-2)W^2_8 = 1 \times -j = -j$

$= 1$

$X_1(7) = (4-1)W^3_8 =$

$= 3(-0.707 - j0.707)$

$= -2.121 - j2.121$

$X_2(0) = 5 + 5 = 10$

$X_2(1) = 5 + 5 = 10$

$X_2(2) = (5-5)W^0_8 = 0$

$X_2(3) = (5-5)W^2_8 = 0$

$X_2(4) = -3 - j$

$X_2(5) = -0.707 + j0.707$

$-2.121 - j2.121$

$= -2.828 - j1.414$

$X_2(6) = -3 + j$

$X_2(7) = ((-0.707 + j0.707)$

$+2.121 + 2.12j) - j$

$= 2.828 + j1.414$

$X_3(0) = 10 + 10 = 20$

$X_3(1) = 10 - 10 = 0$

$X_3(2) = 0 + 0 = 0$

$X_3(3) = 0 + 0 = 0$

$X_3(4) =$

$= -5.828 - 2.414j$

$X_3(5) = -0.172 + 2.414j$

$X_3(6) = -3 + j + 2.828$

$+ j1.414$

$X_3(7) = -3 + j - 2.828$

$+ j1.414$

$= -5.828 + 2.414j$

$X_3(5) = (-3 - j - (-2.828 - j1.414))$

$= -3 - j + 2.828 + j1.414$

$= -0.172 + 0.414j$

$X_3(6) = ((-3 + j) + 2.828 - j1.414) W^0_8$

$= -3 + j + 2.828 - j1.414$

$-0.172 - 0.414j$

$X_3(7) = (-3 + j) - 2.828 + j1.414$

$= -5.828 + j2.414$

$X_3(k) = \{20, 0, 0, 0, -5.828 - j2.414, -0.172 + j0.414, -0.1722 - j0.414, -5.828 + j2.414\}$

$x(0)\ x(4)\ x(2)\ x(6)\ x(1)\ x(5)\ x(3)\ x(7)$

$X(K) = \{X(0), X(4), X(2), X(6), X(1), X(5), X(3), X(7)\}$

$= \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$

Ex DEC 09/10 (30) 14 marks

Compute the Eight point DFT of the sequence

$x(n) = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & \text{otherwise} \end{cases}$ by using DIT & DIF Algo

$x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

Develop decimation in frequency (DIF) FFT Algorithm with all necessary steps and neat signal flow dgm used in computing N-point DFT, X(K) of a N-point sequence x(n). Using the same, compute the DFT of sequence $x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$.

DIT - Algorithm

(1) $N = 8$

(2) I/P sequence with bit reversal order

0	000	000	0	$x(0) = 1$
1	001	100	4	$x(4) = 1$
2	010	010	2	$x(2) = 1$
3	011	110	6	$x(6) = 1$
4	100	001	1	$x(1) = 1$
5	101	101	5	$x(5) = 1$
6	110	011	3	$x(3) = 1$
7	111	111	7	$x(7) = 1$

$W_8^0 = 1$

$W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$

$W_8^3 = -0.707 - j0.707$

(3) No of stages $m = \log_2 N = \log_2 8 = 3$

(4) No of butterflies at each stage $N/2 = 8/2 = 4$ butterfly

(5) I/P / O/P for each butterfly are separated by 2^{m-1}

stage (1) $2^{1-1} = 2^0 = 1$

stage (2) $2^{2-1} = 2^1 = 2$

stage (3) $2^{3-1} = 2^2 = 4$

(6) No of complex multⁿs are $\frac{N}{2}\log_2 N = \frac{8}{2}\log_2 8 = 12$

(7) No of complex addⁿs are $N\log_2 N = 8\log_2 8 = 24$

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(8) No of sets of butterflies at each stage 2^{M-m}

Stage ① " " " " $2^{3-1} = 2^2 = 4$

" " " stage ② $2^{3-2} = 2$.

" " " stage ③ $2^{3-3} = 2^0 = 1$.

(9) Twiddle factor exponents are k

$k = \frac{Nt}{2^m}$ $t = 0, 1 \ldots 2^{m-1} - 1$

stage ① $m = 1$ $t = 0, 1 \ldots 2^{1-1}$ $t = 0$

$t = 0$ $k = 0$ $2^{2-1} - 1$, $t = 0, 1$.

stage ② $m = 2$ $t = 0, 1 \ldots$

$m = 2$ $t = 0$ $k = \frac{8 \times 0}{2^2} = 0$

$t = 1$ $k = \frac{8 \times 1}{2^2} = 2$ $2^{3-1} - 1$, $t = 0, 1, 2, 3$.

stage ③ $m = 3$ $t = 0, 1 \ldots$

$m = 3$ $t = 0$ $k = 0$

$m = 3$ $t = 1$ $\frac{8 \times 1}{2^3} = 1$

$m = 3$ $t = 2$ $\frac{8 \times 2}{2^3} = 2$

$m = 3$ $t = 3$ $\frac{8 \times 3}{2^3} = 3$

(10) Exponent repeat factor 2^{M-m}.

stage ① ERF $= 2^{3-1} = 2^2 = 4$

② ERF $= 2^{3-2} = 2^1 = 2$

③ ERF $= 2^{3-3} = 2^0 = 1$.

$X_1(0) = 2$
$X_1(1) = 0$
$X_1(3) = 2$
$X(4) = 0$
$X_1(5) = 2$
$X_1(6) = 0$
$X_1(7)$.

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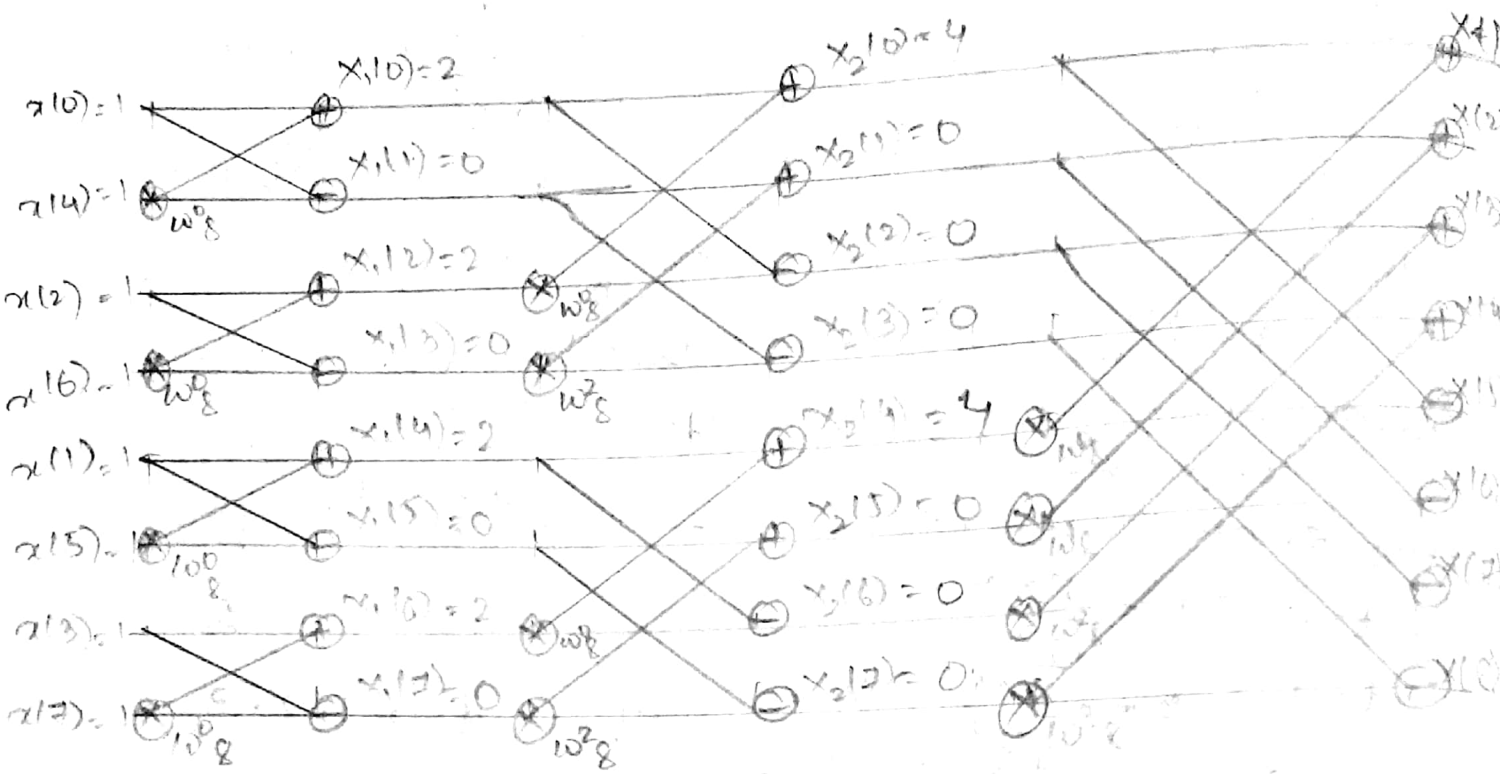


$X(K) = \{8,0,0,0,0,0,0,0\}$

DIF Algorithm

(1) $N=8$

(2) $M = \log_2 N = 3$

I/P/O/P's are separated by 2^{M-m}

$2^{3-1} = 2^2 = 4$

stage (1) $2^{3-2} = 2^1 = 2$

stage (2) $2^{3-3} = 2^0 = 1$

stage (3)

Twiddle factor exponents are $k = \frac{Nt}{2^{M-m+1}}$ $\quad t = 0,1,\ldots,2^{M-m}-1$

stage (1) $m=1$ $\quad 2^{3-1}-1$, $t = 0,1,2,3$

$m=1$ $t=0$ $k=0$ $\quad$ $t=2$ $k = \frac{8\times2}{8} = 2$

$m=1$ $t=1$ $k = \frac{8\times1}{2^{3-1+1}} = 1$ $\quad$ $t=3$ $k = \frac{8\times3}{8} = 3$

stage (2) $t = 0,1,\ldots,2^{3-2}-1 = 1$ $\quad t=0,1$

$m=2$ $t=0$ $k=0$ $\quad$ $m=2$ $t=1$ $k = \frac{8\times1}{2^{3-2+1}} = \frac{8}{4} = 2$

stage (3) $t = 0,1,\ldots,2^{3-3}-1$ $\quad t=0$ $k=0$

No of sets or sections of butterflies in each stage 2^{m-1}

stage (1) $2^{1-1}=$

stage (2) $2^{2-1}=$

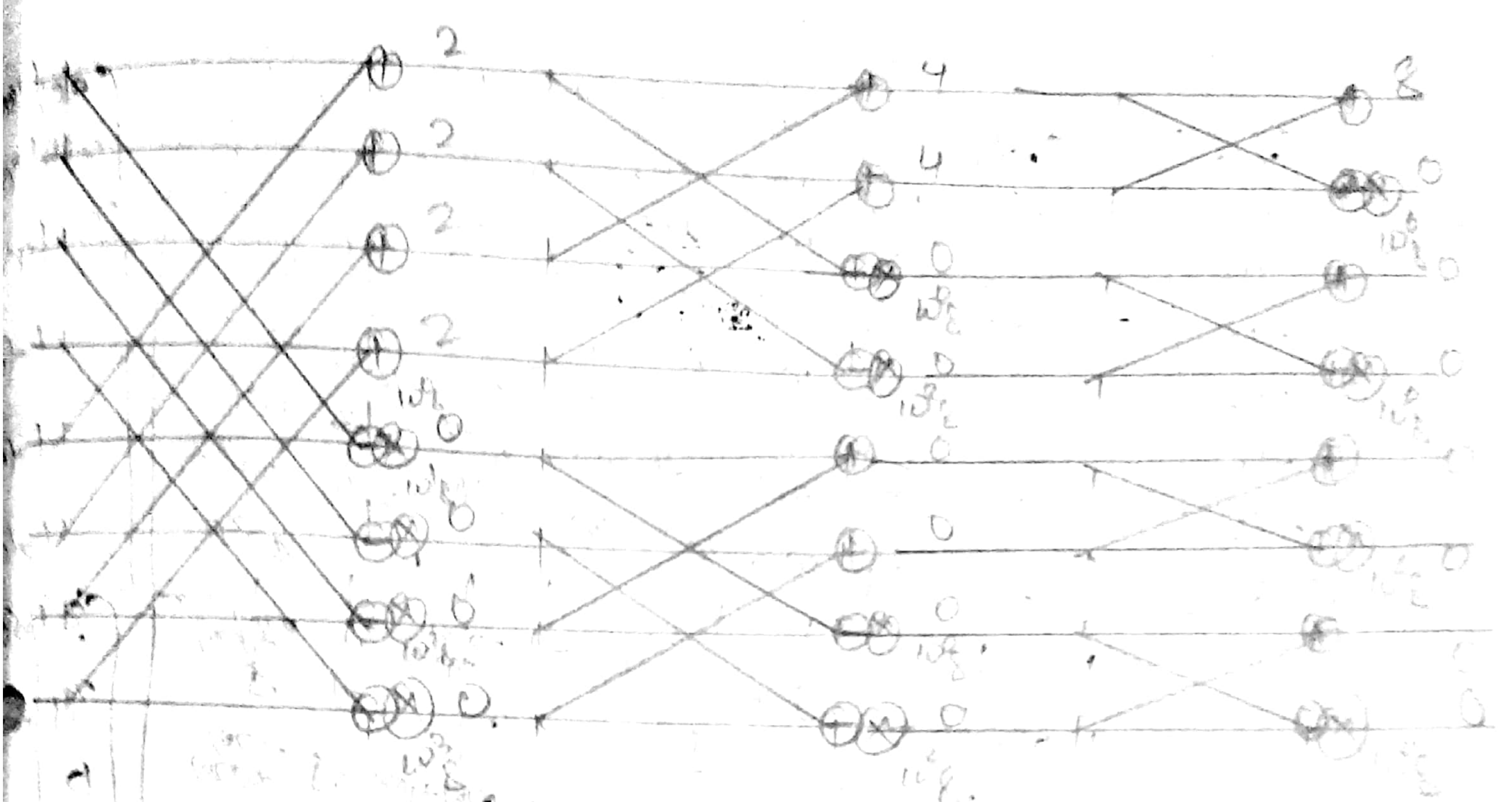
stage (3) $2^{3-1}=2$

ERF factor at each stage 2^{m-1}

stage (1) $2^{1-1} = 2^0 = 1$

(2) $2^{2-1} = 2^1 = 2$

(3) $2^{3-1} = 2^2 = 4$

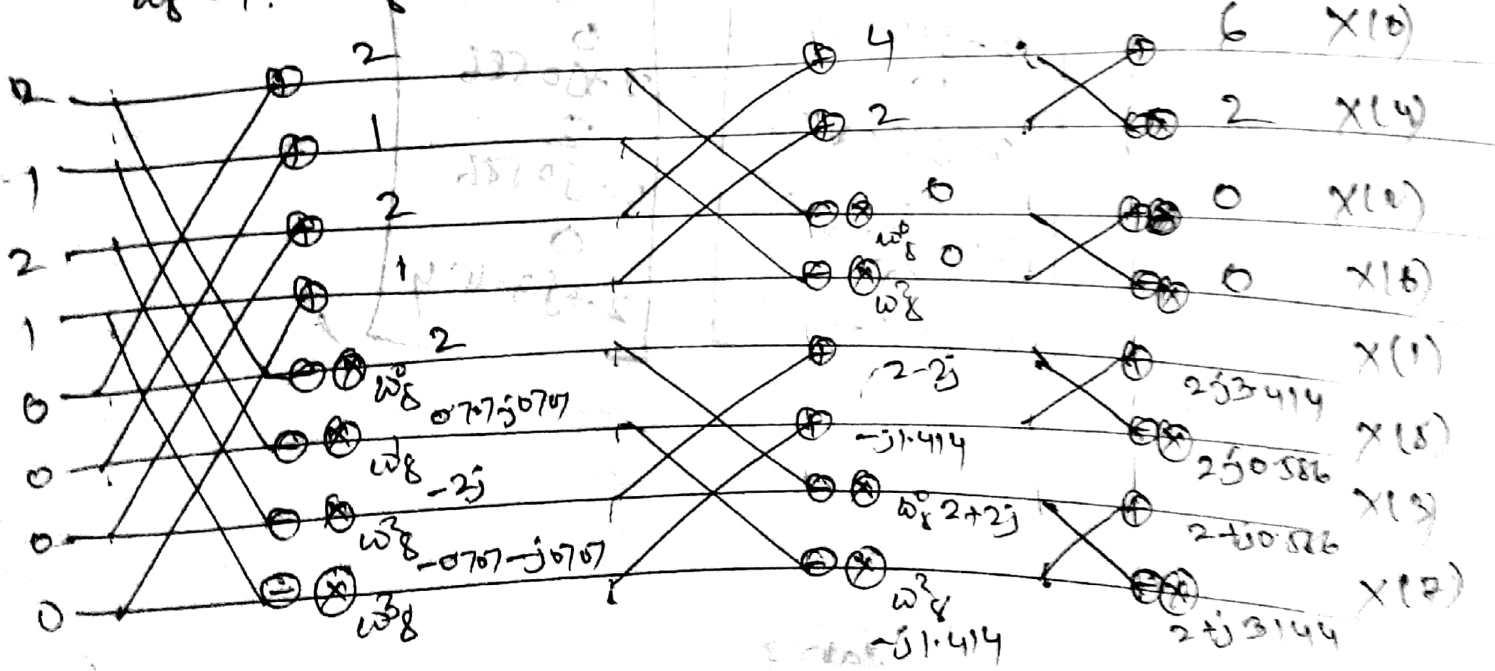


$X(K) = \{8\ 0.0.0.0000\}$

obtain 8 point DFT of the following sequence using Radix-2 DIF-FFT Alg. Show all results along with SFG $x(n) = \{2,1,2,1\}$. Verify your result by direct computation of DFT.

$x(n) = \{2,1,2,1,0,0,0,0\}$

$W_8^0 = 1$, $W_8^1 = 0.707 - j0.707$, $W_8^2 = -j$, $W_8^3 = -0.707 - j0.707$



$X(K) = \{6,\ 2-j3.414,\ 0,\ 2+j0.586,\ 2,\ 2-j0.586,\ 0,\ 2+j3.144\}$

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Direct computation

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W^4 & W^6 & W^8 & W^{10} & W^{12} & W^{14} \\ W_8^0 & W_8^3 & W^6 & W^9 & W^{12} & W^{15} & W^{18} & W^{21} \\ W_8^0 & W_8^4 & W^8 & W^{12} & W^{16} & W^{20} & W^{24} & W^{28} \\ W_8^0 & W_8^5 & W^{10} & W^{15} & W^{20} & W^{25} & W^{30} & W^{35} \\ W_8^0 & W_8^6 & W^{12} & W^{18} & W^{24} & W^{30} & W^{36} & W^{42} \\ W_8^0 & W_8^7 & W^{14} & W^{21} & W^{28} & W^{35} & W^{42} & W^{49} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x. \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.707-j0.707 & -j & -0.707-j0.707 & -1 & -0.707+j0.707 & j & 0.707+j0.707 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.707-j0.707 & j & 0.707-j0.707 & -1 & 0.707+j0.707 & -j & -0.707+j0.707 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.707+j0.707 & -j & 0.707+j0.707 & -1 & 0.707-j0.707 & j & -0.707-j0.707 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.707+j0.707 & j & -0.707+j0.707 & -1 & -0.707-j0.707 & -j & 0.707-j0.707 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 2-j(2+\sqrt{2}) \\ 0 \\ 2+j(2-\sqrt{2}) \\ 2 \\ 2-j(2-\sqrt{2}) \\ 0 \\ 2+j(2+\sqrt{2}) \end{bmatrix} = \begin{bmatrix} 6 \\ 2-j3.414 \\ 0 \\ 2+j0.586 \\ 2 \\ 2-j0.586 \\ 0 \\ 2+j3.414 \end{bmatrix}$$

IDFT using FFT Algorithm

~~IDFT~~ : FFT algorithms can be used to compute an inverse DFT without any change in algorithm. In practice single algorithm can be written to compute DFT as well as IDFT with a very minor difference. & That can be handled easily by common I/O pgm.

The inverse DFT of an N-point sequence $X(k)$, $k=0,1,\ldots N-1$ is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k).W^{-nk} \quad \text{(1)}$$ where $W = e^{-j2\pi/N}$. multiply

By taking complex conjugate to $x(n)$ & multiplying N to above eq-(1) we get

$$N x^*(n) = \sum_{k=0}^{N-1} X^*(k).W^{nk}. \quad \text{— (2)}$$

R.H.S of eq-(2) is DFT of the sequence ~~$X^*(k)$~~. $x^*(n)$. Then o/p sequence $x(n)$ can be obtained by taking complex conjugate of DFT & dividing by N.

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k).W^{nk}\right]^*$$

Eg. Compute IDFT of the sequence

$X(k) = \{7, -0.707-j0.707, -j, 0.707-j0.707, 1, 0.707+j0.707, j, -0.707+j0.707\}$ using DIF Algorithm.

The difference between Radix-2 DIF-FFT & inverse Radix-2 DIF FFT Algorithm

(1) I/P to DIF-FFT is $x(n)$ in bit reversed order but I/P to inverse DIF FFT is $X(k)$ is bit reversed form

(2) Normal phase factors are $W_8^0, W_8^1, W_8^2, W_8^3$ but in inverse DIF-FFT Algorithm phase factors are $W_8^0, W_8^{-1}, W_8^{-2}, W_8^{-3}$

(3) o/p's of final stage are not divided in DIF-FFT but o/p of final stage are divided by N in inverse DIF-FFT

(4) O/P is $X(k)$ in DIF-FFT & o/p is $x(n)$ in inverse DIF-FFT

When the I/p's are $X(k)$ & exponents ~~of~~ all phase factors -ve & o/p are divided by N, then DIF-FFT computes IDFT ~~or~~ hence it is called as inverse DIF-FFT.

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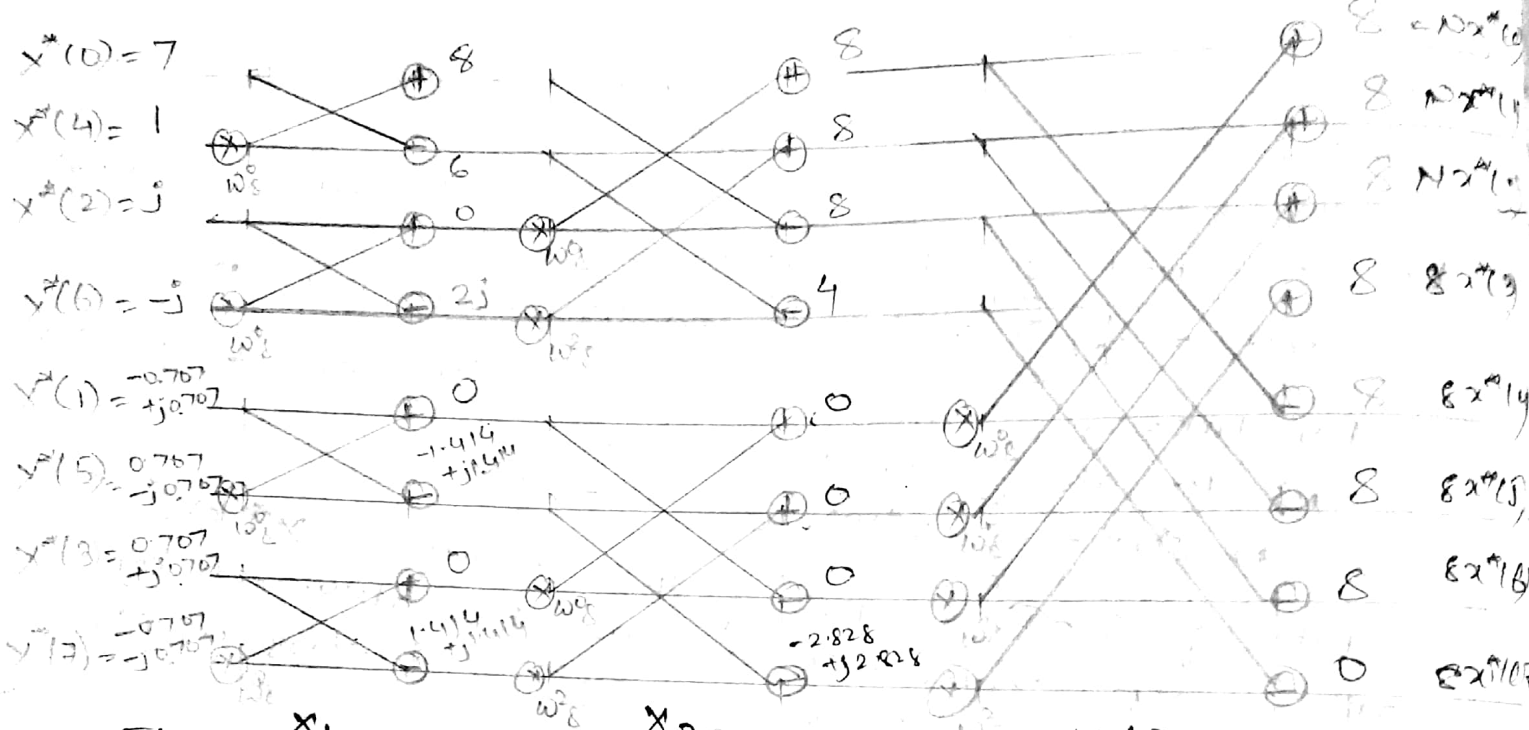
Ex

Compute IDFT of the sequence

$X(K) = \{7, -0.707 - j0.707, -j, 0.707 - j0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$ using DIT algorithm



I/P	X_1	X_2	$X_3(\,)$
7	$7+1=8$	$8+0=8$	$8+0=8$
1	$7-1=6$	$6+2j(-j)=8$	$8+0(W_8^1)=8$
j	$j-j=0$	$8-0=8$	$8+0W_8^2=8$
-j	$j-(-j)=2j$	$6-2j(-j)=4$	$4+(-2.828+j2.828)(-0.707-j0.707)$ $4+2+\cancel{2j}-\cancel{2j}+2=8$
$-0.707+j0.707$	$-0.707+j0.707+0.707-j0.707=0$	$0+0=0$	$8-0=8$
$0.707-j0.707$	$-1.414+j1.414$	$-1.414+j1.414+(1.414+j1.414)(-j)=0$	$8-0=8$
$0.707+j0.707$	0		$8-0=8$
$-0.707-j0.707$	$1.414+j1.414$	$-1.414+j1.414-(1.414+j1.414)(-j)$ $=-1.414+j1.414+1.414j-1.414$ $=-2.828+j2.828$	$4-(-2.828+j2.828)(\ldots)$ $4-2-\cancel{2j}+\cancel{2j}-2=0$

$W_8^0 = 1$, $W_8^1 = 0.707 - j0.707$, $W_8^2 = -j$, $W_8^3 = -0.707 - j0.707$

$N x^*(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$

$x(n) = \{1, 1, 1, 1, 1, 1, 1, 0\}$

2/3 5. 6. 10. ~~18~~. 23. 28. 29. 38. 40. 41. 50. 55. 56. 58.

6/4 2, 10, 15, 16, 17, 19, 22, 24, 25, 28, 29, 31, 35, 61, 63, 64, 65, 67.

4/4 2, 25, 31, 35, 40, 60, 61, 63, 67.

5/4 7, 8, 10, 20, 25, 31, 33, 34, 36, ~~38~~, 39, 42, 61, 63, 65, 67, ~~68~~

7, 9, 10, 12, 13, 15, 18, 19, 24, 25, 25, 29, 35, 36, 37, 38, 4, 43, 45, 58, 61

Mahesh. Yanagimath

DEC 01/09

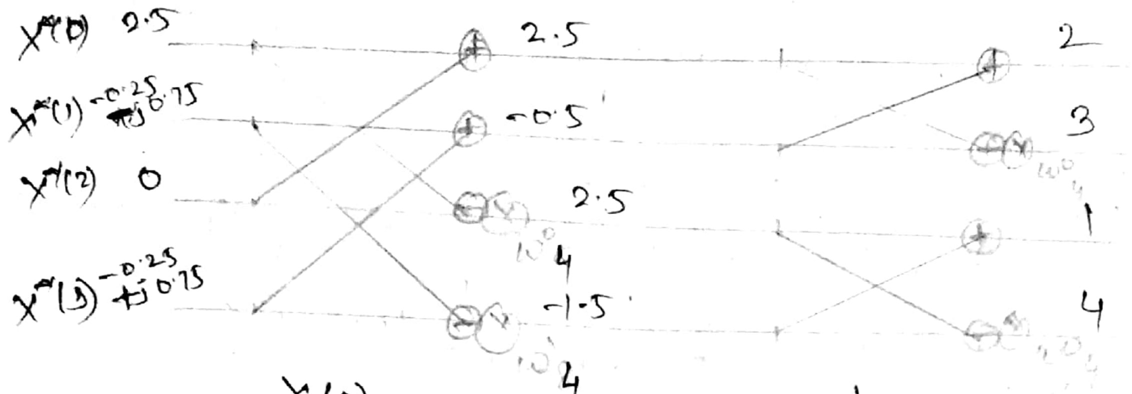
4 marks

Determine 4-point IDFT of $X(k) = \{2.5, -0.25 + j0.75, 0, -0.25 - j0.75\}$ using DIF_FFT Algorithm.

'03

$W_4^0 = e^{-j\frac{2\pi}{4} \times 0} = 0$

$W_4^1 = e^{-j\pi/2} = -j$



$X_1(0)$

$X_1(0) = 2.5 + 0$

$X_1(1) = -0.25 - \cancel{j0.75} - 0.25 + \cancel{j0.75}$

$= -0.5$

$X_1(2) = (2.5 - 0)\, W^0 = 2.5$

$X_1(3) = (-0.25 - j0.75 + 0.25 - j0.75) W^1$

$= (-j1.5)(-j) = 1.5j^2 = -1.5$

$X_2(0) = 2.5 - 0.5 = 2$

$X_2(1) = 2.5 + 0.5 = 3$

$X_2(2) = 2.5 - 1.5 = 1$

$X_2(3) = 2.5 + 1.5 = 4$

$Nx^*(n) = \{2, 3, 1, 4\}$

$Nx^*(n)$ is in bit reversed order

$Nx^*(n) = \{2, 1, 3, 4\}$

$N = 4$

$x(n) = \{0.5, 0.25, 0.75, 1\}$

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