Subject: ELECTROMAGNETIC FIELD THEORY

Subject Code: 15EE45

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VECTOR ALGEBRA

Vector: A vector is a quantity having both magnitude and direction.

Eg: Force, velocity, Electric field, magnetic field etc.

Scalar: It is the quantity having only magnitude.

Eg: Mass, time, speed, electric potential etc.

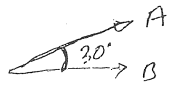
Representation of vectors: A vector can be represented in two ways:

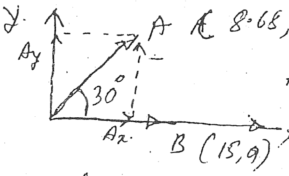
i) Graphical representation

ii) Co-ordinate system representation

Consider two vectors $\bar{A}$ & $\bar{B}$ having magnitudes 10 & 15 respectively with an angular displacement of 30° between them. These vectors can be represented as,

Graphical form; Co-ordinate system





A (8.66, 5) $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y$

$\bar{A} = 8.66\bar{a}_x + 5\bar{a}_y$

$\bar{B} = 15\,\bar{a}_x$

where $\bar{a}_x$ & $\bar{a}_y$ are the unit vectors in x & y directions.

We can see that the analysis of vectors represented in co-ordinate system is easy.

Magnitude of a vector: ($|A|$): It is the length of a given vector.

If $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$, then the magnitude of a vector A is

$$|\bar{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

Unit vector: If we divide the given vector $\vec{A}$ by its magnitude ~~we get~~, we get a vector whose magnitude is unity & its direction is same as that of vector $\vec{A}$, the resulting vector is called unit vector. & its direction is in the direction of $\vec{A}$. It is represented as $\bar{a}_A$

If $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$ then,

$$|\bar{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\therefore \bar{a}_A = \frac{\bar{A}}{|\bar{A}|} = \frac{A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z}{|\bar{A}|}$$

Multiplication of a vector by a constant (K):

If $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$

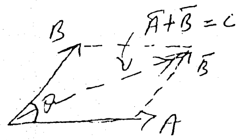
then $K\bar{A} = KA_x \bar{a}_x + KA_y \bar{a}_y + KA_z \bar{a}_z$.

Addition of two vectors: If $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$

& $\bar{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$

Then $\bar{A}+\bar{B} = (A_x + B_x)\bar{a}_x + (A_y+B_y)\bar{a}_y + (A_z+B_z)\bar{a}_z$

(Graphical method):



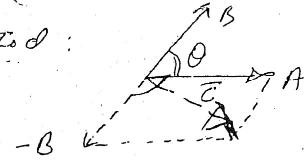
$\bar{C} = \sqrt{|A|^2 + |B|^2 + 2|A||B|\cos\theta}$

Substraction of two vectors: i) Co-ordinate system:

If, $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$ & $\bar{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$.

Then, $\bar{A} - \bar{B} = (A_x - B_x)\bar{a}_x + (A_y - B_y)\bar{a}_y + (A_z - B_z)\bar{a}_z$.

ii) Graphical method:



$\bar{C} = \sqrt{|A|^2 + |B|^2 - 2|A||B|\cos\theta}$

Product of two vectors: There are two products defined for vector

i) Dot product (scalar product)

ii) Cross product (vector product)

i) **Dot product or Scalar product:** It is defined as the product of magnitude of two vectors & the cosine of angle between the two vectors.

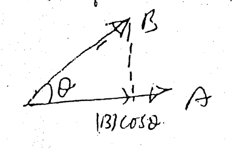
a) Co-ordinate system: If $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$ &

$\bar{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$, then

$$\bar{A}\cdot\bar{B} = (A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z)\cdot(B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z)$$

$$\bar{A}\cdot\bar{B} = A_x B_x + A_y B_y + A_z B_z.$$

b) vector method:



$\bar{A}\cdot\bar{B} = |\bar{A}|\cdot|\bar{B}|\cos\theta$

$= |\bar{A}|\,|\bar{B}|\cos\theta$

Note: 1. Dot product of two vectors results in to a scalar quantity

2. $\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$ (Commutative law hold good)

3. $\bar{A}\cdot\bar{A} = |A|^2$.

4. If $\bar{A}\cdot\bar{B} = 0$ then $\bar{A} \perp \bar{B}$. i

5. If $\bar{A}\cdot\bar{B} = |A||B|$ the $\bar{A}$ & $\bar{B}$ are parallel

6. component or projection of vector $\bar{B}$ on $\bar{A}$ is $|B|\cos\theta = \frac{\bar{A}\cdot\bar{B}}{|A|}$.

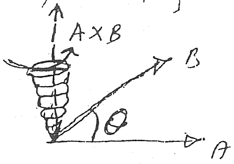
7. llly projection of $\bar{A}$ on $\bar{B}$ is $|\bar{A}|\cos\theta = \frac{\bar{A}\cdot\bar{B}}{|B|}$.

iii) **Cross product of two vectors :** It is the product of magnitude of two vectors & sine of angle between the two vectors. The cross product of two vectors results in a vector quantity & its direction is perpendicular to the plane containing the two vectors & is given by right hand screw rule.

a) Co-ordinate system : If $\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$, $\bar{B} = B_x\bar{a}_x + B_y\bar{a}_y + B_z\bar{a}_z$.

then
$$\bar{A}\times\bar{B} = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

b) Graphical method :



$$\bar{A}\times\bar{B} = |A||B|\sin\theta\,\bar{a}_n$$

$\bar{a}_n \rightarrow$ unit vector normal to the plane containing $\bar{A}$ & $\bar{B}$.

Ex: If $\bar{A}$ & $\bar{B}$ are lieing on xy plane the direction of $\bar{A}\times\bar{B}$ lies ~~on~~ along z-axis.

Note: 1. Corss product of two vectors results in a vector quantity

2. $(\bar{A}\times\bar{B}) = -(\bar{B}\times\bar{A})$

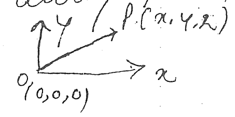
3. $\bar{A}\times\bar{A} = 0$

4. If $\bar{A}\times\bar{B} = 0$, then $\bar{A}$ & $\bar{B}$ are parallel to each other $\xrightarrow{A}$ $\xrightarrow{B}$

5. If $\bar{A}\times\bar{B} = \bar{A}\bar{B}\bar{a}_n$, then $\bar{A}$ & $\bar{B}$ are perpendicular B ⊥ A

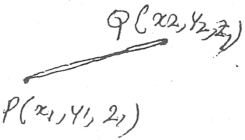
6. $\bar{A}\times\bar{B}$ = Area of the parallelogram having the vectors $\bar{A}$ & $\bar{B}$ as its two sides.

Position vector : If a vector has origin as its one end & directed away from the origin, we call it as position vector.



$$\bar{P} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$$

Note: The vector drawn from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is
$$\overline{PQ} = (x_2-x_1)\bar{a}_x + (y_2-y_1)\bar{a}_y + (z_2-z_1)\bar{a}_z$$
$$|\overline{PQ}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$



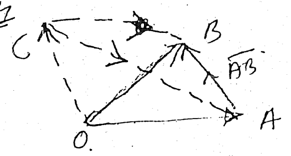
Ex: Find the vector $\bar{A}$ which is directed from (1,-2,3) to (2,4,-3) in cartesian co-ordinate system & find unit vector along $\bar{A}$.

Soln
$$\bar{A} = (2-1)\bar{a}_x + (4-(-2))\bar{a}_y + (-3-3)\bar{a}_z$$
$$\bar{A} = \bar{a}_x + 6\bar{a}_y - 6\bar{a}_z$$

$$\bar{a}_A = \frac{\bar{a}_x + 6\bar{a}_y - 6\bar{a}_z}{\sqrt{1^2+6^2+6^2}} = \frac{1}{\sqrt{73}}(\bar{a}_x + 6\bar{a}_y - 6\bar{a}_z) //$$

Ex: Given the vectors $\bar{A} = -\bar{a}_x - 3\bar{a}_y - 4\bar{a}_z$, $\bar{B} = 2\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z$ & a point $C = (1, 3, 4)$. Find $\bar{AB}$, $\bar{a}_{AB}$ & a unit vector directed from point C to point A.

Soln



From the fig, $\bar{A} + \bar{AB} = \bar{B}$

$\therefore \bar{AB} = \bar{B} - \bar{A} = (2\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z) - (-\bar{a}_x - 3\bar{a}_y - 4\bar{a}_z)$

$= 3\bar{a}_x + 5\bar{a}_y + 6\bar{a}_z$

$$\bar{a}_{AB} = \frac{3\bar{a}_x + 5\bar{a}_y + 6\bar{a}_z}{\sqrt{3^2+5^2+6^2}} = 0.359\bar{a}_x + 0.589\bar{a}_y + 0.717\bar{a}_z.$$

From the above fig, $\bar{C} + \bar{CA} = \bar{A}$

$\therefore \bar{CA} = \bar{A} - \bar{C} = (-\bar{a}_x - 3\bar{a}_y - 4\bar{a}_z) - (\bar{a}_x + 3\bar{a}_y + 4\bar{a}_z)$

$= -2a_x - 6\bar{a}_y - 8\bar{a}_z //$

$$\bar{a}_{CA} = \frac{-2\bar{a}_x - 6\bar{a}_y - 8\bar{a}_z}{\sqrt{(-2)^2 + (-6)^2 + (-8)^2}} = \frac{-2\bar{a}_x - 6\bar{a}_y - 8\bar{a}_z}{\sqrt{104}}$$

$\bar{a}_{CA} = -0.196\bar{a}_x - 0.588\bar{a}_y - 0.784\bar{a}_z.$

Ex: Given two vectors $\bar{A} = \bar{a}_x - 3\bar{a}_y + 2\bar{a}_z$

$\bar{B} = 2\bar{a}_x + 4\bar{a}_y - \bar{a}_z$

Compute $\bar{A}\cdot\bar{B}$, angle between $\bar{A}$ & $\bar{B}$, projection of B on A.

Soln $\bar{A}\cdot\bar{B} = 1\times 2 + (-3)(4) + (2)(-1) = -12.$

$\cos\theta = A\cdot B / |A||B|$

$= -12/(\sqrt{1+9+4})(\sqrt{4+16+1})$

$= -0.6998.$

$\theta = \cos^{-1}(-0.6998) = 134.41$

projection of $\bar{B}$ on $\bar{A}$ $= ||B|\cos\theta| = |\sqrt{4+16+1}\cdot\cos(134.41)|$

$= 3.207.$

OR " $= \frac{|A\cdot B|}{|A|} = \frac{|-12|}{\sqrt{14}} = 3.207$

Ex: If $\bar{A} = 2\bar{a}_x - \bar{a}_z$, & $\bar{B} = 3\bar{a}_x + \bar{a}_y$ & $\bar{C} = -2\bar{a}_x + 6\bar{a}_y - 4\bar{a}_z$ show that C is $\perp$r to both A & B

Soln If $\bar{C}\cdot\bar{A} = 0$, then $C \perp A$.

$\therefore \bar{C}\cdot\bar{A} = (-2\bar{a}_x + 6\bar{a}_y - 4\bar{a}_z)\cdot(2\bar{a}_x - \bar{a}_z) = (-4 + 4) = 0 //$

lly $\bar{C}\cdot\bar{B} = (-2\bar{a}_x + 6\bar{a}_y - 4\bar{a}_z)\cdot(3\bar{a}_x + \bar{a}_y) = (-6 + 6 - 0) = 0 //$

$\therefore \bar{C} \perp \bar{A}$ & $\bar{C} \perp \bar{B}$ Hence the proof.

Ex:- Vector R_{AB} extends from $A(1,2,3)$ to B. If the length of R_{AB} is 10 units & unit vector $a_{AB} = 0.6\bar{a}_x + 0.64\bar{a}_y + 0.48\bar{a}_z$, find the co-ordinates of point B.

Soln

$A(1,2,3)$ —— $\bar{a}_{AB}$ —— $B(x,y,z)$

Given $\bar{a}_{AB} = 0.6\bar{a}_x + 0.64\bar{a}_y + 0.48\bar{a}_z \rightarrow 1$

$|\bar{AB}| = 10$ units

$\bar{AB} = (x-1)\bar{a}_x + (y-2)\bar{a}_y + (z-3)\bar{a}_z$

$$\bar{a}_{AB} = \frac{(x-1)\bar{a}_x + (y-2)\bar{a}_y + (z-3)\bar{a}_z}{|\bar{AB}|} = \frac{(x-1)\bar{a}_x + (y-2)\bar{a}_y + (z-3)\bar{a}_z}{10} \rightarrow 2$$

Comparing 1 & 2

$\frac{x-1}{10} = 0.6 \Rightarrow x-1 = 6 \Rightarrow x = 7$

$\frac{y-2}{10} = 0.64 \Rightarrow y-2 = 6.4 \Rightarrow y = 8.4$

$\frac{z-3}{10} = 0.48 \Rightarrow z-3 = 4.8 \Rightarrow z = 7.8$

Hence the co-ordinates of point B are $B \equiv (7, 8.4, 7.8)$

Ex: If $\bar{A} = 2\bar{a}_y + 3\bar{a}_z$, $\bar{B} = \bar{a}_x - 2\bar{a}_y + \bar{a}_z$, find $\bar{A}\times\bar{B}$, $\bar{B}\times\bar{A}$.
Show that i) $\bar{A} \perp \bar{A}\times\bar{B}$ ii) $\bar{A}\times\bar{B} = -\bar{B}\times\bar{A}$

Soln

$$\bar{A}\times\bar{B} = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 2 & 3 \\ 1 & -2 & 1 \end{bmatrix} = \bar{a}_x(2+6) - \bar{a}_y(-3) + \bar{a}_z(-2) = 8\bar{a}_x + 3\bar{a}_y - 2\bar{a}_z \rightarrow (1)$$

$$\bar{B}\times\bar{A} = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 1 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \bar{a}_x(-8) - \bar{a}_y(3) + \bar{a}_z(2) = -8\bar{a}_x - 3\bar{a}_y + 2\bar{a}_z \quad -(2)$$

i) If $\bar{A} \perp \bar{A}\times\bar{B}$, then $\bar{A}\cdot\bar{A}\times\bar{B} = 0$

$\bar{A}\cdot(\bar{A}\times\bar{B}) = (2\bar{a}_y + 3\bar{a}_z)\cdot(8\bar{a}_x + 3\bar{a}_y - 2\bar{a}_z) = 0 + 6 - 6 = 0$

ii) from (1) & (2) $\bar{A}\times\bar{B} = -(\bar{B}\times A)$

Ex: If $\bar{A} = 2i - 3j - K$, $\bar{B} = i + 4j - 2K$, find a vector perpendicular to both $\bar{A}$ & $\bar{B}$

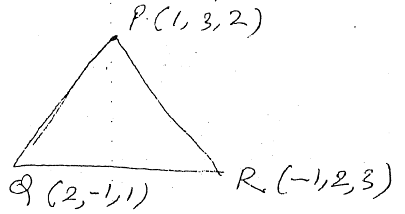
Soln We know that $\bar{A}\times\bar{B}$ is a vector perpendicular to both $\bar{A}$ & $\bar{B}$.

$$\bar{A}\times\bar{B} = \begin{bmatrix} i & j & K \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{bmatrix} = i(10) - j(-3) + K(11) = 10i + 3j + 11K$$

Check: $\bar{A}\cdot(\bar{A}\times\bar{B}) = (2i - 3j - K)(10i + 3j + 11K) = 20 - 9 - 11 = 0$

$\bar{B}\cdot(A\times\bar{B}) = (i + 4K - 2K)\cdot(10i + 3j + 11K) = 10 + 12 - 22 = 0$

Ex: Find the area of the triangle having vertices $P \equiv (1,3,2)$ $Q \equiv (2,-1,1)$ $R \equiv (-1,2,3)$. Also find a unit vector perpendicular to the plane of the triangle.



Sol: Area of the triangle PQR,

$= \frac{1}{2}|\overline{QP} \times \overline{QR}|$

$\overline{QP} = -\bar{a}_x + 4\bar{a}_y + \bar{a}_z$, $\overline{QR} = -3\bar{a}_x + 3\bar{a}_y + 2\bar{a}_z$.

$$\overline{QP} \times \overline{QR} = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ -1 & +4 & +1 \\ -3 & 3 & 2 \end{bmatrix} = \bar{a}_x(+8-3) - \bar{a}_y(-2+3) + \bar{a}_z(-3+12)$$

$$= +5\bar{a}_x + 1\bar{a}_y + 9\bar{a}_z$$

$|\overline{QP} \times \overline{QR}| = \sqrt{25+1+81} = 10.34$ Units

Area of the triangle $= \frac{1}{2}|\overline{QP} \times \overline{QR}| = \frac{1}{2} \times 10.34 = 5.17$ Units

To find a vector perpendicular to the plane of the triangle:

we know that, $\overline{QP} \times \overline{QR}$ is a vector ⊥ to both $\overline{QP}$ & $\overline{QR}$ i.e. it is perpendicular to the plane of the triangle. Hence the unit vector ⊥ to the plane of the triangle is,

$$\frac{\overline{QP} \times \overline{QR}}{|\overline{QP} \times \overline{QR}|} = \frac{5\bar{a}_x - \bar{a}_y + 9\bar{a}_z}{\sqrt{25+1+81}}$$

$$= 0.4834\,\bar{a}_x - 0.0966\,\bar{a}_y + 0.87\,\bar{a}_z$$

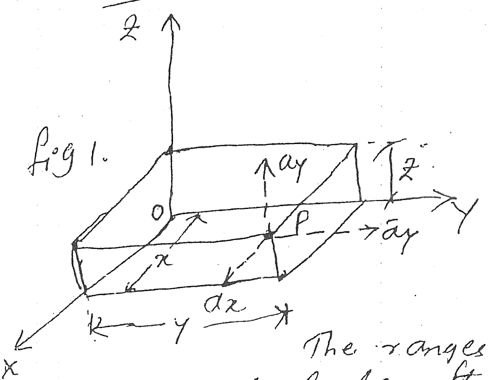
~~VECTOR ALGEBRA~~ Co-ordinate System 1.

In the subject of field theory (Electromagnetic theory), the vector analysis plays an important role. The behaviour of electric & magnetic field can be studied with the knowledge of vector analysis. In order to study the vectors, the co-ordinate systems are used. The various types of co-ordinate systems are:

i) Rectangular co-ordinate system.
ii) Cylindrical co-ordinate system.
iii) Spherical co-ordinate system.

i) Rectangular co-ordinate system: (Cartesian co-ordinate system)

fig 1.

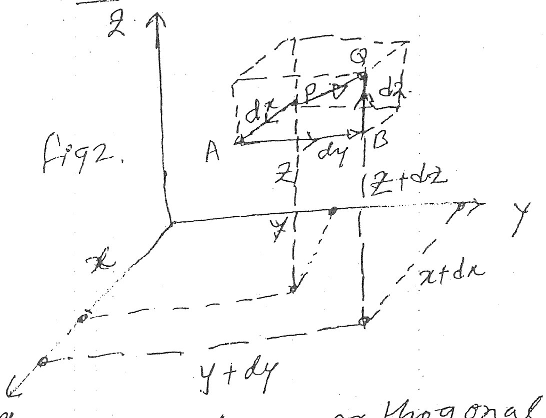


In this system, the three axis are x, y & z which are mutually perpendicular to each other. Any point P is determined by 3 co-ordinates P(x, y, z) as shown in fig 1. The unit vectors are a_x, a_y & a_z. The position vector is given by $\bar{P} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$.

The ranges of x, y & z are $-\infty$ to $+\infty$.

Incremental length in rectangular co-ordinate system:

fig 2.



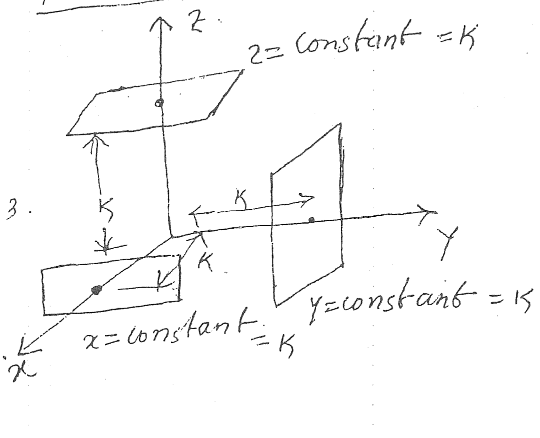
Consider two points P(x, y, z) & Q(x+dx, y+dy, z+dz). The incremental length of volume is given by $\bar{PQ}$ & is denoted by $d\bar{l}$

$\therefore d\bar{l} = \bar{PQ} = \bar{PA} + \bar{AB} + \bar{BQ}$

$\boxed{d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z}$

The three orthogonal surfaces:

fig 3.



A point is defined by the intersection of three orthogonal surfaces. The surfaces are the planes at x = y = z = constant = K. as shown in fig 3.

Incremental surfaces: An incremental surface is a vector with its area magnitude & having direction normal to the surface. The expressions for incremental surfaces are

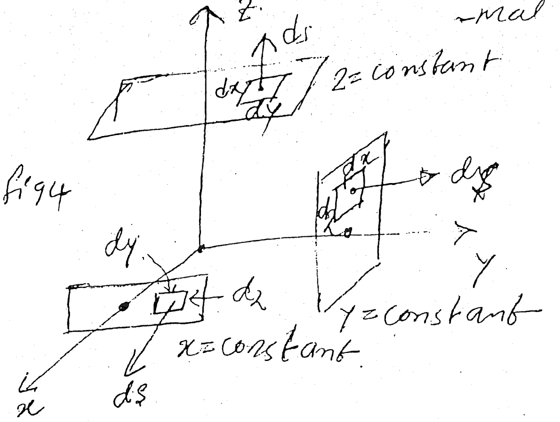


fig 4

i) $x = \text{constant}$

$d\bar{s} = dy\,dz\,\bar{a}_x$ & $|d\bar{s}| = dy \cdot dz$

ii) $y = \text{constant}$

$d\bar{s} = dx\,dz\,\bar{a}_y$ & $|d\bar{s}| = dx\,dz$

iii) $z = \text{constant}$

$d\bar{s} = dy\,dx\,\bar{a}_z$ & $|d\bar{s}| = dx\,dy$

Incremental volume: The incremental volume is given by, $\boxed{dv = dx\,dy\,dz}$. It is a scalar quantity.

Cylendrical co-ordinate system: In this system any point $P(\rho, \phi, z)$ is assumed to lie on the surface of a cylinder. The three mutually perpendicular axes are ρ-axis, ϕ-axis & z-axis. Any point P is determined by $P(\rho, \phi, z)$ where, $\rho \rightarrow$ radius of cylinder.

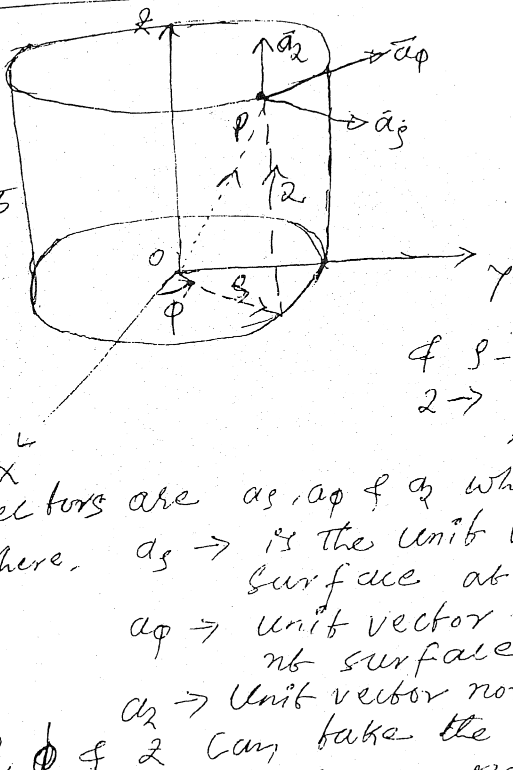


fig.5

$\phi \rightarrow$ angle measured from x-axis & ρ-vector (ρ-axis)

$z \rightarrow$ It is the distance of the point from xy-plane. The three unit vectors are a_ρ, a_ϕ & a_z which are mutually perpendicular where, $a_\rho \rightarrow$ is the unit vector normal to the cylendrical surface at $\rho = \text{Constant}$

$a_\phi \rightarrow$ unit vector normal to the plane at $\phi = \text{constant}$ surface

$a_z \rightarrow$ Unit vector normal to the surface at $z = \text{constant}$.

ρ, ϕ & z can take the values $(0 - \infty)$, $(0 - 2\pi)$ & $(-\infty$ to $+\infty)$ respectively. The position vector is given by,

$\overline{op} = \rho\,\bar{a}_\rho + z\,\bar{a}_z$

Incremental (Elemental length):

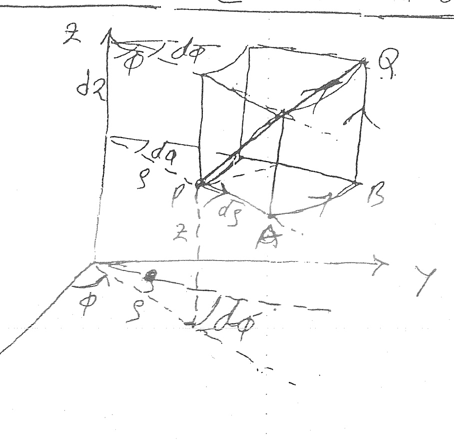


fig6.

Consider two points $P(\rho, \phi, z)$ & $Q(\rho+d\rho,\ \phi+d\phi,\ z+dz)$. The incremental length vector is $\bar{dl}$ & is $\bar{PQ}$ as shown in fig6.

$\therefore\ \bar{PA} + \bar{AB} + \bar{BQ} = \bar{PQ} \rightarrow 1$

$\therefore\ \bar{PA} = d\rho\, \bar{a}_\rho$

$\bar{AB} = (\rho + d\rho)\, d\phi\, \bar{a}_\phi$

$\simeq \rho\, d\phi\, \bar{a}_\phi$

$\bar{BQ} = dz\, \bar{a}_z$ $\therefore$ eqn 1. becomes,

$$\therefore\ \boxed{\bar{dl} = d\rho\, \bar{a}_\rho + \rho\, d\phi\, \bar{a}_\phi + dz\, \bar{a}_z}$$

Orthogonal Surfaces & incremental surface vectors:

The orthogonal surfaces are obtained by varying two co-ordinates constant & keeping one co-ordinate constant.

i) ρ = constant:

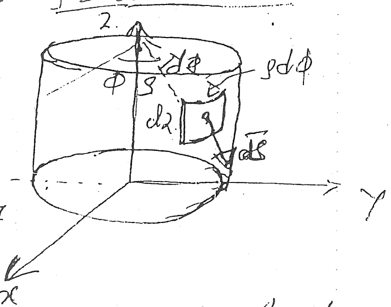


fig 7

Here, ϕ varies between 0 to 2π
& z " " $-\infty$ to $+\infty$.
Consider an incremental surface vector $\bar{ds}$ as shown in fig 7.

$$\boxed{\bar{ds} = \rho\, d\phi\, dz\, \bar{a}_\rho}$$

$|ds| = \rho\, d\phi\, dz$.

ii) ϕ = constant:

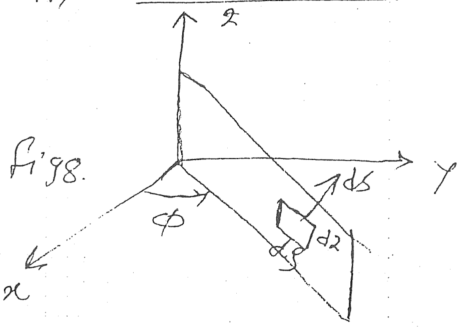


fig8.

$$\boxed{\bar{ds} = d\rho\, dz\, \bar{a}_\phi}$$

$|ds| = d\rho\, dz$.

iii) z = constant:

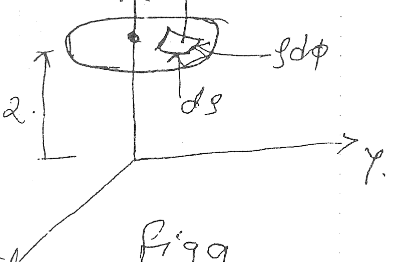


fig9.

$$\boxed{\bar{ds} = \rho\, d\rho\, d\phi\, \bar{a}_z}$$

$|ds| = \rho\, d\rho\, d\phi$ //

Incremental Volume: $\boxed{dv = \rho\, d\rho\, d\phi . dz}$ It is a scalar quantity.

Relationship between Rectangular & cylindrical co-ordinate system:

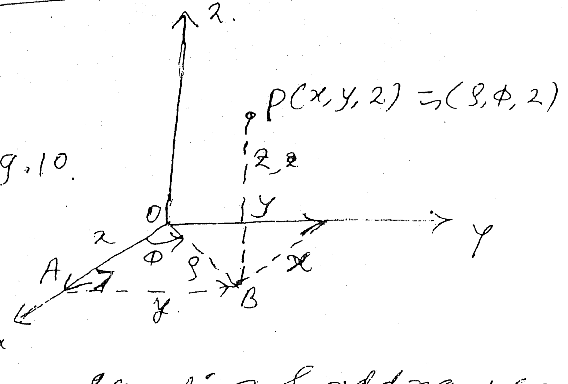


fig.10.

Consider a point $P(x, y, z) \Rightarrow (\rho, \phi, z)$

From the $\triangle OAB$, $\angle OAB = 90°$

$\therefore \cos\phi = OA/OB = x/\rho$

$\therefore \boxed{x = \rho\cos\phi} \longrightarrow 1$

llly $\boxed{y = \rho\sin\phi} \longrightarrow 2$ & $\boxed{z = z}$

Also. Squaring & adding 1 & 2

$x^2 + y^2 = \rho^2(\cos^2\phi + \sin^2\phi) = \rho^2$.

$\therefore \boxed{\rho = \sqrt{x^2 + y^2}}$

$y/x = \dfrac{\rho\sin\phi}{\rho\cos\phi} = \tan\phi$

$\therefore \boxed{\phi = \tan^{-1} y/x}$ & $\boxed{z = z}$

Dot product of two unit vectors in rectangular & cylindrical co-ordinate system: Let $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z \rightarrow 1$ is a vector in cartesian co-ordinate system. The equivalent vector in cylindrical co-ordinates be,

$\bar{A} = A_\rho a_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z \longrightarrow 2.$

Now Let us find A_ρ, A_ϕ & A_z. Here A_ρ, A_ϕ & A_z are the projection of vector $\bar{A}$ on ρ, ϕ & z axes.

$\therefore$ Length of projection of vector = given vector $\cdot$ unit vector in the desired direction.

i.e. $A_\rho = \bar{A} \cdot \bar{a}_\rho \longrightarrow 3$

$A_\phi = \bar{A} \cdot \bar{a}_\phi \longrightarrow 4$ putting 1 in 3, 4, & 5.

$A_z = \bar{A} \cdot a_z \longrightarrow 5.$

$A_\rho = (A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z) \cdot \bar{a}_\rho = A_x \bar{a}_x \cdot \bar{a}_\rho + A_y \bar{a}_y \cdot \bar{a}_\rho + A_z \bar{a}_z \cdot \bar{a}_\rho \rightarrow 6$

$A_\phi = A_x \bar{a}_x \cdot \bar{a}_\phi + A_y \bar{a}_y \cdot \bar{a}_\phi + A_z \bar{a}_z \cdot \bar{a}_\phi \longrightarrow 7$

$A_z = A_x \bar{a}_x \cdot \bar{a}_z + A_y \bar{a}_y \cdot \bar{a}_\phi + A_z \bar{a}_z \cdot \bar{a}_z \longrightarrow 8.$

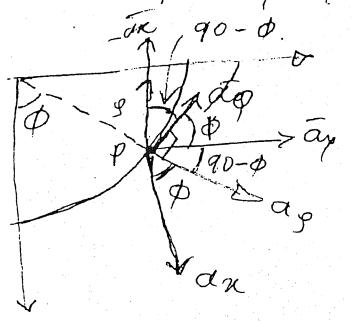


fig 11.

To find the dot product of unit vectors consider the fig shown in fig 11. using dot product,

$\bar{a}_x \cdot \bar{a}_\rho = |\bar{a}_x||a_\rho| \cos\phi = \cos\phi$

$\bar{a}_x \cdot \bar{a}_\phi = |\bar{a}_x||\bar{a}_\phi| \cos(90-\phi)$
$= -1 \cdot 1 \cdot \sin\phi = -\sin\phi$

$\bar{a}_x \cdot \bar{a}_z = |\bar{a}_x||\bar{a}_z| \cos 90 = 0$

$\bar{a_y} \cdot \bar{a_\rho} = 1 \cdot 1 \cdot \cos(90-\phi) = \sin\phi$
$\bar{a_y} \cdot \bar{a_\phi} = 1 \cdot 1 \cos\phi = \cos\phi$
$\bar{a_y} \cdot \bar{a_z} = 1 \cdot 1 \cos 90^\circ = 0$
$\bar{a_z} \cdot \bar{a_\rho} = 1 \cdot 1 \cos 90 = 0$
$\bar{a_z} \cdot \bar{a_\phi} = 1 \cdot 1 \cos 90 = 0$
$\bar{a_z} \cdot \bar{a_z} = 1 \cdot 1 \cdot \cos 0 = 1$

$\Rightarrow$

	a_ρ	a_ϕ	a_z
a_x	$\cos\phi$	$-\sin\phi$	0
a_y	$\sin\phi$	$\cos\phi$	0
a_z	0	0	1

//

$\therefore$ putting these values in 6, 7, 8.

$$\left.\begin{aligned} A_\rho &= A_x\cos\phi + A_y\sin\phi + 0 = A_x\cos\phi + A_y\sin\phi \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi + 0 = -A_x\sin\phi + A_y\cos\phi \\ A_z &= 0 + 0 + A_z = A_z. \end{aligned}\right\} \rightarrow 9$$

putting 9 in (2)

$$\bar{A} = (A_x\cos\phi + A_y\sin\phi)\bar{a_\rho} + (-A_x\sin\phi + A_y\cos\phi)\bar{a_\phi} + A_z\bar{a_z} \rightarrow 10.$$

Eqn. 10 is the equivalent vector in cylendrical co-ordinate.

<u>Spherical - co-ordinate system</u>: In this system any point 'p' is assumed to lie on the surface of a sphere. Its co-ordinates are $p(r, \theta, \phi)$ & the corresponding three mutually pependicular axis are, r- axis, θ- axis & ϕ- axis. where, $r \rightarrow$ Radius of sphere. $\theta \rightarrow$ angle made by r- vector with the z-axis as shown in fig 12.

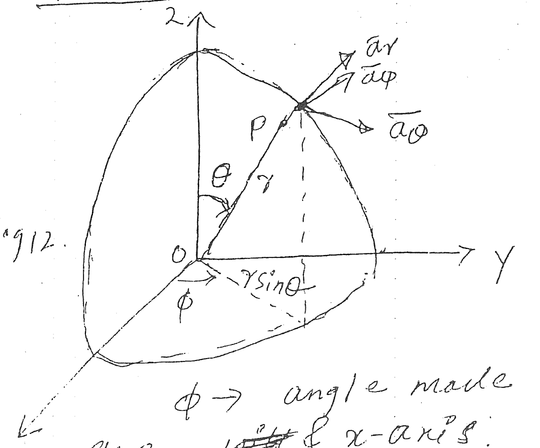


fig 12.

$\phi \rightarrow$ angle made by projection of r- vector on xy-plane & x-axis. The three unit vectors are $\bar{a_r}$, $\bar{a_\theta}$ & $\bar{a_\phi}$ & are mutually perpendicular to each other.

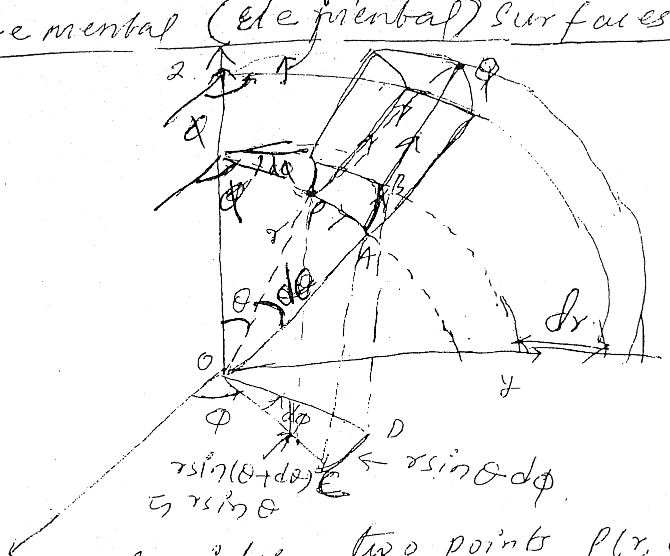
Note: 1. The direction of unit vectors $\bar{a_\theta}$ & $\bar{a_\phi}$ are not fixed. They change from point to point.

2. $\bar{a_\theta}$ & $\bar{a_\phi}$ are tangential to the spherical surface. The unit vector $\bar{a_r}$ is normal to the spherical surface. However all the three unit vectors are mutually perpendicular to each other.

3. The position vector in spherical co-ordinate system is given by $\bar{op} = \bar{r} = r\bar{a_r}$.

<u>Ranges of r, θ, & ϕ</u>: r can take values from 0 to ∞
θ " " " " 0 to π
ϕ " " " " 0 to 2π.

Incremental (Elemental) Surfaces :-



Consider two points $P(r, \theta, \phi)$ & $Q(r+dr, \theta+d\theta, \phi+d\phi)$ in spherical co-ordinates & we wish to find the incremental length between the two points P & Q.

$\therefore \overline{PA} + \overline{AB} + \overline{BQ} = \overline{PQ}$

$\therefore \overline{dl} = \overline{PQ} = \overline{PA} + \overline{PB} + \overline{BQ}$

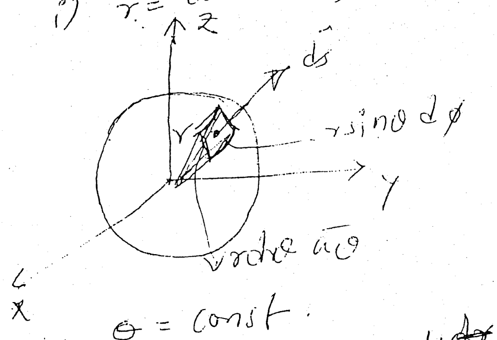
$\overline{PA} = r\, d\theta\, \bar{a}_\theta$, $\overline{AB} = r\sin\theta\, d\phi\, \bar{a}_\phi$, $\overline{BQ} = dr\, \bar{a}_r$.

$\therefore \overline{dl} = dr\,\bar{a}_r + r\sin\theta\, d\phi\, \bar{a}_\phi + d\theta\, \bar{a}_\theta$

$$\boxed{\overline{dl} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\, d\phi\, \bar{a}_\phi}$$

Orthogonal surfaces & Incremental surface expressions :

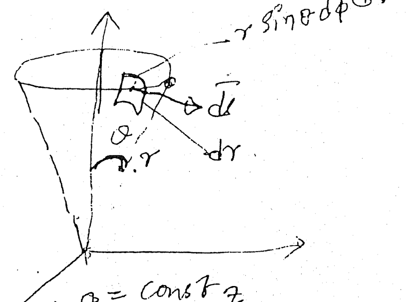
i) r = constant,



$\overline{ds} = r^2 \sin\theta\, d\phi\, d\theta\, \bar{a}_r$

$|ds| = \int_S r^2 \sin\theta\, d\phi\, d\theta$

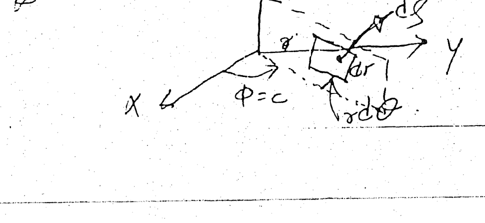
ii) θ = const.



$\overline{ds} = r\sin\theta\, dr\, d\phi\, \bar{a}_\theta$.

$|S| = \int_S r \sin\theta\, dr\, d\phi$.

iii) ϕ = const

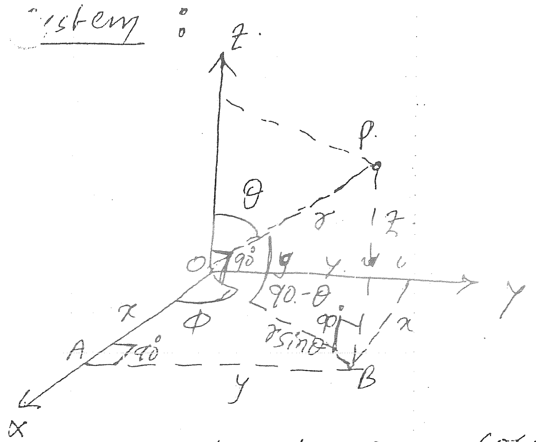


$\overline{ds} = r\,d\theta\, dr\, \bar{a}_\phi$.

$|ds| = r\, d\theta\, dr$

$S = \int_{r=0}^{r} r\,dr \int_{\theta=0}^{\pi} d r$. //

Relation ship between rectangular & cylendrical co-ordinate system :



Let $P \equiv (x, y, z)$
Also $P = (r, \theta, \phi)$
be the point in Rectangular & cylendrical co-ordinates

from the Δle OAB, $\cos\phi = \dfrac{OA}{OB} = \dfrac{x}{r\sin\theta}$

$\therefore \quad x = r\sin\theta\cos\phi$ //

$\sin\phi = AB/OB = y/r\sin\theta$

$\therefore \quad y = r\sin\theta\sin\phi$ //

From the Δle OBP, $\sin(90-\theta) = z/r$

$\cos\theta = z/r \quad (\because \sin(90-\theta) = \cos\theta)$

$\therefore \quad z = r\cos\theta$ //

$$\therefore \boxed{\begin{aligned} x &= r\sin\theta\cos\phi \longrightarrow 1 \\ y &= r\sin\theta\sin\phi \longrightarrow 2 \\ z &= r\cos\theta \longrightarrow 3 \end{aligned}}$$

Also, from 1, 2 & 3

$$\begin{aligned} x^2+y^2+z^2 &= r^2\sin^2\theta\cos^2\phi + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\theta \\ &= r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) + r^2\cos^2\theta) \\ &= r^2\sin^2\theta.1 + r^2\cos^2\theta \qquad (\because \sin^2\theta+\cos^2\theta=1) \\ &= r^2(\sin^2\theta+\cos^2\theta) \\ &= r^2.1 \\ &= r^2 \end{aligned}$$

$\therefore \quad r = \sqrt{x^2+y^2+z^2}$ //

Dividing 2 by 1 $\quad \dfrac{Y}{X} = \dfrac{r\sin\theta\sin\phi}{r\sin\theta\cos\phi} = \tan\phi$

$\phi = \tan^{-1}(Y/X)$ //

from 3. $\quad \theta = \cos^{-1}(r/z) = \cos^{-1}\left(\sqrt{x^2+y^2+z^2}/z\right)$ //

x. Dot product of unit vectors in rectangular & spherical co-ordinate system :

$\bar{a}_x \cdot \bar{a}_r = \sin\theta\cos\phi$. $\quad \bar{a}_x \cdot \bar{a}_\theta = \cos\theta\cos\phi$. $\quad \bar{a}_x \cdot \bar{a}_\phi = -\sin\phi$.

$\bar{a}_y \cdot \bar{a}_r = \sin\theta\sin\phi \quad \bar{a}_y \cdot \bar{a}_\theta = \cos\theta\sin\phi \quad \bar{a}_y \cdot \bar{a}_\phi = \cos\phi$.

$\bar{a}_z \cdot \bar{a}_r = \cos\theta \quad \bar{a}_z \cdot \bar{a}_\theta = -\sin\phi \quad \bar{a}_z \cdot \bar{a}_\phi = 0$.

Note: 1) If we wish to obtain the distance between two points or if we wish to obtain the vector, from one point to other point, when the points are given in cylendrical or spherical co-ordinates, we first convert the points in to rectangular co-ordinates & then proceed.

2. We make use of dot product of unit vectors while converting a vector from one co-ordinate system to other co-ordinate system.

3. Unit vectors may also be represented by the following ways shown below.

a) $\bar{a}_x \ \bar{a}_y \ \bar{a}_z$

$\bar{a}_\rho \ \bar{a}_\phi \ \bar{a}_z$

$\bar{a}_r \ \bar{a}_\theta \ \bar{a}_\phi$

b. $i_x \ i_y \ i_z$

$i_\rho \ i_\phi \ i_z$.

$i_r \ i_\theta \ i_\phi$

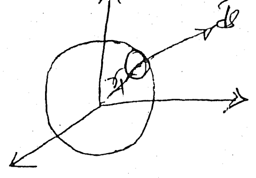
c. $1_x \ 1_y \ 1_z$

$1_\rho \ 1_\phi \ 1_z$

$1_r \ 1_\theta \ 1_\phi$.

x:1 Using the surface integral obtain the surface area of a sphere of radius r_1 mt.

Soln.



Consider an incremental surface area of ~~radius~~ on the surface of a sphere of radius r_1 mt. & this surface is at r = const.

$\therefore \bar{ds} = r^2\sin\theta \, d\theta \, d\phi \ \bar{a}_r$

$|ds| = r^2\sin\theta \, d\theta \, d\phi$

$\therefore$ Area $= \iint\limits_{r=\text{constant}} \bar{ds}$

$= r^2 \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi \Big|_{r=r_1}$

$= r_1^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$.

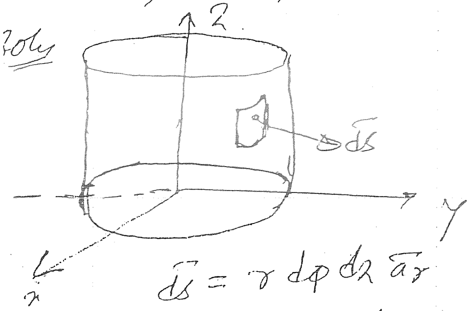
$= r_1^2 (-\cos\theta)_0^{\pi} \ (\phi)_0^{2\pi}$

$= r_1^2 (-\cos\pi + \cos 0) \ (2\pi - 0)$

$= r_1^2 \times 2 \times 2\pi = 4\pi r_1^2$ //

2. Using surface integral obtain the area of the curved surface ~~around~~ of a cylinder of radius r mt & height 'h' mt.

Soln



$\overline{ds} = r\, d\phi\, dz\, \overline{a_r}$

$|ds| = r\, d\phi\, dz.$

$\therefore$ Area $= A = \int_S ds$

$= \int_{\phi=0}^{2\pi}\int_{z=0}^{h} r\, d\phi\, dz \quad (r = \text{const})$

$= r\,(\phi)_0^{2\pi}\,(z)_0^{h}$

$= r\; 2\pi \cdot h$

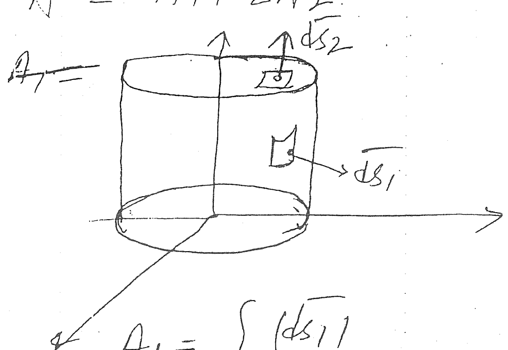
$\boxed{A = 2\pi r h}$

3. Using surface integral obtain the total surface area of a closed cylinder of radius 'r' & height h mt.

Soln Total Surface area, A

= Area of curved Surface + 2x Area of top flat surface

$A = A_1 + 2A_2$



$A_1 = \int_S |\overline{ds_1}|$

$= \int_{z=0}^{h}\int_{\phi=0}^{2\pi} r\, d\phi\, dz \quad (r = \text{const})$

$A_1 = r \int_0^{2\pi} d\phi \int_0^{h} dz.$

$= r \cdot (2\pi)\,(h).$

$\boxed{A_1 = 2\pi r h} \longrightarrow (1)$

$A_2 = \int_S |\overline{ds_2}|$

$= \int_S r\, dr\, d\phi. \quad (z = \text{const})$

$= \int_0^{r} r\, dr \int_0^{2\pi} d\phi$

$= \left(\frac{r^2}{2}\right)_0^{r} (\phi)_0^{2\pi}$

$= \frac{r^2}{2} \times 2\pi = \pi r^2 \longrightarrow 2.$

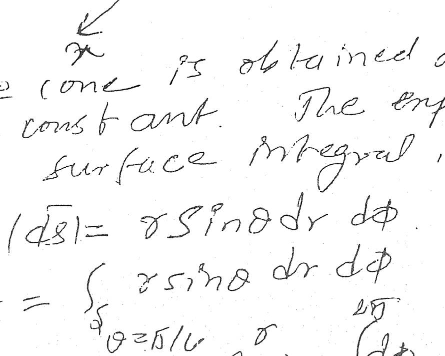
$\therefore\; A = \overline{A_1} + 2\overline{A_2}$

$= 2\pi r^2 + 2\pi r h$

$\boxed{A = 2\pi r\,(r + h)}$

4. Using the surface integral obtain the surface area of the cone of height 'h' & a cone angle of $\pi/2$

Soln



The cone is obtained at θ = constant. The expression for surface integral is,

$|\overline{ds}| = r \sin\theta\, dr\, d\phi.$

$A = \int_{S,\ \theta=\pi/4} r \sin\theta\, dr\, d\phi$

$= \sin\pi/4 \int_{r=0}^{r} r\, dr \int_0^{2\pi} d\phi.$

$= \frac{1}{\sqrt{2}} \left(\frac{r^2}{2}\right)_0^{r} (\phi)_0^{2\pi}$

$\boxed{A = \sqrt{2}\pi h^2.}$

$V = \cancel{113.04\ m^3}\ 523.59\ m^3$

5. Using the volume integral obtain the volume of a sphere of radius 'r' meter.

Soln. The expression for incremental volume is,

$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$V = \int_V r^2 \sin\theta\, dr\, d\theta\, d\phi$

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r} r^2 \sin\theta\, dr\, d\theta\, d\phi$

$= \int_0^r r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$

$= \left(\frac{r^3}{3}\right)_0^R (-\cos\theta)_0^{\pi} (\phi)_0^{2\pi}$

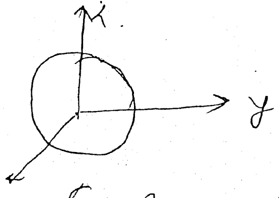
$= \frac{r^3}{3}(-\cos\pi + \cos 0)(2\pi - 0)$

$= \frac{r^3}{3}(1+1)(2\pi)$

$\boxed{V = \frac{4}{3}\pi r^3}\ m^3$

6. Using volume integral obtain the volume of a sphere of radius 5mt.

Soln



$dV = \oint r^2 \sin\theta\, dr\, d\theta\, d\phi$

$V = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{5} r^2 \sin\theta\, dr\, d\theta\, d\phi$

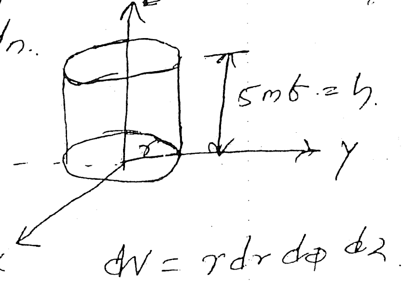
$= \int_0^5 r^2 dr \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$

$= \left(\frac{r^3}{3}\right)_0^5 (-\cos\theta)_0^{\pi} (\phi)_0^{2\pi}$

$= \frac{(125-0)}{3}(2)(2\pi)$

Ex: 7. Using the volume integral, obtain the volume of a cylinder of radius 2mt, height 5mt.

Soln..



$dV = r\, dr\, d\phi\, dz$

$V = \int_V r\, dr\, d\phi\, dz$

$= \int_{z=0}^{5}\int_{\phi=0}^{2\pi}\int_{r=0}^{r=2} r\, dr\, d\phi\, dz$

$= \int_0^5 dz \int_0^{2\pi} d\phi \int_0^2 r\, dr$

$= (z)_0^5 (\phi)_0^{2\pi} \left(\frac{r^2}{2}\right)_0^2$

$= 5 \times 2\pi \times (4-0)/2$

$= 20\pi$

$V = 1633.62\ m^3$

8. What are the co-ordinates of the following points in rectangular co-ordinates system. $P(5, 70°, 2)$ $Q(3, 30°, -1)$.

Soln. The points are given in cylendrical co-ordinate system. We know that

$x = r\cos\phi$

$y = r\sin\phi$

$z = z.$

i) $P(5, 70, 2) \equiv P(5\cos\overset{70}{\phi}, 5\sin 70, 2)$

$\equiv P(1.71, 4.69, 2).$

ii) $Q(3, 30°, -1) \equiv Q(3\cos 30, 3\sin 30, -1)$

$\equiv Q(2.598, 1.5, -1).$

9. What are the co-ordinates of the following points in cylendrical co-ordinates system.

i) $P(3,4,2)$ ii) $Q(3,-1,-2)$.

Soln: we know that

$r = \sqrt{x^2+y^2}$

$\phi = \tan^{-1}(y/x)$

$z = z$

i) $P(3,4,2) \equiv P(\sqrt{3^2+4^2}, \tan^{-1}(\frac{4}{3}), -2)$

$\equiv P(5, 53.13^\circ, -2)$.

ii) $Q(3,-1,-2) \equiv Q(\sqrt{3^2+(-1)^2}, \tan^{-1}(-\frac{1}{3}), -2)$

$\equiv (3.162, -18.43, -2)$.

0. Find a vector PQ in cylendrical co-ordinates system at point P, where $P(5, 60^\circ, 2)$ & $Q(2, 110^\circ, -1)$.

Soln: PQ is in cylendrical co-ordinate system.

i) convert the given points in rectangular co-ordinates

ii) obtain $\overline{PQ}$ in rectangular co-ordinates.

iii) Convert $\overline{PQ}$ in rectangular co-ordinates to cylendrical co-ordinates

i) $P \equiv (5, 60^\circ, 2) \equiv P(5\cos 60^\circ, 5\sin 60^\circ, 2)$

$\equiv (2.5, 4.33, 2)$

$Q \equiv (2, 110^\circ, -1) \equiv (2\cos 110, 2\sin 110, -1)$

$\equiv (-0.684, 1.88, -1)$.

ii) $\overline{PQ} = (-0.684 - 2.5)\bar{a}_x + (1.88 - 4.33)\bar{a}_y + (-1-2)\bar{a}_z$

$\overline{PQ} = -3.184\bar{a}_x - 2.45\bar{a}_y - 3\bar{a}_z$

$|\overline{PQ}| = \sqrt{3.184^2 - 2.45^2 + 3^2} = 5.01\,mt$

iii) Let $\overline{PQ} = A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z = \bar{A}$. 6

in cylendrical co-ordinates.

A_r, A_ϕ & A_z are components of $\overline{PQ}$ along r-axis, ϕ-axis & z-axis respectively.

$A_r = \bar{A}\cdot\bar{a}_r$

$= (-3.184\bar{a}_x - 2.45\bar{a}_y - 3\bar{a}_z)\cdot\bar{a}_r$

$= -3.184\,\bar{a}_x\cdot\bar{a}_r - 2.45\,\bar{a}_y\cdot\bar{a}_r - 3\bar{a}_z\cdot\bar{a}_r$

$= -3.184\cos\phi - 2.45(\sin\phi) - 0$

$A_r = (-3.184\cos\phi + 2.45\sin\phi)$

$A_\phi = \bar{A}\cdot\bar{a}_\phi$

$= (-3.184\bar{a}_x - 2.45\bar{a}_y - 3\bar{a}_z)\cdot\bar{a}_\phi$

$= -3.184\,\bar{a}_x\cdot\bar{a}_\phi - 2.45\,\bar{a}_y\cdot\bar{a}_\phi - 3\bar{a}_z\cdot\bar{a}_\phi$

$A_\phi = -3.184(-\sin\phi) - 2.45\cos\phi - 0$

$= 3.184\sin\phi - 2.45\cos\phi$.

$A_z = \bar{A}\cdot\bar{a}_z$

$= (-3.184\bar{a}_x - 2.45\bar{a}_y - 3\bar{a}_z)\cdot\bar{a}_z$

$= 0 - 0 - 3\bar{a}_z\cdot\bar{a}_z$

$= -3$

$\therefore \bar{A} = \overline{PQ} = (-3.184\cos\phi - 2.45\sin\phi)\bar{a}_r + (3.184\sin\phi - 2.45\cos\phi)\bar{a}_\phi - 3\bar{a}_z$ is the vector in cylendrical co-ordinates.

At point, we have, $\phi = 60^\circ$.

$\overline{PQ} = (-3.184\cos 60 - 2.45\sin 60)\bar{a}_r + (3.184\sin 60 - 2.45\cos 60)\bar{a}_\phi - 3\bar{a}_z$

$\boxed{\overline{PQ} = -3.714\bar{a}_r + 1.5319\bar{a}_\phi - 3\bar{a}_z}$

10. obtain the vector in cylendrical co-ordinates system that extends from $P(0,0,0)$ to $Q(2, 110^\circ, -1)$.

Soln: $P(x_1, y_1, z_1) \equiv (0,0,0)$

$Q(x_2, y_2, z_2) = Q(2\cos 110, 2\sin 110, -1)$

$\equiv Q(-0.684, 1.88, -1)$

∴ $\bar{PQ} = -0.684\bar{a}_x + 1.88\bar{a}_y - \bar{a}_z$.

Let the equivalent vector in cylendrical co-ordinates be,

$\bar{A} = \bar{PQ} = A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z$.

where $\bar{A}_r = \bar{PQ}\cdot\bar{a}_r$

$A_\phi = \bar{PQ}\cdot\bar{a}_\phi$

$A_z = \bar{PQ}\cdot\bar{a}_z$.

$A_r = (-0.684\bar{a}_x + 1.88\bar{a}_y - \bar{a}_z)\cdot\bar{a}_r$

$= -0.684\cos\phi + 1.88\bar{a}_y\cdot\bar{a}_r - \bar{a}_z\cdot\bar{a}_r$

$= -0.684\cos\phi + 1.88\sin\phi - 0$.

$= -0.684\cos\phi + 1.88\sin\phi$.

$A_\phi = (-0.684\bar{a}_x + 1.88\bar{a}_y - \bar{a}_z)\cdot\bar{a}_\phi$

$= -0.684\bar{a}_x\cdot\bar{a}_\phi + 1.88\bar{a}_y\cdot\bar{a}_\phi - \bar{a}_z\cdot\bar{a}_\phi$

$= -0.684(-\sin\phi) + 1.88\cos\phi - 0$.

$= +0.684\sin\phi + 1.88\cos\phi$

$A_z = (-0.684\bar{a}_x + 1.88\bar{a}_y - \bar{a}_z)\cdot\bar{a}_z$

$= -0.684\times 0 + 1.88\times 0 - \bar{a}_z\cdot\bar{a}_z$

$= 0 + 0 - 1$

∴ $\bar{A} = \bar{PQ} = (-0.684\cos\phi + 1.88\sin\phi)\bar{a}_r + (0.684\sin\phi + 1.88\cos\phi)\bar{a}_\phi - \bar{a}_z$ is the vector in cylendrical co-ordinate system.

At point P, we have, $\phi = 110°$

$\cancel{\therefore \bar{A} = \bar{PQ} = (-3.184\cos 60 - 2.45\sin 60)\bar{a}_r + (3.184\sin 60 - 2.45\cos 60)\bar{a}_\phi - 3\bar{a}_z}$

$\cancel{\boxed{\bar{A} = -3.714\bar{a}_r + 1.5319\bar{a}_\phi + 3\bar{a}_z}}$

$\cancel{\therefore \bar{A} = (-0.684\cos 60 + 1.88\sin 60)\bar{a}_r + (0.684\sin 60 + 1.88\cos 60)\bar{a}_\phi - \bar{a}_z}$.

$\bar{A} = (-0.684\cos 110 + 1.88\sin 110)\bar{a}_r + (0.684\sin 110 + 1.88\cos(110))\bar{a}_\phi - \bar{a}_z$

$\bar{A} = 2\bar{a}_r + 0 - \bar{a}_z = 2\bar{a}_r - \bar{a}_z$ //

11. How far is it from A(110, 60°, -20) to B(30°, 125°, 10)

Soln. $A \equiv (x_1, y_1, z_1) = A(110\cos 60, 110\sin 60, -20)$

$\equiv A(55, 95.263, -20)$

$B \equiv (x_2, y_2, z_2)$

$\equiv (30\cos 125, 30\sin 125, 10)$

$\equiv (-17.21, 24.57, 10)$

$\bar{AB} = (-17.21 - 55)\bar{a}_x + (24.57 - 95.26)\bar{a}_y + (10 - (-20))\bar{a}_z$

$\bar{AB} = -72.21\bar{a}_x - 70.69\bar{a}_y + 30\bar{a}_z$ //

$|\bar{AB}| = \sqrt{(72.21)^2 + (70.69)^2 + 30^2}$

$= 105.41$ units

12. Convert the following points in to spherical co-ordinate system. $P(x=3, y=3, z=4)$

Soln. $r = \sqrt{x^2+y^2+z^2}$

$= \sqrt{3^2+3^2+4^2}$

$\boxed{r = 5.83 \text{ mts}}$

$\phi = \tan^{-1}(y/x)$

$= \tan^{-1}(3/3)$

$= 45°$ //

$\theta = \cos^{-1}(z/r) = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right)$

$= \cos^{-1}\left(4/\sqrt{3^2+4^2+3^2}\right)$

$= 46.67°$ //

∴ $P \equiv (r = 5.83, \theta = 46.67°, \phi = 45°)$

13. Find the vector directed from $P(10, 3\pi/4, \pi/6)$ to $Q(5, \pi/4, \pi)$ where the points are given in spherical co-ordinates system.

Soln i) Convert the points in rectangular co-ordinates

$P \equiv (x_1, y_1, z_1)$

$\equiv P(10\sin 3\pi/4 \cdot \cos\pi/6, 10\sin 3\pi/4 \cdot \sin\pi/6, 10\cos 3\pi/4)$

$Q \equiv (x_2, y_2, z_2)$
$= (5\sin\pi/4 \cos\pi, 5\sin\pi/4 \sin\pi, 5\sin\frac{\pi}{4})$
$\equiv (-3.535, 0, 3.535)$

ii) $\overline{PQ} = (-3.535 - 6.124)\bar{a}_x + (0-3.535)\bar{a}_y + (3.535 + 7.07)\bar{a}_z$

$\overline{PQ} = (-9.658\bar{a}_x - 3.535\bar{a}_y + 10.605\bar{a}_z)$

i) To convert $\overline{PQ}$ in spherical co-ordinate system.

Let $\overline{PQ} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi$ be the equivalent vector in spherical co-ordinate system.

$\therefore A_r = \overline{PQ}\cdot\bar{a}_r$

$A_r = (-9.658\bar{a}_x - 3.535\bar{a}_y + 10.605\bar{a}_z)\cdot\bar{a}_r$
$= (-9.658\,\bar{a}_x\cdot\bar{a}_r - 3.535\,\bar{a}_y\cdot\bar{a}_r + 10.605\,\bar{a}_z\cdot\bar{a}_r$.
$= -9.658\sin\theta\cos\phi - 3.535\sin\theta\sin\phi + 10.605\cos\theta$

$A_\theta = \overline{PQ}\cdot\bar{a}_\theta$
$= (-9.658\bar{a}_x - 3.535\bar{a}_y + 10.609\bar{a}_z)\cdot\bar{a}_\theta$
$= -9.658\,\bar{a}_x\cdot\bar{a}_\theta - 3.535\,\bar{a}_y\cdot\bar{a}_\theta + 10.609\,\bar{a}_z\cdot\bar{a}_\theta$
$= -9.658\cos\theta\cos\phi - 3.535\cos\theta\sin\phi - 10.609\cos\theta$ //

$A_\phi = \overline{PQ}\cdot\bar{a}_\phi$
$= (-9.658\bar{a}_x - 3.535\bar{a}_y + 10.609\bar{a}_z)\cdot\bar{a}_\phi$
$= +9.658\sin\phi - 3.535\cos\phi + 0$ //

$\overline{PQ} = (-9.658\sin\theta\cos\phi - 3.535\sin\theta\sin\phi + 10.605\cos\theta)\bar{a}_r$
$+ (-9.658\cos\theta\cos\phi - 3.535\cos\theta\sin\phi - 10.609\cos\theta)\bar{a}_\theta$
$+ (9.658\sin\phi - 3.535\cos\phi)\bar{a}_\phi$

At point P. we have, $\theta = 3\pi/4$, $\phi = \pi/6$,

$\overline{PQ} = -14.66\bar{a}_r - 0.335\bar{a}_\theta + 1.767\bar{a}_\phi$

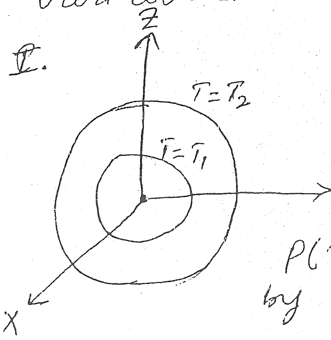
Scalar & vector fields :

In the previous section we studied vector algebra applied to constant vectors, which have constants as the co-efficients of unit vectors. The constant vectors in different co-ordinates systems are given in general form as,

$\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$
or $\bar{A} = A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z$
or $\bar{A} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi$

In this section we shall discuss the scalar & vector fields which are the functions of axes variables.

I.

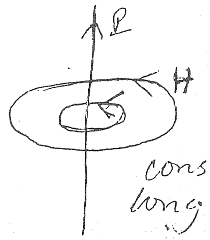


Consider a heat source at the origin. The temperature at any point $P(x,y,z)$ is given by

$$T = \frac{H}{4\pi(x^2+y^2+z^2)}$$

We see that the temperature is different at different points. Temperature T is a scalar field because it is a function of (x,y,z) having no direction.

II. vector field :



$$\bar{H} = \frac{I}{2\pi(x^2+y^2)}(x\bar{a}_x - y\bar{a}_y)$$

consider an infinitely long conductor carrying current I, then the ~~magnitude~~ of magnetic field at any point is given by

$$\bar{H} = \frac{I}{2\pi(x^2+y^2)}(x\bar{a}_x - y\bar{a}_y)$$

As $\bar{H}$ varies both in magnitude & direction at different points, ~~the~~ it is a vector field.

<u>conversion of vector field from one co-ordinate system to another co-ordinate system</u>

we use dot products of unit vectors to convert the vector fields from one co-ordinate system to other co-ordinate system. The co-efficients of unit vectors are also to be transformed in to the converted co-ordinate system.

Ex: obtain the vector in cylendrical & spherical co-ordinate system. Given the vector in Rectangular co-ordinates

$\bar{A} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$.

soln Let $\bar{A} = A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z$ be the vector in cylendrical co-ordinates,

$A_r = \bar{A}\cdot\bar{a}_r$

$= \cancel{(A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z)\cdot\bar{a}_r}$

$= \cancel{A_x\,\bar{a}_x\cdot\bar{a}_r}$

$A_r = (x\bar{a}_x + y\bar{a}_y + z\bar{a}_z)\cdot\bar{a}_r$

$= x\,\bar{a}_x\cdot\bar{a}_r + y\,\bar{a}_y\cdot\bar{a}_r + z\bar{a}_z\cdot\bar{a}_r$

$= x\cos\phi + y\sin\phi + 0.$

$A_r = x\cos\phi + y\sin\phi$

$= r\cos\phi\cdot\cos\phi + r\sin\phi\cdot\sin\phi$

$= r(\cos^2\phi + \sin^2\phi) = r$

($\because x = r\cos\phi$ & $y = r\sin\phi$).

lly $A_\phi = \bar{A}\cdot\bar{a}_\phi$

$= (x\bar{a}_x + y\bar{a}_y + z\bar{a}_z)\cdot\bar{a}_\phi$

$= x\,\bar{a}_x\cdot\bar{a}_\phi + y\,\bar{a}_y\cdot\bar{a}_\phi + z\,\bar{a}_z\cdot\bar{a}_\phi$

$= x(-\sin\phi) + y\cdot\cos\phi + z\cdot 0.$

$= r\cos\phi\cdot(-\sin\phi) + r\sin\phi\cos\phi$

$= -r\cos\phi\sin\phi + r\sin\phi\cos\phi$

$= 0$

$A_z = (x\bar{a}_x + y\bar{a}_y + z\bar{a}_z)\cdot\bar{a}_z$

$= x\,\bar{a}_x\cdot\bar{a}_z + y\,\bar{a}_y\cdot\bar{a}_z + z\cdot\bar{a}_z\cdot\bar{a}_z$

$= 0 + 0 + z\cdot 1$

$A_z = z.$

$\therefore \boxed{\bar{A} = r\bar{a}_r + z\bar{a}_z}$

Ex: Give the vector $\bar{A} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$ in Rectangular co-ordinates obtain the equivalent vector in spherical co-ordinates.

Soln $\bar{A} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$.

Let the equivalent vector in spherical co-ordinate system be, $\bar{A} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi$

$A_r = \bar{A}\cdot\bar{a}_r$

$= (x\bar{a}_x + y\bar{a}_y + z\bar{a}_z)\cdot\bar{a}_r$

$= x\,\bar{a}_x\cdot\bar{a}_r + y\,\bar{a}_y\cdot\bar{a}_r + z\,\bar{a}_z\cdot\bar{a}_r$

$= x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cdot\cos\theta$

Also, W.K.T. $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$A_r = r\sin\theta\cos\phi\cdot\sin\theta\cos\phi + r\sin\theta\sin\phi\cdot\sin\theta\sin\phi + r\cdot\cos\theta\cos\theta$

$= r(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)$

$= r(\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta)$

$= r(\sin^2\theta + \cos^2\theta) = r$

$A_r = r(\sin^2\theta \cdot 1 + \cos^2\theta)$.

$= r \cdot 1$

$A_r = r$

$\bar{A}_\theta = \bar{A} \cdot \bar{a}_\theta$

$= (x\bar{a}_x + y\bar{a}_y + z\bar{a}_z) \cdot \bar{a}_\theta$

$= x\,\bar{a}_x \cdot \bar{a}_\theta + y\,\bar{a}_y \cdot \bar{a}_\theta + z\,\bar{a}_z \cdot \bar{a}_\theta$

$= r\cos\phi\sin\theta \cdot \cos\theta\cos\phi$
$+ r\sin\theta\sin\phi\cos\theta\sin\phi$
$+ r\cos\theta(-\sin\theta)$

$= r(\cos^2\phi\sin\theta\cos\theta + \sin^2\phi\sin\theta\cos\theta$
$+ \sin\theta\cos\theta)$

$= r((\cos^2\phi + \sin^2\phi)\sin\theta\cos\theta$
$- \sin\theta\cos\theta)$

$= r(1 \cdot \sin\theta\cos\theta - \sin\theta\cos\theta)$

$A_\theta = 0$ //

$A_\phi = \bar{A} \cdot \bar{a}_\phi$

$= (x\bar{a}_x + y\bar{a}_y + z\bar{a}_z) \cdot \bar{a}_\phi$

$= x\,\bar{a}_x \cdot \bar{a}_\phi + y\,\bar{a}_y \cdot \bar{a}_\phi + z\,\bar{a}_z \cdot \bar{a}_\phi$

$= r\sin\theta\cos\phi(-\sin\phi)$
$+ r\sin\theta\sin\phi\cos\phi$
$+ r\cos\theta \cdot 0$

$= -r\sin\theta\cos\phi\cancel{\sin\phi}$
$+ r\sin\theta\sin\phi\cancel{\cos\phi}$

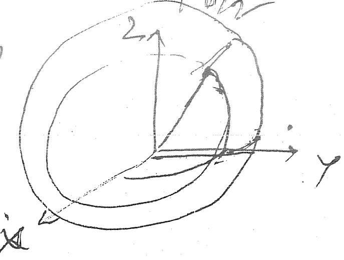
$= 0$ //

$\therefore \bar{A} = r\bar{a}_r + 0 + 0$.

$\boxed{\bar{A} = r\bar{a}_r}$

Ex: using spherical coordinates to express the differential volume, integrate to obtain the volume defined by $1 \le r \le 2m$, $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$.

Soln



$dv = r^2\sin\theta\,dr\,d\theta\,d\phi$

$v = \int dv$

$= \int_1^2 r^2\,dr \int_{\theta=0}^{\pi/2} \sin\theta\,d\theta \int_0^{90} d\phi$

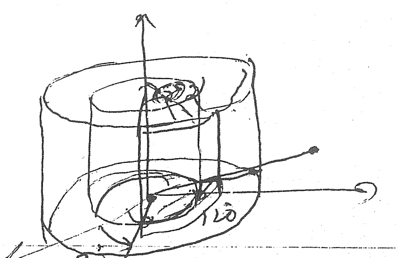
$= \left(\frac{r^3}{3}\right)_1^2 (-\cos\theta)_0^{\pi/2} (\phi)_0^{\pi/2}$

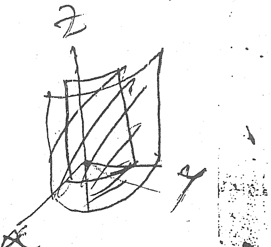
$= \left(\frac{2^3 - 1^3}{3}\right)(-\cos\pi/2 + \cos 0)(\pi/2)$

$v = \cancel{4.188\ m^3}\ 3.665\ m^3$.

Ex: Use cylindrical coordinate system & integrate to obtain the volume of right circular cylinder

Ex: use cylindrical coordinate system to find the area of curved surface of a right circular cylinder where $r = 2m$, $h = 5m$ & $30° \le \phi \le 120°$





At the end of this chapter the student should be ~~able to~~.

i. able to differentiate between vectors & scalars.

ii. able to perform mathematical operations on vectors.

iii.

MS →

BS

BE → DDB
RMHP
MGM

JCB →

BE

Shrikant Akandkar →

Rad (EI)

BRU — MAT + BEE
JMR → CMOS + EI
DDB → AC + BE
~~JMR~~
RMH → BE + LD
Bme → PE + NA
SSIn → RTC + MWR
PSVe → ME + DSPA
VLH → ~~DSPA~~ + ~~RES~~ DSD
PLG → ESD + AEC
MKRM → BE + OFC

Mahantshetty

28 1, 2, 6, 15, 31, 38, 40, 44;

24 :- 6, 7, 8, 25, 28, 31.

31/7 : 1, 2, 3, 6, 14, 9, 23, 38, 8, 40, 4

01/8 : 6, 19, 28, 35, 40, 2

15/2 : 3, 6, 13, 7, 8, 8, 34, 8, 41, 3, 52, 3, 4, 60, 4, 8,

UNIT-I-9. ELECTROSTATICS

The branch of science which deals with the study of charges at rest is known as electrostatics.

Laws of Electrostatics :

First law : Like charges of electricity repel each other & unlike charges attract each other.

Second law : Coulomb's law : Inverse square law of electrostatics:

It states that the force of attraction or repulsion between two point charges is directly proportional to the product of magnitude of the charges & inversely proportional to the square of the distance between them. ~~two point~~

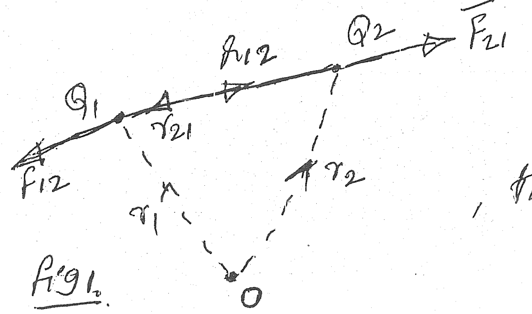


fig1.

Consider two point charges Q_1 & Q_2 are placed at a distance of r_1 & r_2 from the origin. By coulomb's law, the force on Q_2 due to Q_1 is given as

$$\bar{F}_{21} \propto Q_1 Q_2$$

$$\bar{F}_{21} \propto \frac{1}{d_{r12}^2}$$

$\therefore \quad \bar{F}_{21} = k \dfrac{Q_1 Q_2}{d_{r12}^2}$, where K is a proportionality constant & its value depends upon the type of units used. In SI Unit $K = 1/4\pi\epsilon$

where ϵ = permittivity of the medium
$= \epsilon_0 \epsilon_r$.

ϵ_0 = Absolute permittivity.
ϵ_r = Relative permittivity & its value depends upon the type of medium.

$$\therefore \quad \bar{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon |\bar{r}_{12}|^2}$$

In vector form

$$\bar{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon |\bar{r}_{12}|^2} \bar{a}_{r12} \longrightarrow 1.$$

Where $\bar{a}_{r12}$ is the unit vector in the direction of Q_1 to Q_2.

From the above fig1., $\bar{r}_1 + \bar{r}_{12} = \bar{r}_2 \Rightarrow \bar{r}_{12} = \bar{r}_2 - \bar{r}_1$

& $\bar{a}_{r12} = \dfrac{\bar{r}_{12}}{|\bar{r}_{12}|} = \dfrac{\bar{r}_2 - \bar{r}_1}{\sqrt{r_2^2 + r_1^2}}$

$$\therefore \quad \bar{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon (\sqrt{r_1^2 + r_2^2})^2} \cdot \frac{\bar{r}_2 - \bar{r}_1}{\sqrt{r_2^2 + r_1^2}}$$

$$\boxed{\bar{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon (r_1^2 + r_2^2)^{3/2}} (\bar{r}_2 - \bar{r}_1)} \longrightarrow 2$$

Eqn. 2 is the coulombs law in vector form.

Ily, the force exerted on Q_1 due to Q_2 is

$$\bar{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon\,|\bar{r}_{21}|^2}\,(\bar{r}_1 - \bar{r}_2)$$

$$\boxed{\bar{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon\,(r_1^2 + r_2^2)^{3/2}}\,(\bar{r}_1 - \bar{r}_2)}$$

Units : Q_1 & Q_2 are measured in coulomb (C)

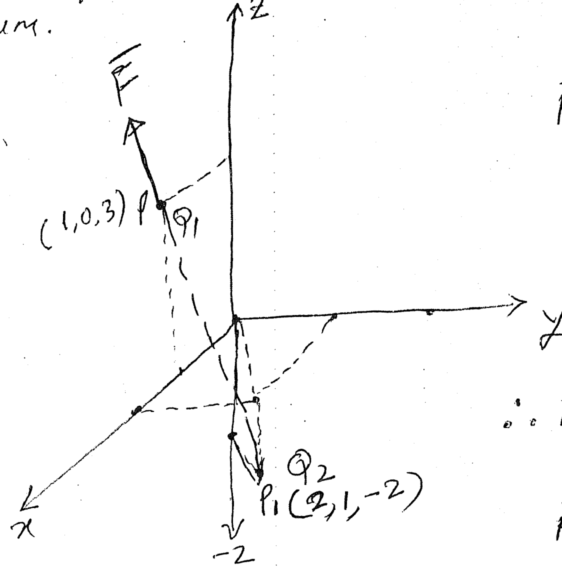
$|\bar{r}_{12}| \Rightarrow |\bar{r}_{21}|$ is measured in ~~mtr~~ meter (mt)

$\epsilon \rightarrow$ measured in farad/meter. (F/mt)

$F \rightarrow$ measured in Newton. (N)

Ex:1. obtain the force experienced by a 20 μC charge at P(1,0,3) due to a point charge of −100 μC at (2,1,−2) mt placed in air medium.

Soln.



Force on Q_1 is given by.

$$\bar{F} = \frac{Q_1 Q_2}{4\pi\epsilon\,|\overline{P_1P}|^2}\,\bar{a}_{P_1P}$$

using distance formula

$$\overline{P_1P} = -\bar{a}_x - \bar{a}_y + 5\bar{a}_z$$

$$|\overline{P_1P}| = \sqrt{1^2+1^2+5^2} = 5.196 \text{ mt}$$

$$\therefore \bar{F} = \frac{20\times10^{-6}\times(-100\times10^{-6})}{4\pi\epsilon_0\times(5.196)^2}\,\frac{-\bar{a}_x - \bar{a}_y + 5\bar{a}_z}{5.196}$$

$$\bar{F} = -0.128(-\bar{a}_x - \bar{a}_y + 5\bar{a}_z)$$

$$= 0.128(\bar{a}_x + \bar{a}_y - 5\bar{a}_z) \text{ N.}$$

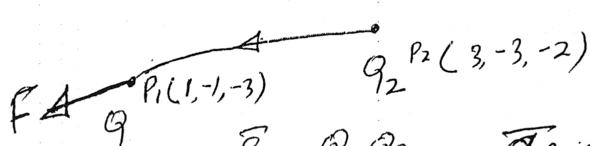
$$|\bar{F}| = \sqrt{(0.128)^2(1+1+25)}.$$

$$\boxed{|F| = 0.665 \text{ Newton}}$$

Ex:2 A point charge of $Q_1 = 300\,\mu C$ located at (1,−1,−3) experiences a force $\bar{F} = 8\bar{a}_x - 8\bar{a}_y + 4\bar{a}_z$ Newton. due to a point charge Q_2 at (+3,−3,−2). what is the charge of Q_2.

Soln. Given. $\bar{F} = 8\bar{a}_x - 8\bar{a}_y + 4\bar{a}_z$

$Q_1 = 300\,\mu C$ at $P_1(1,-1,-3)$ mt. $Q_2 = ?$ placed at $P_2(+3,-3,-2)$



$$\bar{F} = \frac{Q_1 Q_2}{4\pi\epsilon\,|\overline{P_2P_1}|^2}\,\bar{a}_{P_2P_1}$$

$$\overline{P_2P_1} = -2\bar{a}_x + 2\bar{a}_x - \bar{a}_z \qquad |\overline{P_2P_1}| = \sqrt{2^2+2^2+1} = 3 \text{ mt.}$$

$$\bar{F} = \frac{300\times10^{-6}\times Q_2}{4\pi\epsilon_0 \times 3^2} \times \frac{-2\bar{a}_x+2\bar{a}_y-\bar{a}_z}{3}$$

$$8\bar{a}_x - 8\bar{a}_y + 4\bar{a}_z = 9.98\times10^{4}\, Q_2 \times (-2\bar{a}_x+2\bar{a}_y-\bar{a}_z)$$

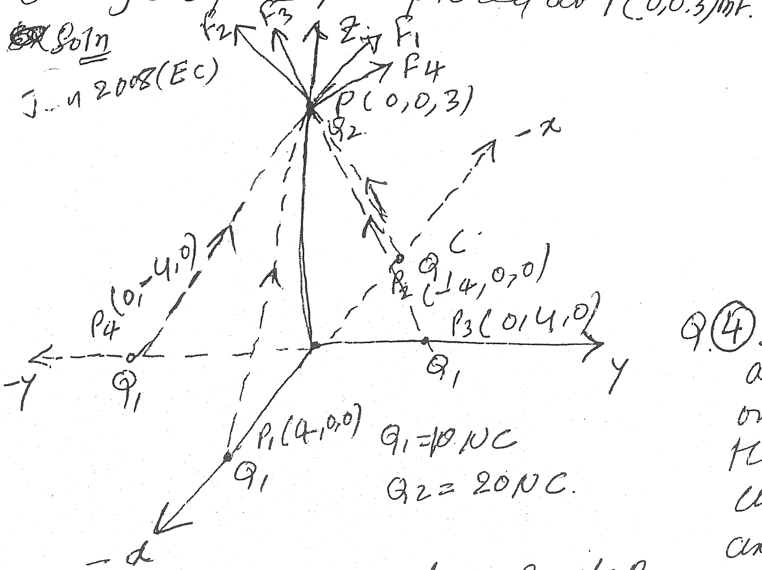
$$-4(2\bar{a}_x + 2\bar{a}_y - \bar{a}_z) = 9.98\times10^{4}\, Q_2 (-2\bar{a}_x+2\bar{a}_y+\bar{a}_z)$$

$$Q_2 = \frac{-4}{9.98\times10^{4}} = \underline{-40\,\mu C}$$

Ex:3 Four point charges of 10 nC each are placed on each of the x & y axis at a distance of ±4 mt. from the origin. Obtain the force on a charge of 20 nC placed at P(0,0,3) mt.

Soln

Jan 2008 (EC)



$\bar{F}_1$ = Force on Q_1 due Q_1 at P_1

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi\epsilon_0 |\bar{P_1P}|^2}\,\bar{a}_{P_1P}$$

$$= \frac{10\times20\times10^{-12}}{4\pi\epsilon_0 (5)^2}\,\frac{(-4\bar{a}_x+3\bar{a}_z)}{5}$$

$$= +0.0143(-4\bar{a}_x+3\bar{a}_z)$$

$$= +0.0574\,\bar{a}_x + 0.0431\,\bar{a}_z \rightarrow (1)$$

$$|\bar{F}_1| = \sqrt{0.0574^2 + 0.0431^2}$$

$$|\bar{F}_1| = \underline{0.0717 \text{ Newton}}$$

From the symmetry, we can write that, the force due to Q_1 at P_2 is given by

$$\bar{F}_2 = 0.0574\,\bar{a}_x + 0.0431\,\bar{a}_z \rightarrow 2.$$

The force exerted at Q_2 due to Q_1 at P_3 is,

$$\bar{F}_3 = \frac{Q_1 Q_2}{4\pi\epsilon_0 |\bar{P_3P_1}|^2}\,\bar{a}_{P_3P_1}$$

$$\bar{F}_3 = \frac{200\times10^{-12}}{4\pi\epsilon_0 (\sqrt{4^2+3^2})^2}\,\frac{(-4\bar{a}_y+3\bar{a}_z)}{\sqrt{4^2+3^2}}$$

$$= -0.057\,\bar{a}_y + 0.044\,\bar{a}_z \rightarrow 3.$$

Due to symmetry, we can write the force on Q_2 due to Q_1 at P_4 is

$$\bar{F}_4 = 0.057\,\bar{a}_y + 0.044\,\bar{a}_z \rightarrow 4.$$

Adding eqn. (1) to (4)

$$\bar{F} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$= 4\times0.044\,\bar{a}_z$$

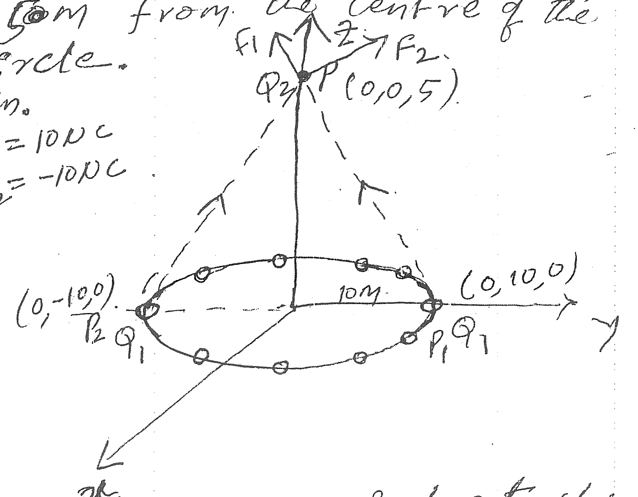
$$\boxed{\bar{F} = 0.176\,\bar{a}_z} \text{ Newton.}$$

Q(4). 10 point charges of 10 μC each are placed at equal distance on a circle of radius 10 mt. Find the force experienced by a charge of −10 μC placed on the axis of the circle at a distance of 5 m from the centre of the circle.

Soln.

$Q_1 = 10\,\mu C$

$Q_2 = -10\,\mu C$



The force F_1 at P due to charge Q_1 at P_1 is given by,

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi\epsilon_0 |\bar{P_1P}|^2}\,\bar{a}_{P_1P}$$

$$\bar{F}_1 = \frac{-100\times10^{-12}}{4\pi\epsilon_0(\sqrt{10^2+5^2})^2}\frac{(-10\bar{a}_y+5\bar{a}_z)}{\sqrt{10^2+5^2}}$$

$$\bar{F}_1 = -6.43\times10^{-4}(-10\bar{a}_y+5\bar{a}_z)$$

$$= (+64.3\times10^{-4}\bar{a}_y - 32.15\times10^{-4}\bar{a}_z) \rightarrow (1)$$

The force F_2 at 'p' due to a charge at P_2 is,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0|\bar{P_2P_1}|^2}\bar{a}_{P_2P_1}$$

$$= \frac{-100\times10^{-12}}{4\pi\epsilon_0(\sqrt{125})^2}\frac{(10\bar{a}_y+5\bar{a}_z)}{\sqrt{125}}$$

$$\bar{F}_2 = -6.43\times10^{-4}(10\bar{a}_y+5\bar{a}_z)$$

$$= -6.43\times10^{-4}(10\bar{a}_y+5\bar{a}_z)$$

$$\bar{F}_2 = -64.3\times10^{-4}\bar{a}_y - 32.15\times10^{-4}\bar{a}_z \rightarrow (2)$$

By symmetry the components of forces parallel to xy plane due to the two radially opposite ~~elements~~ charges cancel out & their components along z axis addup.

$\therefore$ The net force on $-10\,\mu C$ charge is,

$$\bar{F} = 10\times\bar{F}_2$$

$$= 10\times32.15\times10^{-4}\bar{a}_z$$

$$\boxed{\bar{F} = 32.15\times10^{-3}\bar{a}_z \text{ Newton}}$$

5. Two small conducting spheres have charges of 2nc & -0.5nc respectively, when they are placed 4 cm apart, what is the force between them? If they are brought in to contact & then seperated, by 4cm, what is the force between them.

Soln. Given, $Q_1 = 2nc$, $Q_2 = -0.5nc$

$d = 5cm = 5\times10^{-2}$ mt.

$$F = \frac{Q_1 Q_2}{4\pi\epsilon r^2}$$

$$= \frac{4\times-0.5\times10^{-18}}{4\pi\epsilon_0(5\times10^{-2})^2}$$

$$\boxed{F = -7.19\times10^{-6}\ N}$$

When these two conducting spheres are brought in to contact & then are seperated by equal amount, the charge on each sphere is given by,

$$Q = \frac{Q_1+Q_2}{2} = \frac{4\times10^{-9}+(-0.5\times10^{-9})}{2}$$

$$Q = 1.75\times10^{-9}C.$$

$$\therefore F = \frac{Q\times Q}{4\pi\epsilon r^2} = \frac{1.75\times1.75\times10^{-18}}{4\pi\epsilon(5\times10^{-2})^2}$$

$$\boxed{F = 1.1009\times10^{-5}\text{ Newton}}$$

6. 2003 July

The force on a point charge situated 10 cm away from another charge of the same magnitude in a dielectric medium, of relative permittivity 81 is 0.1 Newton. Determine the magnitude of charge.

Soln Given: $Q_1 = Q_2 = Q = ?$

$d = 10cm = 10\times10^{-2}$ mt, $\epsilon_r = 81$, $F = 0.1N$.

$$F = \frac{Q_1Q_2}{4\pi\epsilon d^2} = \frac{Q^2}{4\pi\epsilon_0(10\times10^{-2})^2\times81}$$

$$0.1 = 1.11\times10^{18}\,Q^2$$

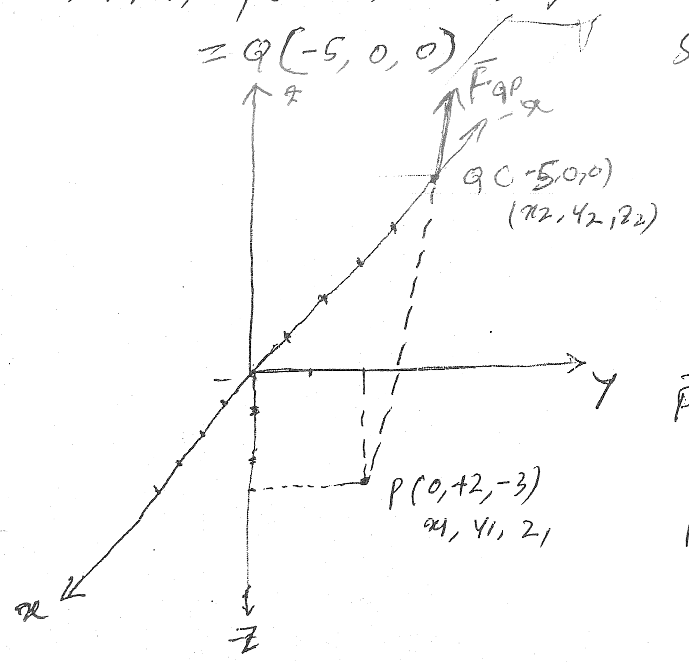
$$Q = \sqrt{\frac{0.1}{1.11\times10^{10}}} = 3.0\,nC$$

7. A charge of 5nc is located at P(2, π/2, 3) & another charge of -10nc is situated at Q(5, π, 0), calculate the force exerted by one charge on the other.

Soln. ~~Pt~~

Soln. $P(x_1, y_1, z_1) = P(2\cos\pi/2,\ 2\sin\pi/2, -3)$
$= P(0,\ 2, -3)$

$Q(x_2, y_2, z_2) = Q(5\cos\pi,\ 5\sin\pi,\ 0)$
$= Q(-5, 0, 0)$



$$\bar{F}_{QP} = \frac{5\times10^{-9}\times(-10\times10^{-9})}{4\pi\epsilon_0\left(\sqrt{5^2+2^2+3^2}\right)^2}\ \frac{(-5\bar{a}_x - 2\bar{a}_y + 3\bar{a}_z)}{\sqrt{5^2+2^2+3^2}}$$

$\bar{F}_{QP} = -1.98\times10^{-9}(-5\bar{a}_x - 2\bar{a}_y + 3\bar{a}_z)\,N.$
$= 1.98\times10^{-9}(5\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z)$

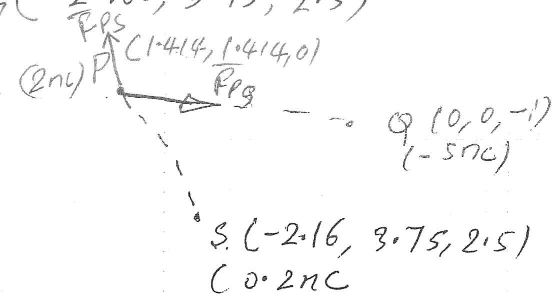
The nature of force is attractive.

8. Three charges of 2nC, −5nC & 0.2nC are situated at $P(2, \frac{\pi}{2}, \frac{\pi}{4})$ $Q(1, \pi, \pi/2)$ & $S(5, \pi/3, 2\pi/3)$ respectively. Find the force acting on 2nC charge at point P. Is this force a force of attraction or repulsive.

Soln. P, Q & S are given in spherical co-ordinates & its equivalent rectangular co-ordinates are

$P(x_1, y_1, z_1) = P(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$
$= P(2\sin\pi/2\cos\pi/4,\ 2\sin\pi/2\sin\pi/4,\ 2\cos\pi/2)$
$= P(1.414,\ 1.414,\ 0)$

$Q \equiv (1\sin\pi\cos\pi/2,\ 1\sin\pi\sin\pi/2,\ 1\cos\pi)$ 3.
$Q \equiv (0, 0, -1)$
$S \equiv (5\sin\pi/3\cos2\pi/3,\ 5\sin\pi/3\sin2\pi/3,\ 5\cos\pi/3)$
$S \equiv (-2.165,\ 3.75,\ 2.5)$



$$\bar{F}_{PQ} = \frac{2\times5\times10^{-18}(1.414\bar{a}_x + 1.414\bar{a}_y + \bar{a}_z)}{4\pi\epsilon_0\,(1.414^2 + 1.414^2 + 1)^{3/2}}$$

$\bar{F}_{PQ} = 8.052\times10^{-9}(1.414\bar{a}_x + 1.414\bar{a}_y + \bar{a}_z)$
$= (11.38\,\bar{a}_x + 11.38\,\bar{a}_y + 8.052\,\bar{a}_z)\times10^{-9}\,N.$

$$\bar{F}_{PS} = \frac{2\times0.2\times10^{-18}(3.574\bar{a}_x - 2.336\bar{a}_y - 2.5\bar{a}_z)}{4\pi\epsilon_0\,(3.574^2 + 2.336^2 + 2.5^2)^{3/2}}$$

$\bar{F}_{PS} = 0.0148(3.574\bar{a}_x - 2.336\bar{a}_y - 2.5\bar{a}_z)\times10^{-9}$
$= (0.0528\,\bar{a}_x - 0.0345\,\bar{a}_y - 0.037)\times10^{-9}\,N$

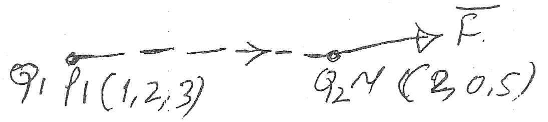
$\bar{F} = \bar{F}_{PS} + \bar{F}_{PQ}$
$= (11.432\,\bar{a}_x - 11.345\,\bar{a}_y + 8.015\,\bar{a}_z)\times10^{-9}$

$|\bar{F}| = \sqrt{11.432^2 + 11.34^2 + 8.015^2}\ \times10^{-9}$

$\boxed{|\bar{F}| = 17.98\times10^{-9}\ \text{newton}}$

9. Find the force exerted on Q_2 by Q_1 if $Q_1 = 3\times10^{-4}C$, $Q_2 = -10^{-4}C$ at $N(2, 0, 5)$ in vacuum.
June 2010 (EC44)

Soln



$$\bar{F} = \frac{Q_1Q_2}{4\pi\epsilon\,|P_1N|^2}\ \bar{a}_{P_1N}$$

$$\bar{F} = \frac{3\times10^{-4}\times(-1\times10^{-4})\times(-\bar{a}_x-2\bar{a}_y+2\bar{a}_z)}{4\pi\epsilon_0\,(1^2+2^2+2^2)^{3/2}}$$

$$\bar{F} = \frac{-30\times10^{-8}(-\bar{a}_x-2\bar{a}_y+2\bar{a}_z)}{4\pi\epsilon_0\,(27)}$$

$$\bar{F} = -10(-\bar{a}_x-2\bar{a}_y+2\bar{a}_z)$$

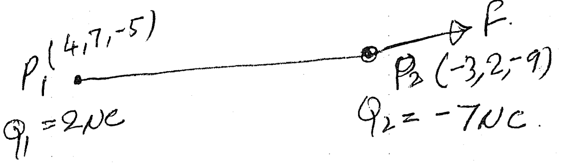
$$\bar{F} = 10\bar{a}_x + 20\bar{a}_y - 20\bar{a}_z \text{ Newton.}$$

$$|F| = \sqrt{10^2+20^2+20^2}$$

$$\boxed{F = 30 \text{ Newton.}} \text{ Newton.}$$

10. Two point charges of magnitude 2nC & -7nC are located at places $P_1(4,7,-5)$ & $P_2(-3,2,-9)$ respectively in free space, evaluate the vector force on charge P_2. (July-2008)(EC)

Soln.



Assuming repulsive force,

$$\bar{F} = \frac{2\times10^{-6}\times(-7\times10^{-6})(-7\bar{a}_x-5\bar{a}_y-4\bar{a}_z)}{4\pi\epsilon_0\,(7^2+5^2+4^2)^{3/2}}$$

$$\bar{F} = -1.475\times10^{-4}(-7\bar{a}_x-5\bar{a}_y-4\bar{a}_z)$$

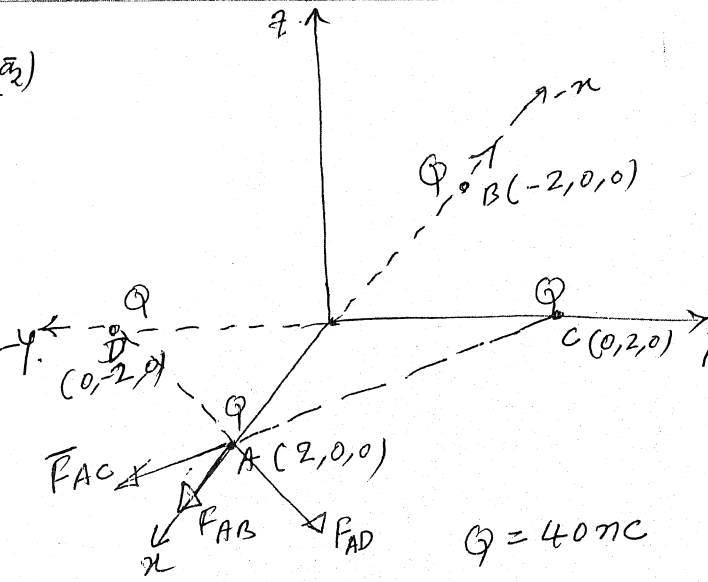
$$= 10.325\,\bar{a}_x + 7.375\,\bar{a}_y + 5.9\,\bar{a}_z$$

$$|F| = \sqrt{10.325^2+7.375^2+5.9^2}$$

$$|F| = 13.993 \text{ Newton.}$$

11. Four positive point charges of 40nC each are located at $A(2,0,0)$, $B(-2,0,0)$, $C(0,2,0)$ & $D(0,-2,0)$. Find the force $\bar{F}$ & electric field intensity $\bar{E}$ at A. July08 (EC)

Soln.



$Q = 40nC$

$$\bar{F}_{AB} = \frac{40\times40\times10^{-18}}{4\pi\epsilon_0(\sqrt{4^2})^2}\left(\frac{4\bar{a}_x}{\sqrt{4^2}}\right)$$

$$\bar{F}_{AB} = 0.9\,\bar{a}_x\times10^{-6}$$

$$\bar{F}_{AC} = \frac{40\times40\times10^{-18}}{4\pi\epsilon_0(\sqrt{2^2+2^2})^2}\,\frac{(2\bar{a}_x-2\bar{a}_y)}{\sqrt{2^2+2^2}}$$

$$= 6.363\times10^{-7}(2\bar{a}_x-2\bar{a}_y)$$

$$= (1.272\,\bar{a}_x - 1.272\,\bar{a}_y)\times10^{-6}$$

$$\bar{F}_{AD} = \frac{40\times40\times10^{-18}}{4\pi\epsilon_0}\,\frac{(2\bar{a}_x+2\bar{a}_y)}{(2^2+2^2)^{3/2}}$$

$$= (1.272\,\bar{a}_x + 1.272\,\bar{a}_y)\times10^{-6}.$$

$$\bar{F} = \bar{F}_{AB} + \bar{F}_{AC} + \bar{F}_{AD}$$

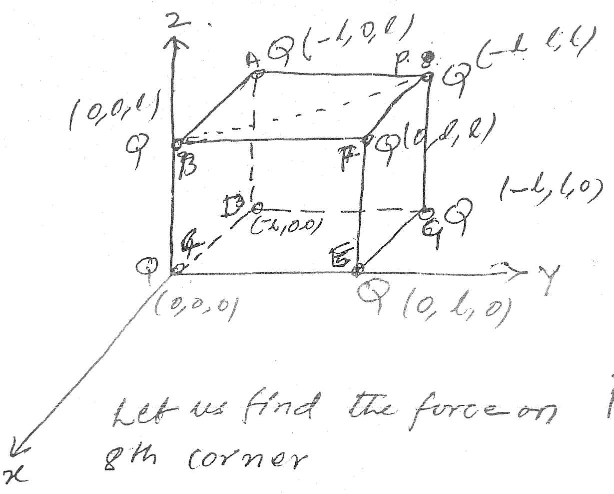
$$= (0.9\,\bar{a}_x + 1.272\,\bar{a}_x - 1.272\,\bar{a}_y + 1.272\,\bar{a}_x + 1.272\,\bar{a}_y)\times10^{-6}$$

$$\boxed{\bar{F} = 3.444\,\bar{a}_x \text{ Newton}}$$

$$|F| = 3.444 \text{ Newton}$$

12. Identical charges of Q coulomb are located at the corners of a cube with side 'l' mt. Show that the coulomb force on each charge has a magnitude of $3.29\,Q^2/4\pi\epsilon_0 l^2$ Newton.

Soln



Let us find the force on 8th corner

$\bar{F}_{PA}$ = force on charge at φ due to charge at A.

$$= \frac{Q^2}{4\pi\epsilon_0} \frac{l a_y}{l^2 \cdot l} = \frac{Q^2}{4\pi\epsilon_0 l^2}\bar{a}_y \rightarrow 1$$

$$\bar{F}_{PB} = \frac{Q^2}{4\pi\epsilon_0} \cdot \frac{(-l\bar{a}_x + l\bar{a}_y)}{(l^2+l^2)^{3/2}} = \frac{Q^2(-l a_x + l a_y)}{4\pi\epsilon_0 2\sqrt{2} l^3} \rightarrow 2.$$

$$\bar{F}_{PE} = \frac{Q^2}{4\pi\epsilon_0} \frac{(-l\bar{a}_x + l\bar{a}_y + l\bar{a}_z)}{(l^2+l^2+l^2)^{3/2}} = \frac{Q^2(-l\bar{a}_x + l\bar{a}_y + l\bar{a}_z)}{4\pi\epsilon_0 3\sqrt{3} l^3} \rightarrow 4$$

$$\bar{F}_{PD} = \frac{Q^2}{4\pi\epsilon_0} \frac{(l\bar{a}_y + l\bar{a}_z)}{(l^2+l^2)^{3/2}}$$

$$= \frac{Q^2}{4\pi\epsilon_0 2\sqrt{2} l^3}(l\bar{a}_y + l\bar{a}_z) \rightarrow 5$$

$$\bar{F}_{PE} = \frac{Q^2}{4\pi\epsilon_0 2\sqrt{2} l^3}(-l\bar{a}_x + l\bar{a}_z) \rightarrow 6$$

$$\bar{F}_{PF} = \frac{Q^2}{4\pi\epsilon_0 l^2}(-\bar{a}_x) \rightarrow 7.$$

$$\bar{F}_{PG} = \frac{Q^2}{4\pi\epsilon_0 l^2}\bar{a}_z \rightarrow 3.$$

Adding eqn. 1 to 7.

$$\bar{F} = \frac{Q^2}{4\pi\epsilon_0}\left[\frac{\bar{a}_y}{l^2} - \frac{l\bar{a}_x + l\bar{a}_y}{2\sqrt{2} l^3} + \frac{(-l\bar{a}_x + l\bar{a}_y + l\bar{a}_z)}{3\sqrt{3} l^3} + \frac{l\bar{a}_y + l\bar{a}_z}{2\sqrt{2} l^3} \frac{-l\bar{a}_x + l\bar{a}_z}{2\sqrt{2} l^3} - \frac{\bar{a}_x}{l^2} + \frac{\bar{a}_z}{l^2}\right]$$

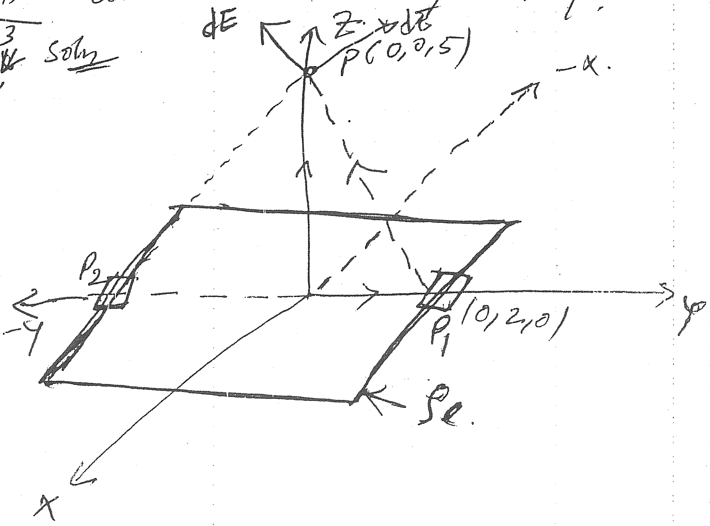
$$\bar{F} = \frac{Q^2}{4\pi\epsilon_0 l^2}\left[\bar{a}_x\left(-\frac{1}{2\sqrt{2}} - \frac{1}{3\sqrt{3}} - \frac{1}{2\sqrt{2}} - 1\right) + \bar{a}_y\left(1 + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}}\right) + \bar{a}_z\left(\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{2\sqrt{2}} + 1 + \frac{1}{2\sqrt{2}}\right)\right]$$

$$\bar{F} = \frac{Q^2}{4\pi\epsilon_0 l^2}\left[-1.899\,\bar{a}_x + 1.8976\,\bar{a}_y + 1.899\,\bar{a}_z\right]$$

$$\boxed{\bar{F} = 3.29\, Q^2/4\pi\epsilon_0 l^2}$$

13. Find the force on a charge of 30 nC at P(0,0,5) m due to a 4 m square in the z=0 plane between x = ±2 m & y = ±2 m with a total charge of 500 pC, distributed uniformly.

Soln



Total charge on the square is 500 nC.

$\therefore$ charge density, $\rho_l = \frac{500}{16} = 31.25\,nC/m$

Consider a small incremental length of 'dx' mt. at (0,2,0) & the charge on this elemental length = $\rho_l \cdot dx$.

The force at 'p' due to charge at P_1 is

$$d\bar F_1 = \frac{dq \cdot q}{4\pi\epsilon (P_1P)^2}\,\bar a_{P_1P}$$

$$P_1P = -2\bar a_y + 5\bar a_z$$

$$|P_1P| = \sqrt{2^2+5^2} = 5.385 \text{ mt}$$

$$\bar a_{P_1P} = \frac{-2\bar a_y + 5\bar a_z}{5.385} \quad \& \quad dq = \rho_l (dx)$$

$$\therefore d\bar F_1 = \frac{q\rho_l\, dx}{4\pi\epsilon(5.385)^2}\,\frac{(-2\bar a_y+5\bar a_z)}{5.385}$$

$$d\bar F_1 = q\,\frac{31.25\times10^{-6}\times9\times10^{9}}{(5.385)^3}(-2\bar a_y+5\bar a_z)\,dx$$

$$= q\,1.8\times10^{3}(-2\bar a_y+5\bar a_z)\,dx$$

$$\bar F_1 = q\,1.89\times10^{3}(-2\bar a_y+5\bar a_z)\int_{-2}^{2} dx$$

$$= q\,1.89\times10^{3}(-2\bar a_y+5\bar a_z)(4)$$

$$= q\,7.203\times10^{+3}(-2\bar a_y+5\bar a_z)\text{ N.} \rightarrow 1$$

lly. the force at P due to elemental charge at P_2 is

$$\bar F_2' = 7.203\times10^{3}q(+2\bar a_y+5\bar a_z) \rightarrow 2$$

From 1 & 2 it is clear that y-component of force due to two conductors at $y=\pm2$ cancels each other & z-component will ~~The total force~~ add up.

∴ The net force

$$F_1^P = 2\times F_{1z}$$

$$= 2\times 7.203\times10^{3}q(5\bar a_z)$$

$$\boxed{\bar F_1 = 72.03\times10^{3}q\,\bar a_z \text{ newton}}$$

$$\bar F_1 = 30\times10^{-6}\times72.03\times10^{3}\,\bar a_z$$

$$\boxed{F_1 = 2.16\,\bar a_z \text{ newton}}$$

lly the force F_2 at P due to the two sides of a square at $x = \pm 2$ mt is

$$\bar F_2 = 2.16\,\bar a_z \text{ Newton}$$

∴ The resultant force at P due to the square is

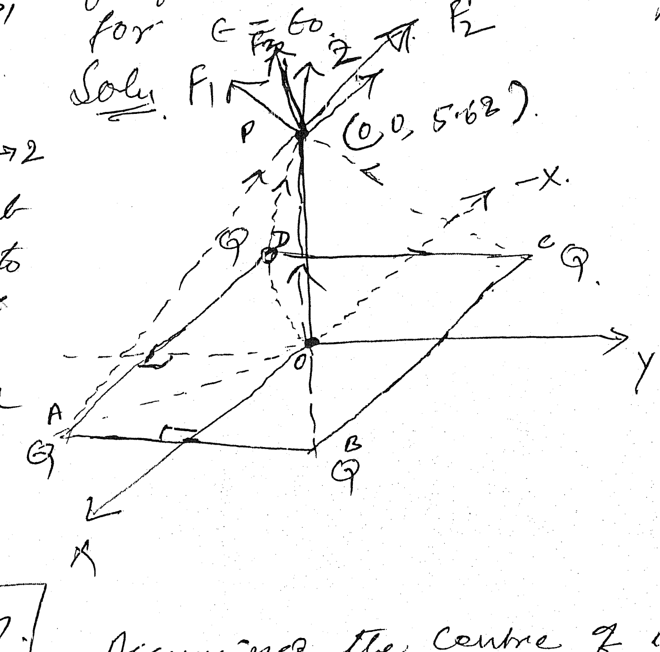
$$\bar F = F_1 + F_2$$

$$= 2\times 2.16\,\bar a_z$$

$$\boxed{\bar F = 4.32\,\bar a_z \text{ Newton}}$$

Ex 14. Jan 2010. Four like positive charges are located in the z=0 plane at the corners of a square 8cm on a side. A fifth like positive charge is located at a point 8cm distant from the other charges, calculate the magnitude of the force on this fifth charge for $\epsilon = \epsilon_0$.

Soln.



Assuming the centre of the square is at the origin.

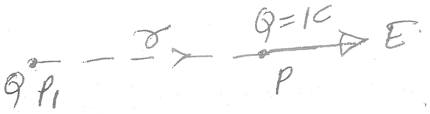
$$OA = \sqrt{(4\times10^{-2})^2 + (4\times10^{-2})^2}$$

$$= 5.65\times10^{-2} \text{ mt}$$

$$OP = \sqrt{(8\times10^{-2})^2 - (5.65\times10^{-2})^2}$$

$$= 5.66\times10^{-2} \text{ mt}.$$

<u>Electric field intensity</u> ($\bar{E}$): Electric field intensity at a point is defined as the force experienced by unit positive charge placed at that point. It is a vector quantity & its unit is Newton/coulomb (N/C) or volts/mt. (V/m).



Consider a point charge Q placed at point P_1. The electric field intensity at point P due to a point charge of Q coulombs placed at point P_1 is,

$$\bar{E} = \frac{Q}{4\pi\epsilon|P_1P|^2}\bar{a}_{PP_1}$$

$$\boxed{E = \frac{Q}{4\pi\epsilon r^2}\bar{a}_r}$$ Newton/coulomb.

where $\bar{a}_r$ is the unit vector along the line joining the points P_1 & P.

Note: If we know the electric field intensity at a point, then the force experienced by a charge of Q coulombs placed at that point is

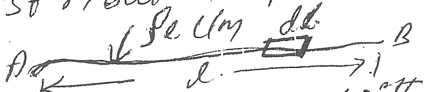
$$\boxed{F = Q\bar{E}}$$ //

<u>Types of charge configuration</u>: (charge distributions)

In electrostatics we deal with point charges & different types of charge distributions

i) Line charge distribution
ii) Surface charge distribution
iii) Volume charge distribution.

i) <u>Line charge distribution</u>: is visualized as a thin sharp beam in a cathode-ray tube. It can also be considered as charge distribution on a very thin conductor. The line charge distribution is ~~defined~~ described in terms of line charge density & is ~~d~~ defined as the charge distributed per unit length. It is denoted by ρ_l



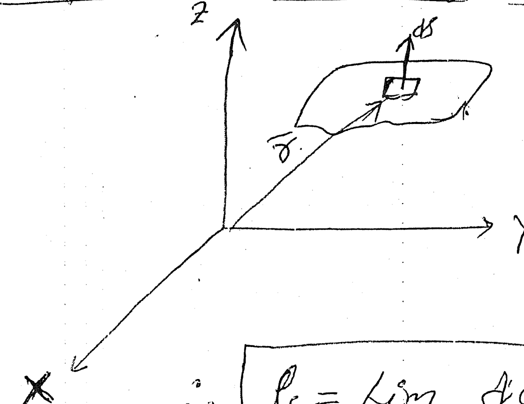
Consider a conductor AB of length l meter with a line charge density of ρ_l C/m.

Then $$\boxed{\rho_l = \lim_{dl \to 0}\frac{dQ}{dl}}$$ coulombs/meter.

or $dQ = \rho_l \cdot dl$.

$$\boxed{Q = \int_l \rho_l\, dl}$$ coulombs

Surface (Sheet) charge distribution : It is visualised as charge distributed uniformly on a conducting sheet like i.e Charges on the plates of a capacitor. It is defined as the ~~d~~ charge distributed per unit surface area. It is denoted as ρ_s & its unit is coulombs/meter^2 (C/m^2).

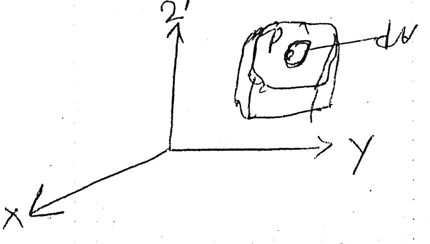


$$\therefore \boxed{\rho_s = \lim_{ds \to 0} \frac{dq}{ds} \ C/m^2}$$

where $dq \to$ charge stored in an incremental surface ds as shown above. The total charge is given by.

$$\boxed{Q = \int_S \rho_s \cdot ds} \quad \text{coulombs.}$$

Volume charge distribution : It is visualised as a region of space filled with very large number of discrete particles (electrons or atoms) seperated by finite atomic distances. ~~The assumption~~ of The volume charge distribution is expressed in terms of volume charge density & is defined as the charge distributed per unit volume. It is denoted as ρ_v.



$$\therefore \boxed{\rho_v = \lim_{dv \to 0} \frac{dq}{dv}} \quad C/m^3.$$

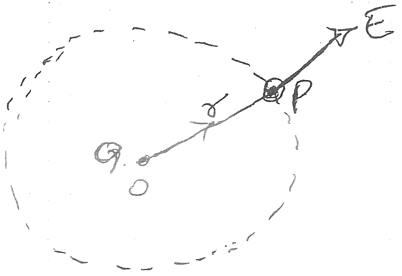
dq is the charge stored in an incremental volume 'dV' m^3

Now, $\therefore \boxed{Q = \int_v \rho_v \cdot dv}$ coulombs. is the total charge with in defined volume

ex: The space between the control grid & the cathode in the electron-gun assembly of a cathode-ray tube operating with space charge.

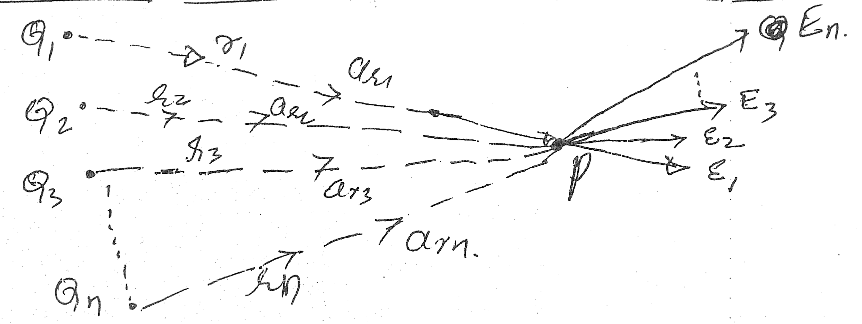
Expression for electric field intensity due to point charge.

Consider a point charge of Q coulombs placed at 'O'. We wish to find the electric field intensity at a point 'p' at a distance of 'r' meter from the charge Q in spherical co-ordinate system. So we assume that the point p lies ~~assumed~~ on the surface of a sphere of radius r meter & then the unit vector in the direction of $\bar{E}$ is $\bar{a}_r$



Hence $$\boxed{\bar{E} = \frac{Q}{4\pi\epsilon r^2}\bar{a}_r} \quad N/C.$$

Electric field due to multiple charges



Consider a point charges of $Q_1, Q_2, Q_3 \ldots\ldots Q_n$ coulombs are placed at a distance $r_1, r_2, r_3 \ldots r_n$ from the point "p", wherein we wish to find the electric field intensity.

$\bar{E}_1$ = electric field intensity due to Q_1

$$\bar{E}_1 = \frac{Q_1}{4\pi\epsilon r_1^2}\bar{a}_{r_1}$$

$\bar{E}_2$ = Electric field intensity due to Q_2

$$= \frac{Q_2}{4\pi\epsilon r_2^2}\bar{a}_{r_2}$$

$\bar{E}_3$ = Electric field intensity due to Q_3

$$= \frac{Q_3}{4\pi\epsilon r_3^2}\bar{a}_{r_3}$$

$||^{ly}$ Electric field intensity due to Q_n is

$$\bar{E}_n = \frac{Q_n}{4\pi\epsilon r_n^2}\bar{a}_{r_n}$$

The resultant electric field intensity is,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + \cdots\cdots + \bar{E}_n$$

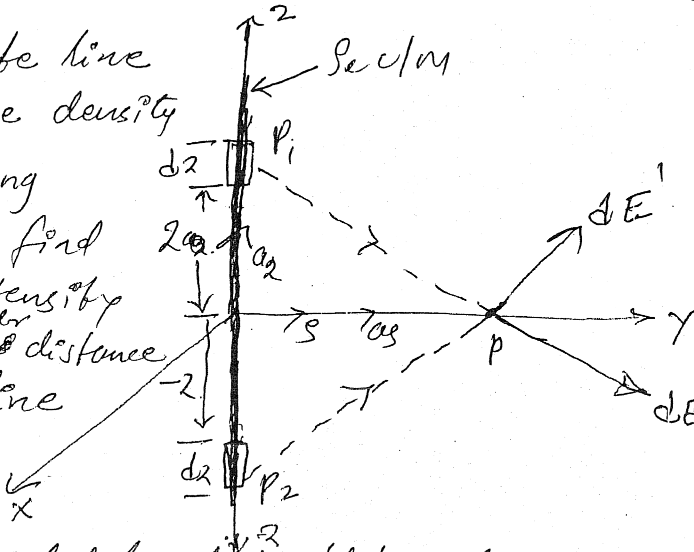
$$= \frac{Q_1}{4\pi\epsilon r_1^2}\bar{a}_{r_1} + \frac{Q_2}{4\pi\epsilon r_2^2}\bar{a}_{r_2} + \frac{Q_3}{4\pi\epsilon r_3^2}\bar{a}_{r_3} + \cdots\cdots + \frac{Q_n}{4\pi\epsilon r_n^2}\bar{a}_{r_n}$$

$$= \frac{1}{4\pi\epsilon}\left[\frac{Q_1}{r_1^2}\bar{a}_{r_1} + \frac{Q_2}{r_2^2}\bar{a}_{r_2} + \frac{Q_3}{r_3^2}\bar{a}_{r_3} + \cdots\cdots + \frac{Q_n}{r_n^2}\bar{a}_{r_n}\right]$$

$$\boxed{\bar{E} = \frac{1}{4\pi\epsilon}\sum_{i=1}^{n}\frac{Q_i}{r_i^2}\bar{a}_{r_i}}$$ Newton/Coulomb.

Electric field intensity due to infinite line charge.

Consider an infinite line conductor with charge density of ρ_L C/m placed along z-axis. We wish to find the electric field intensity at a point at a ⊥r distance 'ρ' meter from the line charge.



Consider an incremental length of 'dz' meter at P_1 at a distance of 'z' meter from the origin.

By Coulomb's law, the electric field intensity at point 'P' due to an incremental charge at P_1 is,

$$\bar{dE} = \frac{dQ}{4\pi\epsilon|\bar{P_1P}|^2}\bar{a}_{P_1P} \longrightarrow 1$$

$$dQ = \rho_L \cdot dz, \quad \bar{P_1P} = \rho\bar{a}_\rho - z\bar{a}_z, \quad |\bar{P_1P}| = \sqrt{\rho^2 + z^2}$$

$$\bar{a}_{P_1P} = (\rho\bar{a}_\rho - z\bar{a}_z)/\sqrt{\rho^2+z^2}$$

putting above values in eqn 1,

$$\bar{dE} = \frac{\rho_L dz}{4\pi\epsilon(\sqrt{\rho^2+z^2})^2} \cdot \frac{\rho\bar{a}_\rho - z\bar{a}_z}{\sqrt{\rho\bar{a}_\rho - z\bar{a}_z}}$$

$$\bar{dE} = \frac{\rho_L \cdot dz}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \cdot (\rho\bar{a}_\rho - z\bar{a}_z) \longrightarrow 2.$$

Similarly, due to symmetry, the electric field intensity at point 'p' due to an elemental charge at P_2 is.

$$\overline{dE'} = \frac{\rho_l . dz}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \cdot (\rho\bar{a}_\rho + z\bar{a}_z) \longrightarrow 3.$$

From eqn. 2 & 3 it is clear that, due to symmetry z-component of electric field at p cancels each other & ρ-component will add up. Hence the net electric field is

$$\overline{dE} = \frac{\rho_l . dz}{4\pi\epsilon(\rho^2+z^2)} \rho\, \bar{a}_\rho$$

$$\therefore \quad \bar{E} = \int_{-\infty}^{\infty} \frac{\rho_l . dz}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \rho\, \bar{a}_\rho$$

$$= \frac{\rho_l . \cancel{dz} . \rho\, \bar{a}_\rho}{4\pi\epsilon} \int_{-\infty}^{\infty} \frac{dz}{(\rho^2+z^2)^{3/2}}$$

put, $z = \rho \tan\theta$, $dz = \rho \tan^2\theta$.

if $z = -\infty \Rightarrow \theta = \theta_1 = \tan^{-1}(-\infty/\rho) = -\pi/2$

& $z = \infty \Rightarrow \theta = \theta_2 = \tan^{-1}(+\infty/\rho) = \pi/2$.

$$\therefore \quad \bar{E} = \frac{\rho_l . \rho\, \bar{a}_\rho}{4\pi\epsilon} \int_{-\pi/2}^{\pi/2} \frac{\rho \tan^2\theta\, d\theta}{(\rho^2 + \rho^2\tan^2\theta)^{3/2}}$$

$$= \frac{\rho_l \cdot \rho^2 \bar{a}_\rho}{4\pi\epsilon \rho^3} \int_{-\pi/2}^{\pi/2} \frac{\tan^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$$

$$= \frac{\rho_l\, \bar{a}_\rho}{4\pi\epsilon} \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sec^3\theta}$$

$$= \frac{\rho_l}{4\pi\epsilon} \bar{a}_\rho \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{\rho_l}{4\pi\epsilon} \bar{a}_\rho \left(+\sin\theta\right)_{-\pi/2}^{\pi/2}$$

$$= \frac{\rho_l}{4\pi\epsilon} \bar{a}_\rho \left(\sin\pi/2 - \sin(-\pi/2)\right)$$

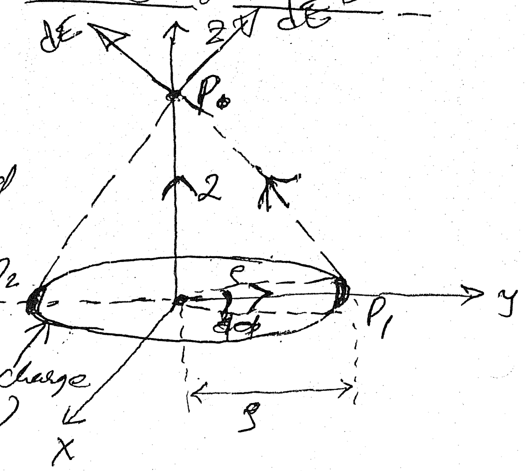
$$\bar{E} = \frac{\rho_l}{\underset{2}{\cancel{4}}\pi\epsilon} \bar{a}_\rho\, (\cancel{2})$$

$$\therefore \boxed{\bar{E} = \frac{\rho_l}{2\pi\epsilon\rho}\,\bar{a}_\rho} \ \text{N/C}.$$

where, $\rho \rightarrow$ Perpendicular distance of the point p' from the line charge.

<u>Electric field intensity due to ring of charge</u>

Consider a ring of charge ρ_l C/m with radius of ρ mt placed at $z=0$ plane centered at the origin. We wish to find the electric field intensity at a point p' at $(0,0,z)$ m from the axis of the ring.



Consider an elemental length at P_1. The electric field intensity at p due to an elemental charge at P_1 is

$$d\bar{E} = \frac{dq}{4\pi\epsilon\,(P_1P)^2}\,\bar{a}_{P_1P}$$

From the above fig, $\rho\bar{a}_\rho + \overline{P_1P} = z\bar{a}_z$

$$\therefore \overline{P_1P} = z\bar{a}_z + \rho\bar{a}_\rho$$

$$|\overline{P_1P}| = \sqrt{z^2+\rho^2}$$

$$\bar{a}_{P_1P} = \frac{z\bar{a}_z - \rho\bar{a}_\rho}{\sqrt{\rho^2+z^2}} \qquad \text{also } dq = \rho_l\,dl = \rho_l\,\rho\,d\phi.$$

$\therefore$ equation ① becomes

$$d\bar{E} = \frac{\rho_l\cdot\rho\,d\phi}{4\pi\epsilon\left(\sqrt{\rho^2+z^2}\right)^2}\,\frac{(z\bar{a}_z-\rho\bar{a}_\rho)}{\sqrt{\rho^2+z^2}}.$$

$$d\bar{E} = \frac{\rho_l\cdot\rho\,d\phi}{4\pi\epsilon(\rho^2+z^2)^{3/2}}\,(z\bar{a}_z - \rho\bar{a}_\rho) \longrightarrow 1$$

~~Hly~~ The electric field intensit at p due to an elemental charge at P_2 is

$$\overline{dE'} = \frac{\rho_L \cdot \rho\, d\phi}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \cdot (z\bar{a}_z + \rho\bar{a}_\rho) \longrightarrow 2.$$

From equation 1 & 2 it is clear that, due to symmetry ρ-component of electric field intensity cancels each other & Z-component will add up. Therefore the net electric field intensity is

$$\bar{E} = \int_0^{2\pi} \frac{\rho_L \cdot \rho\, d\phi}{4\pi\epsilon(\rho^2+z^2)^{3/2}}\, z \cdot \bar{a}_z$$

$$\therefore \bar{E} = \frac{\rho_L \cdot \rho z\, \bar{a}_z}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \cdot \int_0^{2\pi} d\phi$$

$$= \frac{\rho_L \cdot \rho z\, \bar{a}_z}{4\pi\epsilon(\rho^2+z^2)^{3/2}} (\phi)_0^{2\pi}$$

$$= \frac{\rho_L \cdot \rho \cdot z\, \bar{a}_z}{\cancel{4\pi}\epsilon(\rho^2+z^2)^{3/2}} \cancel{2\pi}.$$

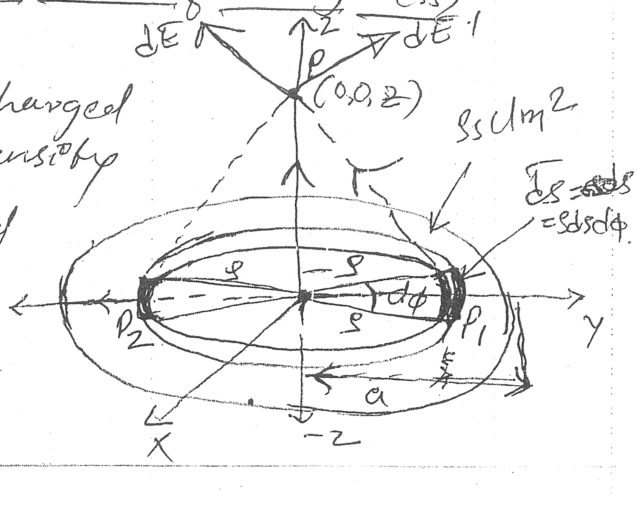
$$\boxed{\bar{E} = \frac{\rho_L \cdot \rho z}{2\epsilon(\rho^2+z^2)^{3/2}} \cdot \bar{a}_z \quad C/M.} \longrightarrow 3.$$

Electric field intensity at the centre of the ring can be obtained by putting $z=0$ in the above eqn.

$\therefore \boxed{\bar{E} = 0}$ at the centre of the ring.

Electric field intensity due to a disc of charge (ρ_s)

Consider a disc ~~of charge~~ having radius of 'a' mt, charged with a surface charge density of ρ_s C/m2. We wish to find the electric field intensity at point P(0,0,z) from the axis of the disc.



The electric field intensity at 'p' due to an elemental charge at P_1 is,

$$\bar{dE} = \frac{dQ}{4\pi\epsilon |\bar{P_1P}|^2} \bar{a}_{P_1P} \longrightarrow 1.$$

Now, $dQ = \rho_s \cdot |ds|$

$= \rho_s \cdot \rho d\rho d\phi.$

$\bar{P_1P} = z\bar{a}_z - \rho\bar{a}_\rho$

$|\bar{P_1P}| = \sqrt{z^2+\rho^2}$

$\bar{a}_{P_1P} = (z\bar{a}_z - \rho\bar{a}_\rho)/(\sqrt{\rho^2+z^2}.$

$$\therefore \bar{dE} = \frac{\rho_s \cdot \rho d\rho d\phi}{4\pi\epsilon(\sqrt{\rho^2+z^2})^2} \cdot \frac{z\bar{a}_z - \rho\bar{a}_\rho}{\sqrt{\rho^2+z^2}}$$

$$\bar{dE} = \frac{\rho_s \cdot \rho d\rho\, d\phi}{4\pi\epsilon(\rho^2+z^2)^{3/2}} (z\bar{a}_z - \rho\bar{a}_\rho) \longrightarrow 1$$

lly The the electric field intensity at p due to an incremental charge at P_2 is,

$$\bar{dE'} = \frac{\rho_s \cdot \rho d\rho d\phi}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \cdot (z\bar{a}_z + \rho\bar{a}_\rho) \longrightarrow 2.$$

From 1 & 2, due to symmetry, ρ-component of electric field intensity cancels each other & z-component will add up.

$$\therefore \quad \bar{dE} = \frac{\rho_s \cdot \rho d\rho\, d\phi}{4\pi\epsilon(\rho^2+z^2)^{3/2}} (z \cdot \bar{a}_z)$$

The net electric field intensity due to entire disc is

$$\bar{E} = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \frac{\rho_s \cdot \rho d\rho\, d\phi \; z}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \bar{a}_z$$

$$= \frac{\rho_s \cdot z\, \bar{a}_z}{4\pi\epsilon} \int_0^a \frac{\rho d\rho}{(\rho^2+z^2)^{3/2}} \int_0^{2\pi} d\phi.$$

$$\overline{dE} = \frac{\rho_s \cdot z\, \bar{a}_z}{4\pi\epsilon} \cdot 2\pi \int_0^a \frac{\rho\, d\rho}{(\rho^2+z^2)^{3/2}} \longrightarrow 3.$$

$$= \frac{\rho_s \cdot z}{2\epsilon} \int_0^a \frac{\rho\, d\rho}{(\rho^2+z^2)^{3/2}}$$

put $\rho^2 + z^2 = t$

$2\rho\, d\rho = dt$

$\rho\, d\rho = dt/2$

when, $\rho = 0 \Rightarrow z^2 = t$

& $\rho = a \Rightarrow t = z^2 + a^2$.

$\therefore$ Equation 3 becomes

$$\overline{dE} = \frac{\rho_s \cdot z}{2\epsilon} \bar{a}_z \int_{z^2}^{z^2+a^2} \frac{dt/2}{(t)^{3/2}}$$

$$= \frac{\rho_s z}{4\epsilon} \bar{a}_z \cdot \left(\frac{t^{-3/2+1}}{-3/2+1} \right)_{z^2}^{z^2+a^2}$$

$$= \frac{\rho_s \cdot z}{4\epsilon} \bar{a}_z \left(\frac{t^{-1/2}}{-1/2} \right)_{z^2}^{z^2+a^2}$$

$$= -\frac{\rho_s \cdot z}{2\epsilon} \bar{a}_z \left[(z^2+a^2)^{-1/2} - (z^2)^{-1/2} \right]$$

$$= \frac{\rho_s z}{2\epsilon} \bar{a}_z \left[\frac{1}{z} - \frac{1}{z^2+a^2} \right]$$

$$= \frac{\rho_s \cdot}{2\epsilon} \bar{a}_z \left[\frac{z}{z} - \frac{z}{z^2+a^2} \right]$$

$$\bar{E} = \frac{\rho_s}{2\epsilon} \bar{a}_z \left[1 - \frac{z}{z^2+a^2} \right]$$

$$\boxed{\bar{E} = \frac{\rho_s}{2\epsilon} \left[1 - \frac{z}{z^2+a^2} \right] \bar{a}_z} \quad N/C.$$

Electric field due to an infinite sheet of charge

We know that the electric field intensity at a point due to a finite sheet charge (described as a disc of radius 'a' m) is

$$\bar{E} = \frac{\rho_s}{2\epsilon}\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]\bar{a}_z \longrightarrow (1)$$

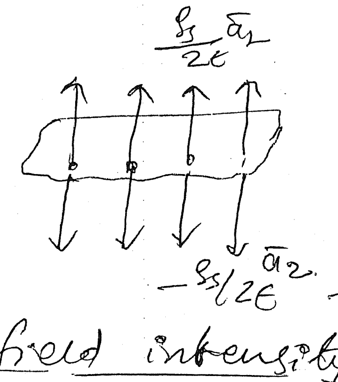
We can consider an infinite sheet of charge as a disc of charge with radius tending to ∞. Therefore equation becomes,

$$\bar{E} = \lim_{a\to\infty} \frac{\rho_s}{2\epsilon}\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]\bar{a}_z$$

$$\bar{E} = \frac{\rho_s}{2\epsilon}\left[1 - 0\right]\bar{a}_z$$

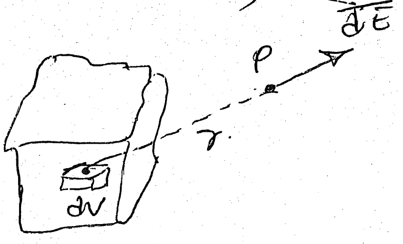
$$\boxed{\bar{E} = \frac{\rho_s}{2\epsilon}\bar{a}_z} \quad N/C \longrightarrow 2.$$

Note: From equation 2 we see that, the electric field intensity at a point due to an infinite sheet of charge is indendent of the distance of the point. Therefore its electric field is uniform everywhere & is equal to $\rho_s/2\epsilon$ & is shown below.



Electric field intensity due to volume charge

Let us consider a region of any configuration as shown above in which distribution of charge is specified by the volume charge distribution. Let the volume charge distribution be ρ_v C/m³.



In a differential element of volume dv, the differential the differential charge is given by

$$dq = \rho_v \, dv$$

The electric field intensity at point 'p' due to an elemental charge of dq Coulombs is

$$\overline{dE} = \frac{dq}{4\pi\epsilon r^2}\,\overline{a_r}$$

$$= \frac{\rho_v \cdot dv}{4\pi\epsilon r^2}\,\overline{a_r}$$

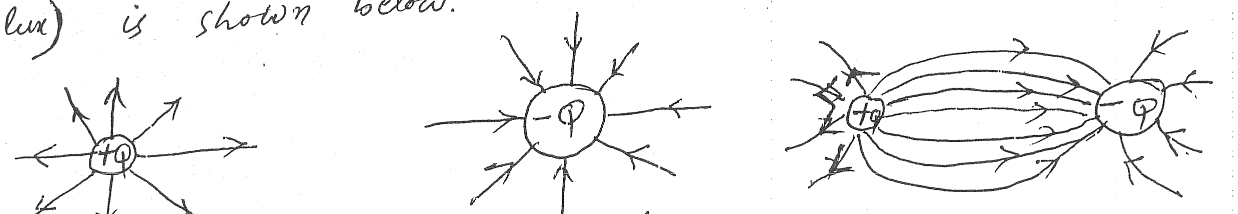
$$\therefore \overline{E} = \int_v \frac{\rho_v \, dv}{4\pi\epsilon r^2}\,\overline{a_r}$$

$$\boxed{\overline{E} = \frac{1}{4\pi\epsilon}\,\overline{a_r}\int_v \frac{\rho_v \, dv}{r^2}} \quad N/C.$$

Electric flux density, Gauss law & Divergence (UNIT-I-b)

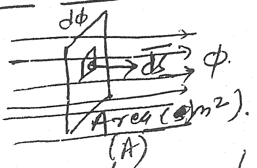
Electric flux: The effect of electric force around the charged body is assumed to be consisting of an imaginary electric lines of force. The total no. of electric flux around the charged body is known as electric flux. It is denoted by ϕ & its unit is coulomb. An electric charge ~~flux~~ of Q coulombs gives rise to Q coulombs of electric flux. The electric lines of force will emanate from the positive charge & terminate in to a negative charge.

The distribution of electric lines of force (electric flux) is shown below.



(positive charge) (negative charge)

Electric flux Density or Displacement density (D):



The flux density at a point in an electric field is defined as, the total normal flux passing per unit area. It is a vector quantity. It is denoted by D & its unit is c/m².

Consider a total flux ϕ passing normal to the surface area of 'A' m², then the magnitude of flux density is given by,

$$\boxed{\bar{D} = \frac{\phi}{A}} \quad c/m^2.$$

In vector form it can be written as,

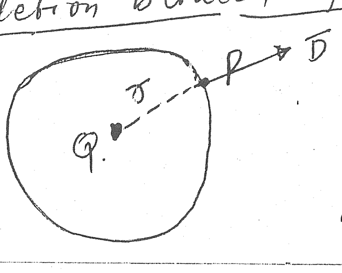
$$\boxed{\bar{D} = \frac{d\phi}{ds}\,\bar{a}_n} \quad c/m^2.$$

$d\phi \to$ Total flux crossing normal to surface area ds.

$ds \to$ Incremental surface area.

$\bar{a}_n \to$ Unit vector in the direction normal to the incremental surface.

Relation between flux density (D) & Electric field intensity (E)



Consider a point charge of Q coulombs is placed at the centre of an imaginary sphere of radius 'r' mt. Let 'p' be any point on the ~~circut~~ circumference of an imaginary sphere. The flux density at any point 'p' on the surface of a sphere is,

$$\bar{D} = \frac{\phi}{area}\,\bar{a}_r = \frac{\phi}{4\pi r^2}\,\bar{a}_r \longrightarrow (1)$$

where, $\bar{a}_r \rightarrow$ unit vector directed radially outwards on the sphere. Indicating the direction of flux density.

The electric field intensity at 'p' at a distance of 'r' mt. from the point charge Q is given by,

$$E = \frac{Q}{4\pi\epsilon r^2}\,\bar{a}_r \longrightarrow (2)$$

putting (1) in (2)

$$\bar{E} = \left(\frac{Q}{4\pi r^2}\bar{a}_r\right)\frac{1}{\epsilon}$$

$$\bar{E} = \frac{\bar{D}}{\epsilon}$$

$$\boxed{\therefore \bar{D} = \epsilon \bar{E}}$$

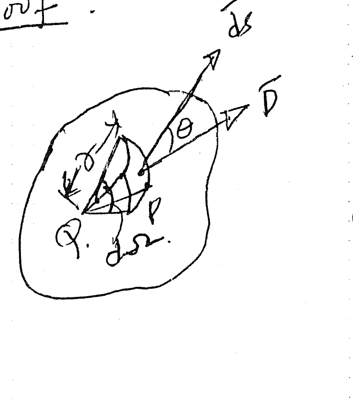
or $\boxed{\bar{D} = \epsilon_0 \epsilon_r \bar{E}}$ → for any medium.

$\boxed{\bar{D} = \epsilon_0 \bar{E}}$ → for free space.

<u>Gauss's law</u> : Gauss's law states that, the total normal electric flux through any closed surface surrounding an electric charge is equal to the total charge enclosed by the surface.

Mathematically, $\boxed{\oint_s \bar{D}\cdot\bar{ds} = Q_{enclosed}}$

<u>Proof</u> :



Consider an arbitrary closed surface enclosing a point charge Q. Consider an elemental surface area 'ds' at point P. The total normal flux passing through this elemental area is,

$$\bar{D}\cdot\bar{ds} = |D||ds|\cos\theta$$
$$= \epsilon E\, ds\cos\theta \rightarrow (1) \quad (\because \bar{D} = \epsilon\bar{E})$$

But $|E| = \frac{Q}{4\pi\epsilon r^2}$

$\therefore$ eqn (1) becomes

$$\bar{D}\cdot\bar{ds} = \cancel{\epsilon}\frac{Q}{4\pi\cancel{\epsilon} r^2}\,ds\cos\theta$$
$$= \frac{Q}{4\pi r^2}\,ds\cos\theta.$$

∴ The total electric flux through the entire closed surface is,

$$\oint_S \bar{D}\cdot \bar{ds} = \oint_S \frac{Q}{4\pi r^2}\, ds\cos\theta$$

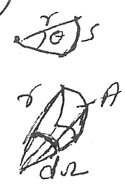
(2)

$$= \frac{Q}{4\pi}\oint_S \frac{ds\cos\theta}{r^2}$$

$$= \frac{Q}{4\pi}\oint_S d\Omega$$

Where $d\Omega$ is the solid angle subtended by the elemental surface & is equal to 4π.

$$\therefore \oint_S \bar{D}\cdot\bar{ds} = \frac{Q}{4\pi}\, 4\pi$$



$\theta = \frac{s}{r}$

$d\Omega = \frac{A}{r^2}$

$$\boxed{\oint_S \bar{D}\cdot\bar{ds} = Q} \rightarrow (2)$$ Hence the proof.

* Eqn. (2) is known as Gauss' law in <u>integral form</u>.

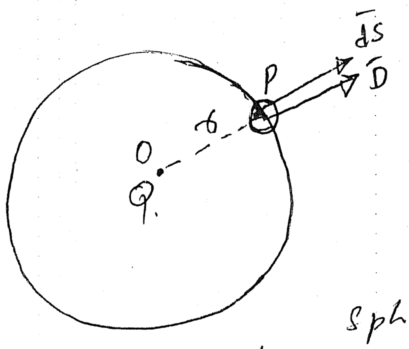
<u>Conditions to apply Gauss law (or Nature of Gaussian surface)</u> :

- The surface around the charge is should be closed surface. This closed surface is called as <u>Gaussian surface</u>.
- The electric field or flux density should be either parallel to or perpendicular to the Gaussian surface at any point.

<u>Applications of Gauss' law</u> :

1. It allows us to find the net charge enclosed by the closed surface from the knowledge of only the flux comming out of the closed surface.
2. It is easier to compute electric field intensity using Gauss's law for the symmetric charge distributions such as point charge, line charge, sheet charge, as compared to the methods using coulombs law.

Electric field intensity due to a point charge :



Consider a point charge of Q coulombs placed at 'O'. We wish to find the electric field intensity at point p at a distance of r meter from point charge. As the point charge has spherical symmetry, we choose the Gaussian surface as a sphere of radius r meter passing through the point p.

From Gauss' law,

$$\oint_s \bar{D}\cdot\bar{ds} = Q$$

$$\oint_s |D|\,|ds| \cos 0 = Q$$

$$\oint_s D\ r^2 \sin\theta\, d\theta\, d\phi = Q$$

$$D r^2 \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = Q$$

$$D r^2 \left(-\cos\theta\right)_0^{\pi} \left(\phi\right)_0^{2\pi} = Q$$

$$D r^2 (-\cos\pi + \cos 0)(2\pi - 0) = Q$$

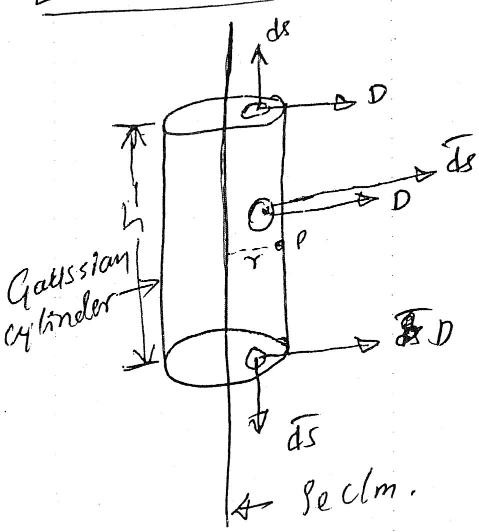
$$D\, 4\pi r^2 = Q$$

$$\epsilon E\, 4\pi r^2 = Q \qquad (\bar{D} = \epsilon E)$$

$$E = \frac{Q}{4\pi\epsilon r^2}$$

$$\boxed{\bar{E} = \frac{Q}{4\pi\epsilon r^2}\,\bar{a}_r}$$

Electric field due to an infinite line charge :



Consider an infinite line charge with charge density ρ_l c/m. We wish to find the electric field intensity at point p at a distance of 'r' meter from the line charge. We assume closed hollow cylinder of radius 'r' meter & height h meter & this cylinder is called Gaussian cylinder.

From Gauss' law,

$$\oint \bar{D}\cdot\bar{ds} = Q_{enclosed}.$$

As the Gaussian cylinder's includes 3 closed surface, hence above eqn. becomes.

$$\oint_{\text{top surface}} \bar{D}\cdot\bar{ds} + \oint_{\text{curved surface}} \bar{D}\cdot\bar{ds} + \oint_{\text{Bottom surface}} \bar{D}\cdot\bar{ds} = \rho_l\cdot h \rightarrow (1)$$ ($\because Q = \rho_l\cdot h$ = Charge enclosed by the Gaussian surface) (3)

As shown in the fig, the angle between $\bar{ds}$ & $\bar{D}$ on top & bottom surface is $90°$. $\therefore \oint_{\text{top surface}} \bar{D}\cdot\bar{ds} = 0$ & $\oint_{\text{Bottom surface}} \bar{D}\cdot\bar{ds} = 0$.

$\therefore$ eqn. (1) becomes

~~$0 + \oint_{\text{curved surface}} D\, ds \cos 0 = \rho_l \cdot h$~~

$$0 + \oint_{\text{curved surface}} D\, ds \cos 0 + 0 = \rho_l \cdot h.$$

$$\oint D \cdot r\, d\phi\, dz = \rho_l \cdot h.$$

$$D\cdot r \cdot \int_0^{2\pi} d\phi \int_0^h dz = \rho_l \cdot h.$$

$$D r \cdot (2\pi)(h) = \rho_l \cdot h.$$

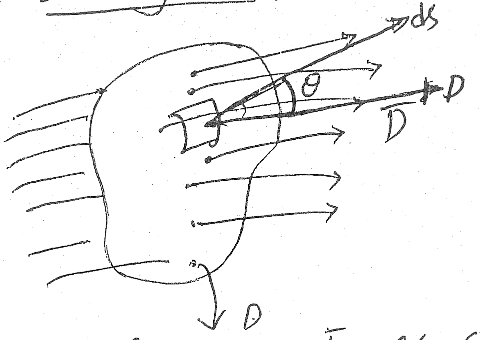
$$D \cdot 2\pi r \cancel{h} = \rho_l \cdot \cancel{h}$$

$$D = \rho_l / 2\pi r$$

$$E = \rho_l / 2\pi\epsilon r \qquad (\because D = \epsilon E)$$

$$\boxed{\bar{E} = \frac{\rho_l}{2\pi\epsilon r}\cdot \bar{a}_r} \quad N/C \text{ or volts/meter.}$$

Divergence of a vector field at a point:



Consider a vector field, $\bar{D} = D_x \bar{a}_x + D_y \bar{a}_y + D_z a_z$ as shown above. Let us compute the total outward flux from the closed surface as shown above.

Consider an elemental surface of area $\bar{ds}$ as shown above. The net flux can be resolved in to two components namely normal to the surface & tangential to the surface. The tangential component of the ~~field~~ flux does not contribute for the outward flux. Therefore the normal component of the field 'D' is $D\cos\theta$. Hence the net normal flux out of the elemental surface is $D\, ds \cos\theta = \bar{D}\cdot\bar{ds}$.

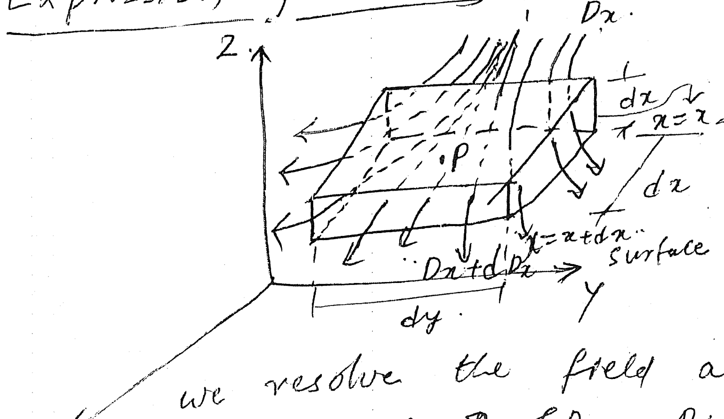
$\therefore$ The total normal flux out of the entire closed surface $= \oint_s \bar{D}\cdot\bar{ds} \rightarrow (1)$

If we integrate in the amount of flux diverging from a point, we obtain this surface integral when the volume of the closed surface shrinks to zero.

So, the <u>Divergence of a vector field</u> at a point is <u>defined</u> as the net outward flux from a closed surface around the point per unit volume as the volume of the closed surface shrinks to zero. It is denoted by $\nabla\cdot\bar{D}$ or $\operatorname{div}\bar{D}$

$$\therefore \boxed{\nabla\cdot\bar{D} = \lim_{dv\to 0} \frac{\oint_s \bar{D}\cdot\bar{ds}}{dv}}$$

Expression for divergence in Rectangular co-ordinate system



Consider a rectangular volume element $dv = dx\,dy\,dz$. Let the flux density at point 'P' be D C/m². The net outward flux at x=x surface can be computed as follows.

We resolve the field at x=x surface into three components, D_x, D_y & D_z. But D_y & D_z are tangential to the surface at x=x & hence donot contribute for the normal flux.

Hence the outward flux through x=x surface $= -D_x\,dy\,dz$. (−ve sign ∵ the flux is inward to the surface)

Ily The net outward flux through x=x+dx surface $= (D_x + dD_x)\,dy\,dz$.

The net outward flux in the +ve X-direction
$$= (D_x+dD_x)\,dy\,dz - D_x\,dy\,dz$$
$$= (\cancel{D_x} + dD_x - \cancel{D_x})\,dy\,dz$$
$$= dD_x\,dy\,dz$$
$$= \frac{\partial D_x}{\partial x}\,dx\,dy\,dz \longrightarrow (1)$$

Ily The net outward flux in y-direction $= \dfrac{\partial D_y}{\partial y}\,dx\,dy\,dz \longrightarrow 2.$

& The net outward flux in z-direction $= \dfrac{\partial D_z}{\partial z}\,dx\,dy\,dz \longrightarrow 3.$

Adding eqn 1, 2 & 3 gives net outward flux from entire rectangular volume element.

$\therefore$ we can write $\oint_S \bar{D}\cdot d\bar{s} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dx\,dy\,dz \rightarrow (4)$

By Defn,

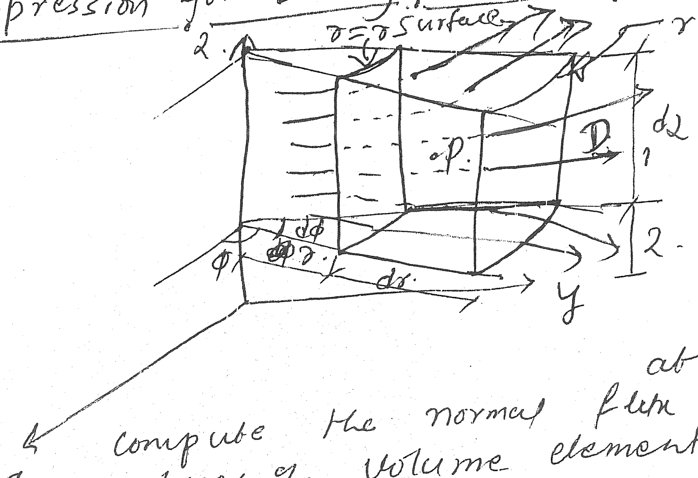
$\nabla\cdot\bar{D} = \lim_{dv\to 0} \frac{\oint_S \bar{D}\cdot d\bar{s}}{dv} \rightarrow (5)$

putting 4 in 5.

$\nabla\cdot\bar{D} = \lim_{dv\to 0} \frac{\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] \cancel{dx\,dy\,dz}}{\cancel{dx\,dy\,dz}}$

$$\boxed{\nabla\cdot\bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}.}$$

Expression for Divergence in cylendrical co-ordinate system.



consider cylendrical volume element, $dv = r\,dr\,d\phi\,dz$. we wish to find the divergence of fluxdensity at point 'p'. first we compute the normal flux passing from all the surfaces of volume element.

Net outward flux in r-direction $= -D_r\, r\,d\phi\,dz + (D_r + dD_r)(r+dr)d\phi\,dz$.

$= -D_r\, r\,d\phi\,dz + (rD_r + D_r dr + r\,dD_r + dD_r\,dr)d\phi\,dz$

$= (-\cancel{D_r r} + \cancel{D_r r} + D_r\,dr + r\,dD_r + dD_r\cdot dr)\,d\phi\,dz$

$= \left(D_r\,dr + r\cdot\frac{\partial D_r}{\partial r}dr + \frac{\partial D_r}{\partial r}dr^2\right) d\phi\,dz.$

$= \left(D_r\cdot dr + r\frac{\partial D_r}{\partial r}dr + 0\right) d\phi\,dz.$ ($\because$ dr is very small $\therefore dr^2 << 0$)

$= \left(D_r + r\frac{\partial D_r}{\partial r}\right) dr\,d\phi\,dz.$

$= \frac{\partial}{\partial r}(r D_r)\,dr\,d\phi\,dz \longrightarrow (1)$

lly the net outward flux in ϕ direction $= -D_\phi\, dr\,dz + (D_\phi + dD_\phi)\,dr\,dz.$

$= (-\cancel{D_\phi} + \cancel{D_\phi} + dD_\phi)\,dr\,dz$

$= dD_\phi\, dr\,dz.$

$= \frac{\partial D_\phi}{\partial \phi}\, dr\,d\phi\,dz \longrightarrow 2.$

& Net outward flux in z-direction $= ~~-D_z \, r\, d\phi\, dz + (D_z + dD_z) r\, d\phi\, dz~~$

$$~~= (-D_z + D_z + dD_z) r\, d\phi\, dz~~$$

$$~~= dD_z\, r\, d\phi\, dz~~$$

$$= -D_z\, r\, d\phi\, dr + (D_z + dD_z) r\, d\phi\, dr$$

$$= (-D_z + D_z + dD_z) r\, d\phi\, dr$$

$$= dD_z\, r\, d\phi\, dr$$

$$= \frac{\partial D_z}{\partial z} r\, dr\, d\phi\, dz \longrightarrow 3.$$

Adding eqn. 1, 2 & 3 gives net outward normal flux.

∴ By Gauss law.

$$\oint_s \bar{D} \cdot \bar{ds} = \frac{\partial (r D_r)}{\partial r} dr\, d\phi\, dz + \frac{\partial D_\phi}{\partial \phi} dr\, dz\, d\phi + \frac{\partial D_z}{\partial z} r\, dr\, d\phi\, dz.$$

$$= \left(\frac{\partial}{\partial r}(r D_r) + \frac{\partial D_\phi}{\partial \phi} + r \frac{\partial D_z}{\partial z} \right) dr\, d\phi\, dz \cdot \longrightarrow (4)$$

Now, $\nabla \cdot \bar{D} = \lim_{dv \to 0} \frac{\oint_s \bar{D} \cdot \bar{ds}}{dv}$.

$$= \lim_{dv \to 0} \frac{\left(\frac{\partial}{\partial r}(r D_r) + \frac{\partial D_\phi}{\partial \phi} + r \frac{\partial D_z}{\partial z} \right) \cancel{dr\, d\phi\, dz}}{r\, \cancel{dr\, d\phi\, dz}}$$

$$\boxed{\nabla \cdot \bar{D} = \frac{1}{r} \left[\frac{\partial}{\partial r}(r D_r) + \frac{\partial D_\phi}{\partial \phi} + r \frac{\partial D_z}{\partial z} \right]}$$ //.

Expression for divergence in spherical co-ordinate system:

$$\boxed{\nabla \cdot \bar{D} = \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r}(r^2 \sin\theta\, D_r) + \frac{\partial}{\partial \theta}(r \sin\theta\, D_\theta) + \frac{\partial}{\partial \phi}(r D_\phi) \right]}$$ //.

Gauss law in point form or Differential form or Maxwell's first equation: It states that, the divergence of electric flux density at a point in a medium is equal to charge per unit volume at that point.

i.e. $\boxed{\nabla \cdot \bar{D} = \rho_v}$

Proof: Consider an arbitrary volume element $dv\, m^3$ & let dq be the total charge in the volume element.

By Gauss law, $\oint_S \bar{D}\cdot\bar{ds} = dq.$

Dividing both sides by dv, we get;

$$\oint_S \frac{\bar{D}\cdot\bar{ds}}{dv} = \frac{dq}{dv}$$

Applying the limit $dv \to 0$ on both sides,

$$\lim_{dv\to 0} \oint_S \frac{\bar{D}\cdot\bar{ds}}{dv} = \lim_{dv\to 0} \frac{dq}{dv}$$

By Defn. of divergence, $\lim_{dv\to 0} \oint_S \frac{\bar{D}\cdot\bar{ds}}{dv} = \nabla\cdot\bar{D}$

$$\therefore \boxed{\nabla\cdot\bar{D} = \rho_v}$$

<u>Divergence theorem</u> : <u>(Gauss strokes theorem)</u>.

It states that, integral of flux density, D over a closed surface is equal to the integral of divergence of D flux density, D through the volume enclosed by the surface.

$$\text{i.e.}\quad \oint \bar{D}\cdot\bar{ds} = \oint_v (\nabla\cdot\bar{D})\, dv.$$

<u>Proof</u>. From Gauss's law integral form, we have

$$\oint_S \bar{D}\cdot\bar{ds} = Q. \longrightarrow (1)$$

But $\rho_v = \frac{dq}{dv}.$

$\therefore dq = \rho_v \cdot dv$

$Q = \oint_v \rho_v\, dv. \to (2)$

putting (2) in (1)

$$\oint_S \bar{D}\cdot\bar{ds} = \oint_v \rho_v\, dv \longrightarrow 3$$

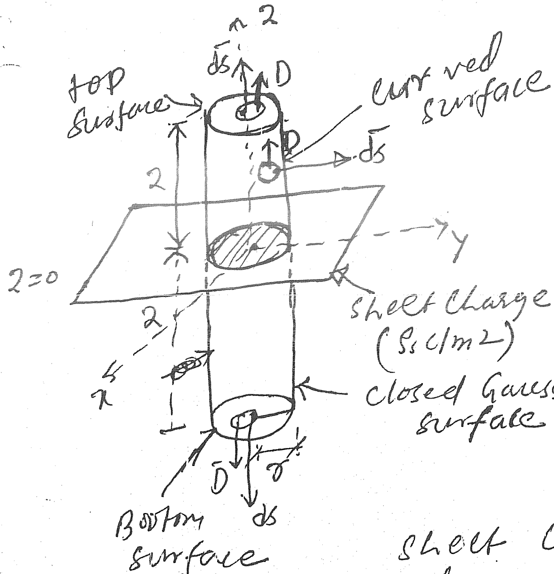
By Gauss law in point form, $\nabla\cdot\bar{D} = \rho_v \to (4)$

putting 4 in (3)

$$\boxed{\oint_S \bar{D}\cdot\bar{ds} = \oint_v (\nabla\cdot\bar{D})\, dv.}$$ Hence the proof.

This theorem is also known as <u>Gauss Divergence</u> theorem.

Electric field due to an infinite sheet of charge



Consider an infinite sheet of charge carrying a surface charge density of ρ_s C/m². Kept at z=0. plane. It is required to find EFI at point at a distance 'z' mt. from the sheet charge. Consider a hollow cylinder of radius 'r' mt & height of 'z' mt on either of sheet charge.

From Gauss's law,

$$\oint_s \bar{D}\cdot\bar{ds} = Q_{enclosed}.$$

$$\underset{\text{top surface}}{\oint_s \bar{D}\cdot\bar{ds}} + \underset{\text{Bottom surface}}{\oint_s \bar{D}\cdot\bar{ds}} + \underset{\text{curved surface}}{\oint_s \bar{D}\cdot\bar{ds}} = Q_{enclosed}$$

From the above fig $D\bar{a}_z$ parallel (tangential) to the curved surface. $\therefore \underset{\text{curved surface}}{\oint \bar{D}\cdot\bar{ds}} = 0.$

$$\therefore \oint_{s\,t.s} D_z\bar{a}_z \cdot r\,dr\,d\phi\,\bar{a}_z + \oint_{s\,B.S} -D_z\bar{a}_z\cdot r\,dr\,d\phi\,(-\bar{a}_z) + 0 = Q_{enclosed}.$$

$$D_z \cdot \int_{r=0}^{r} r\,dr \cdot \int_{\phi=0}^{\phi=2\pi} d\phi + D_z \cdot \int_{r=0}^{r} r\,dr \cdot \int_{\phi=0}^{2\pi} d\phi = \rho_s \cdot \pi r^2$$

$$D_z \cdot \left(\frac{r^2}{2}\right)_0^r (\phi)_0^{2\pi} + D_z \cdot \left(\frac{r^2}{2}\right)(\phi)_0^{2\pi} = \rho_s \pi r^2$$

$$D_z\left(\frac{r^2}{2}\right)(2\pi) + D_z\left(\frac{r^2}{2}\right)(2\pi) = \rho_s \pi r^2$$

$$D_z \cdot \cancel{r^2}\, 2\cancel{\pi} = \rho_s \cdot \cancel{\pi}\cancel{r^2}$$

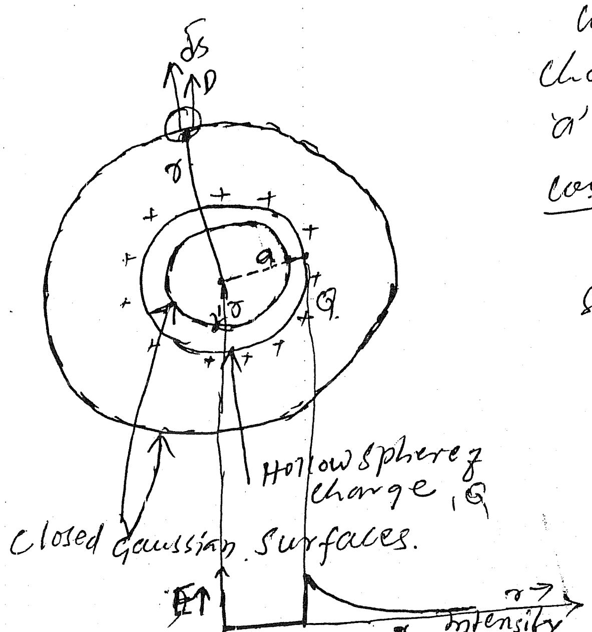
$$|D_z| = \rho_s/2.$$

$$\therefore \bar{D} = \frac{\rho_s}{2}\bar{a}_z$$

$$\therefore \bar{E} = \bar{D}/\epsilon$$

$$\boxed{\bar{E} = \rho_s/2\epsilon\ \bar{a}_z}\ \text{V/m}.$$

Electric field due to hollow sphere of charge:



Consider a hollow sphere of charge Q coulombs & radius 'a' mt.

case i) EFI inside the hollow sphere:

Consider a closed surface of radius 'r' mt $(r<a)$ as the Gaussian surface.

From Gauss law,

$$\oint_S \bar{D}\cdot\bar{ds} = Q_{enclosed}$$

$$\oint_S \bar{D}\cdot\bar{ds} = 0$$

$$\therefore \bar{D}=0 \Rightarrow \boxed{\bar{E}=0}$$

case-II) Electric field outside the hollow sphere $(r>a)$

Consider a sphere of radius 'r' mt $(r>a)$ as the Gaussian surface.

From Gauss law,

$$\oint_S \bar{D}\cdot\bar{ds} = Q_{enclosed}.$$

$$\oint_S D_r \bar{a_r}\cdot r^2 \sin\theta\, d\theta\, d\phi\, \bar{a_r} = Q.$$

$r =$ const.

$$\therefore D_r \cdot r^2 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta\, d\theta = Q.$$

$$D_r\, r^2 (\phi)_0^{2\pi} (-\cos\theta)_0^{\pi} = Q.$$

$$D_r \cdot r^2 \cdot (2\pi)(2) = Q.$$

$$D_r = Q/4\pi r^2.$$

$$\bar{D} = \frac{Q}{4\pi r^2}\bar{a_r}$$

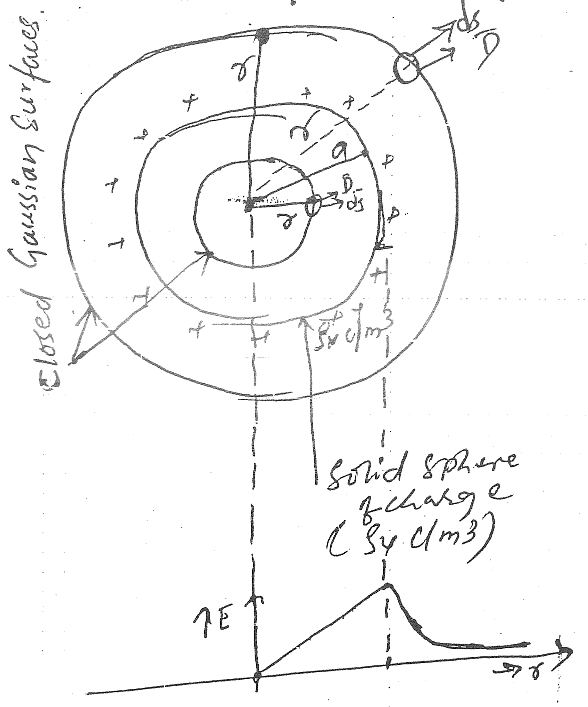
Now, $\bar{D} = \epsilon\bar{E}$

$$\therefore \bar{E} = \bar{D}/\epsilon$$

$$\boxed{\bar{E} = \frac{Q}{4\pi\epsilon r^2}\bar{a_r}} \quad V/m$$

variation of EFI with r is as shown above.

Electric field intensity due to solid sphere of charge



Consider a sphere of charge of radius 'a' mt with charge density of ρ_v C/m³.

Case-I: EFI inside the sphere (r<a)

Consider a closed surface of radius 'r' mt (r<a) as the Gaussian surface.

From Gauss's law.

$$\oint_s \bar{D}\cdot\bar{ds} = Q_{enclosed}$$

$$\oint_s D_r \bar{a_r}\cdot r^2 \sin\theta\, d\theta\, d\phi\, \bar{a_r} = Q_{enclosed}$$

$$D_r \cdot 4\pi r^2 = \rho_v \cdot \frac{4}{3}\pi r^3$$

$$D_r = \rho_v \cdot r/3 \qquad \bar{D} = \frac{\rho_v \cdot r}{3}\bar{a_r}$$

$$\therefore \boxed{\bar{E} = \frac{\rho_v \cdot r}{3\epsilon}\bar{a_r} \text{ V/m}}$$

Case II: EFI outside the sphere (r>a):

Consider closed Gaussian sphere of radius 'r' mt (r>a). From Gauss's law.

$$\oint_s \bar{D}\cdot\bar{ds} = Q_{enclosed}.$$

$$\oint_s D_r \bar{a_r}\cdot r^2 \sin\theta\, d\theta\, d\phi\, \bar{a_r} = \rho_v \cdot \frac{4}{3}\pi a^3$$

$$D_r \cdot r^2 \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = \rho_v \cdot \frac{4}{3}\pi a^3$$

$$D_r \cdot r^2 (-\cos\theta)_0^{\pi} (\phi)_0^{2\pi} = \rho_v \cdot \frac{4}{3}\pi a^3$$

$$\cancel{D_r = \rho_v} \quad D_r \cdot r^2 (2)(2\pi) = \rho_v \cdot \frac{4}{3}\pi a^3$$

$$D_r = \frac{\rho_v\, a^3\, \bar{a_r}}{3r^2}$$

$$\therefore \bar{E} = \bar{D}/\epsilon$$

$$\boxed{\bar{E} = \frac{\rho_v\, a^3}{3\epsilon r^2}\bar{a_r}} \text{ V/m}$$

or

$$\bar{E} = \frac{\frac{Q}{\frac{4}{3}\pi a^3}\cdot a^3}{3\epsilon r^2}\bar{a_r}$$

$$\boxed{\bar{E} = \frac{Q}{4\pi\epsilon r^2}\bar{a_r}} \text{ V/m.}$$

But $Q = \rho_v \cdot \frac{4}{3}\pi a^3$

$\therefore \rho_v = Q/\frac{4}{3}\pi a^3$

The variation of EFI with 'r' is as shown above.

UNIT-II : WORK ENERGY & POTENTIAL

Work done in moving a point charge in an electric field

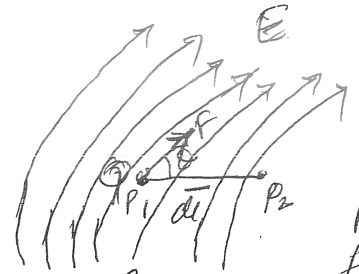


fig 1.

Consider an electric field as shown in fig ①. Let us find the work done by an external agency in moving a point charge from P_1 to P_2. The electric force exerted on a point charge Q at P_1 is given by $\bar{F} = Q.\bar{E}$ → 1

Now the component of electric force along P_1P_2 is $F\cos\theta$.

∴ The work done by the external agency in moving point charge Q from P_1 to P_2 = $F\cos\theta \times dl$

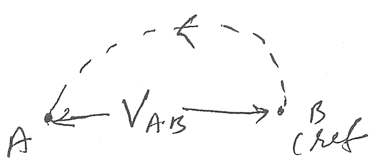
$$dw = \bar{F}\cdot\overline{dl} \qquad [\because \bar{A}\cdot\bar{B} = A\cdot B\cos\theta]$$

$$\therefore\ dw = Q\cdot\bar{E}\cdot\overline{dl} \qquad [\text{from } ①]$$

∴ Total work done is, $\boxed{W = -Q\int_l \bar{E}\cdot\overline{dl}}$ Joules.

-ve sign indicates that Q is moved against to the direction of electric field.

Electric potential between two points (potential diffn.)



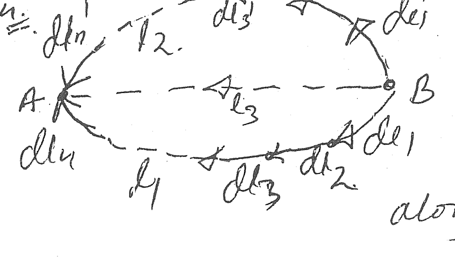
The electric potential at point 'A' with respect to a reference point 'B' is defined as the work done in moving a unit positive charge from reference point B to the point A along any path. It is denoted by 'V' & its unit is volts.

$$\therefore V_{AB} = W/Q = -\cancel{Q}\int_l \bar{E}\cdot\overline{dl}\,/\,\cancel{Q} = -\int_l \bar{E}\cdot\overline{dl}$$

$$\therefore \boxed{V_{AB} = -\int_l \bar{E}\cdot\overline{dl}}$$

Question: Prove that the work done in moving a point charge from B to A is independent of the path between A & B.

Soln:



Consider three different paths l_1, l_2 & l_3 between the points 'A' & 'B'. We wish to find the work done in moving a point charge Q from B to A along the path l_1 & l_2.

Work done in moving point charge along the path l_1 $= W_1 = -Q \int_l \bar{E} \cdot \bar{dl}$

$$W_1 = -Q \int_l \bar{E} \cdot (\bar{dl_1} + \bar{dl_2} + \bar{dl_3} + \cdots + \bar{dl_n})$$

$$= -Q \int_l \bar{E} \cdot \bar{BA} \longrightarrow 1$$

Ily the work done in moving point charge from B to A along path l_2 is $\Big\} = W_2 = -Q \int_l \bar{E} \cdot \bar{dl}$

$$W_2 = -Q \int_l \bar{E} \cdot [\bar{dl_1}' + \bar{dl_2}' + \bar{dl_3}' + \cdots + \bar{dl_n}']$$

$$= -Q \int_l \bar{E} \cdot \bar{BA} \longrightarrow (2)$$

From equation (1) & (2) $\boxed{W_1 = W_2 = -Q \int_l \bar{E} \cdot \bar{BA}}$

Hence the proof.

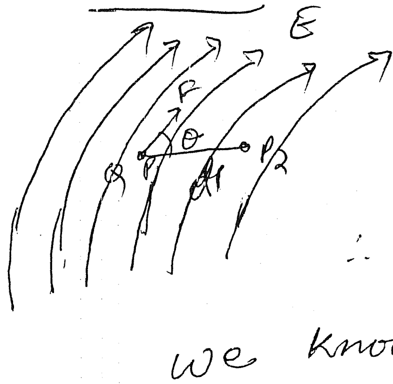
Gradient of a scalar field: Gradient is an operator which when operated on a scalar field results in to a vector field. Gradient of a scalar field 'V' is denoted by grad V or ∇V. The ∇V in different coordinate system are

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z \dashrightarrow \text{Rectangular co-ordinates}$$

$$= \frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z \dashrightarrow \text{Cylendrical co-ordinates.}$$

$$= \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi \rightarrow \text{Spherical co-ordinates.}$$

Relation between electric field intensity & electric potential: :→



Consider a point charge 'Q' coulombs placed at point P_1. Work done in moving point charge Q from P_1 to P_2 is

$$dw = -Q\,\bar{E} \cdot \bar{dl}$$

$$\therefore dV = \frac{dw}{Q} = \frac{-Q \int_l \bar{E} \cdot \bar{dl}}{Q} = -\int_l \bar{E} \cdot \bar{dl}$$

We know that from vector calculus

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

$$\therefore dV = \left(\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right)\cdot(dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z) \rightarrow (2)$$

By definition of gradient,

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z \rightarrow (3)$$ & the incremental length

$$\bar{dl} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z \rightarrow (4)$$

putting 3 & 4 in (2)

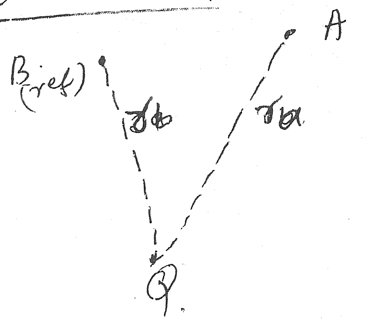
$$dV = \nabla V\cdot \bar{dl} \rightarrow (5)$$

Comparing 1 & 5

$$\nabla V\cdot \bar{dl} = -\bar{E}\cdot\bar{dl} \Rightarrow \nabla V = -\bar{E}$$

$$\therefore \boxed{\bar{E} = -\nabla V}$$

Electric potential due to a point charge :



Let A & B be the two points at radial distances r_a & r_b from a point charge Q. We wish to find the potential at point "A" w.r.t "B"

V_{AB} = potential at point 'A' w.r.t "B"

$$V_{AB} = -\int_{B}^{A} \bar{E}\cdot\bar{dl} \longrightarrow (1)$$

We know that, $\bar{E} = \frac{Q}{4\pi\epsilon r}\bar{a}_r \rightarrow (2)$ is the electric field intensity at any point due to a point charge.

As the point charge has spherical symmetry,

$$\vec{dl} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi \rightarrow (3)$$

putting (2) & (3) in (1)

$$V_{AB} = -\int_{r_b}^{r_a} \frac{Q}{4\pi\epsilon r^2}\bar{a}_r\cdot(dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi)$$

$$= -\int_{r_b}^{r_a} \frac{Q}{4\pi\epsilon r^2}\,dr$$

$$= -\frac{Q}{4\pi\epsilon}\int_{r_b}^{r_a}\frac{dr}{r^2}$$

$$= -\frac{Q}{4\pi\epsilon}\left[-\frac{1}{r}\right]_{r_b}^{r_a}$$

$$\boxed{V_{AB} = \frac{Q}{4\pi\epsilon}\left[\frac{1}{r_a} - \frac{1}{r_b}\right] \text{ volts}}$$

Note: If the reference point 'B' is at ∞, then $r_b = \infty$.

$\therefore$ The above equation modifies to,

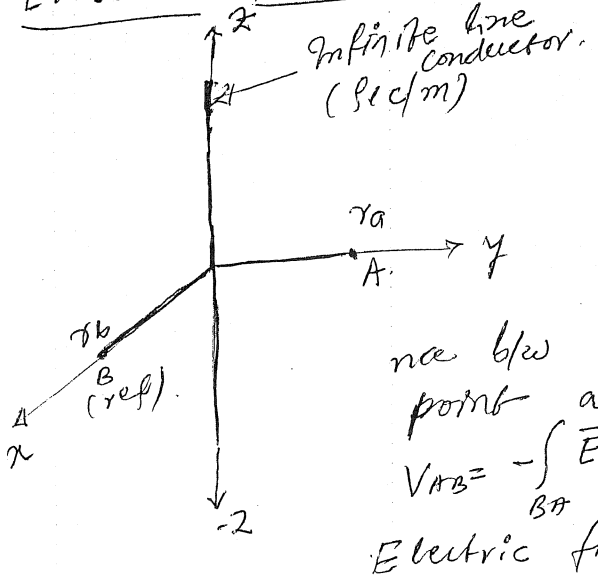
$$V_A = \frac{Q}{4\pi\epsilon}\left[\frac{1}{r_a} - \frac{1}{\infty}\right]$$

$$\boxed{V_A = \frac{Q}{4\pi\epsilon r_a}} \longrightarrow 4.$$

such potential is called absolute potential.

Absolute potential: The potential at a point w.r.t reference point is called absolute potential. It is defined as the work done in moving a unit positive charge from infinity to the point against the direction of electric field.

Electric potential due to infinite line charge:



consider two points A & B at radial distance of r_a & r_b mt from an infinite line charge with charge density of ρ_l c/m placed along z-axis. we wish to find the potential difference b/w points A & B w.r.t. reference point at 'B'.

$$V_{AB} = -\int_{B}^{A} \bar{E} \cdot \bar{dl} \longrightarrow ①$$

Electric field intensity due to ∞ line charge is $\bar{E} = \frac{\rho_l}{2\pi\epsilon r}\bar{a}_r \longrightarrow ②$

The distribution of electric field due to line charge has ~~sph~~ cylendrical symmetry,

$\therefore \bar{dl} = dr\bar{a}_r + r d\phi \bar{a}_\phi + dz\bar{a}_z \longrightarrow ③$

putting 2 & 3 in ①

$$V_{AB} = -\int_{r_b}^{r_a} \frac{\rho_l}{2\pi\epsilon r}\bar{a}_r \cdot (dr\bar{a}_r + r d\phi\bar{a}_\phi + dz\bar{a}_z)$$

$$\left[\because \bar{a}_r\cdot\bar{a}_\phi = \bar{a}_r\cdot\bar{a}_z = 0\right].$$

$$= -\int_{r_b}^{r_a} \frac{\rho_l}{2\pi\epsilon r} dr.$$

$$= -\frac{\rho_l}{2\pi\epsilon}(\ln r)_{r_b}^{r_a}.$$

$$V_{AB} = -\frac{\rho_l}{2\pi\epsilon}(\ln r_a - \ln r_b)$$

$$= \frac{\rho_l}{2\pi\epsilon}\{\ln(r_b) - \ln(r_a)\}$$

$$\boxed{V_{AB} = \frac{\rho_l}{2\pi\epsilon}\ln(r_b/r_a)} \text{ volts.}$$

③

Energy density in the electrostatic field :

We know that, when a unit positive charge is moved from infinity to a point in an electric field, work is said to be done by the external source & the energy is expended. If the external source is removed, then the unit positive charge will be subjected to a force exerted by the field & will be moved in the direction of force. Thus to hold a charge at a point in the electric field, an external source has to do work. This energy gets stored in the form of potential energy, when the charge is held at a point in a field. When external source is removed, the potential energy gets converted to kinetic energy. Let us determine the expression for such potential energy.

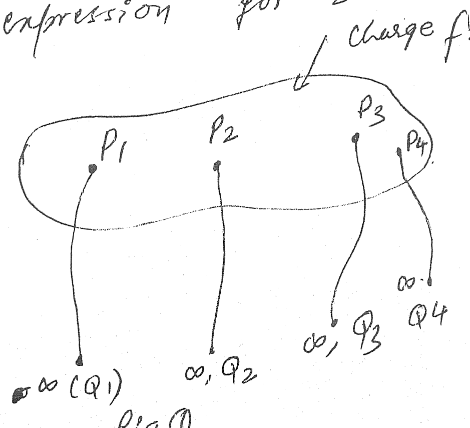


fig ①

Consider a region shown in fig ① where in, there is no electric field. When the point charge Q_1 is moved from ∞ to P1, it requires no work as there is no electric field exist. When Q_2 is moved to P_2, it has to overcome the field produced by Q_1. So work is done to move Q_2 to P_2 against the field of Q_1.

$\therefore$ work done to move Q_2 to $P_2 = Q_2 V_{21}$ $\left[\because V = \frac{\text{work done}}{\text{charge}},\ V = W/Q,\ \therefore W = QV\right]$

where V_{21} = potential at P_2 due to P_1.

Illy work done to move Q_3 to $P_3 = Q_3 V_{31} + Q_3 V_{32}$

" " Q_n to $P_n = Q_n V_{n1} + Q_n V_{n2} + Q_n V_{n3} + \cdots + Q_n V_{n\,n-1}$

Total work done in positioning all charges is,

$$W_E = Q_2 V_{21} + Q_3 V_{31} + Q_3 V_{32} + Q_4 V_{41} + Q_4 V_{42} + Q_4 V_{43} + \cdots \quad ①.$$

The total work done is nothing but the potential energy in the system of charges. If the charges are placed in the reverse order, then,

$$W_E = Q_3 V_{34} + Q_2 V_{23} + Q_2 V_{24} + Q_1 V_{12} + Q_1 V_{13} + Q_1 V_{14} + \cdots \quad (2)$$

Adding eqn. (1) & (2)

$$2W_E = Q_1 (V_{12} + V_{13} + V_{14} + \cdots V_{1n}) + Q_2 (V_{21} + V_{23} + V_{24} \cdots + V_{2n})$$
$$+ Q_3 (V_{31} + V_{32} + V_{34} + \cdots V_{3n}) + Q_4 (V_{41} + V_{42} + V_{43} + \cdots V_{4n}) \quad (3)$$

Each sum of the potential is the resultant potential due to all the charges, except for the charge at the point at which the potential is obtained.

$\therefore V_1 =$ potential at P_1 due to all charges at $P_2, P_3 \cdots$

$= V_{12} + V_{13} + V_{14} + \cdots V_{1n}$.

Illy $V_2 = V_{21} + V_{23} + V_{24} + \cdots V_{2n}$

$V_3 = V_{31} + V_{32} + V_{34} + \cdots V_{3n}$

$V_4 = V_{41} + V_{42} + V_{43} + \cdots V_{4n}$

$\therefore$ eqn (3) becomes

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + Q_4 V_4 + \cdots$$

$$\boxed{W_E = \frac{1}{2} \sum_{m=1}^{n} Q_m V_m} \text{ Joules.} \quad (4)$$

Eqn. 4. gives the potential energy stored in the system of 'n' charges.

The above equation can be written in integral form due to different charge configuration as,

$$W_E = \frac{1}{2} \int_l \rho_L \cdot dl \; V. \qquad (\because Q = \rho_L \, dl \text{ for line charge})$$

$$= \frac{1}{2} \int_v \rho_v \cdot dv \cdot V \qquad \{ Q = \rho_v \, dv \text{ for volume charge}; \; dv \to \text{volume element} \}$$

Expression for energy stored in terms of D & E :

Consider the volume charge distribution having uniform charge density of ρ_v C/m³. Here the total energy stored is given by,

$$W_E = \frac{1}{2} \int_{vol} \rho_v dv \cdot V \longrightarrow (1).$$

$$= \frac{1}{2} \int_v ((\nabla \cdot \bar{D}) V) dv \longrightarrow (1) \qquad (\because \nabla \cdot \bar{D} = \rho_v)$$

For any scalar 'V' & vector $\bar{A}$ there is vector identity.

$$\nabla \cdot V\bar{A} = \bar{A} \cdot \nabla V + V(\nabla \cdot \bar{A})$$

$$\therefore (\nabla \cdot \bar{A}) V = \nabla \cdot V\bar{A} - \bar{A} \cdot \nabla V$$

$$\text{i.e. } (\nabla \cdot \bar{D}) V = \nabla \cdot V\bar{D} - \bar{D} \cdot \nabla V \longrightarrow (2)$$

putting 2 in (1)

$$W_E = \frac{1}{2} \int_v (\nabla \cdot V\bar{D} - \bar{D} \cdot \nabla V)\, dv$$

$$= \frac{1}{2} \int_v \nabla \cdot V\bar{D}\, dv - \frac{1}{2} \int_v \bar{D} \cdot \nabla V\, dv \longrightarrow (3)$$

By divergence theorem,

$$\int_v (\nabla \cdot \bar{D})\, dv = \oint_s \bar{D} \cdot \bar{ds}$$

$$\text{i.e. } \int_v (\nabla \cdot V\bar{D})\, dv = \oint_s V\bar{D} \cdot \bar{ds}$$

$\therefore$ Eqn. 3 becomes,

$$W_E = \frac{1}{2} \oint_s V\bar{D} \cdot \bar{ds} - \frac{1}{2} \int_v \bar{D} \cdot (-\bar{E})\, dv \longrightarrow (4) \quad [\because E = -\nabla V]$$

We know that $V \propto \frac{1}{r}$ & $D \propto \frac{1}{r^2}$ $\therefore VD \propto \frac{1}{r^3}$ for point charge

& $ds \propto r^2$

$$\therefore V\bar{D} \cdot ds \propto \frac{1}{r^3} \times r^2 \propto \frac{1}{r}$$

As the surface becomes ∞ i.e. $r = \infty$ $1/r = 0$.

$$\therefore \frac{1}{2} \oint_s V\bar{D} \cdot \bar{ds} = 0.$$

$\therefore$ eqn (4) becomes, $\boxed{W_E = \frac{1}{2} \int_v \bar{D} \cdot \bar{E}\, dv}$

$$W_E = \frac{1}{2} \int_v (\bar{D} \cdot \bar{E})\, dv$$

$$= \frac{1}{2} \int_v \epsilon \bar{E} \cdot \bar{E}\, dv \qquad (\because \bar{D} = \epsilon \bar{E})$$

$\boxed{W_E = \frac{1}{2} \int_v \epsilon E^2\, dv}$ in terms of E

$$W_E = \frac{1}{2} \int_v \bar{D} \cdot \frac{\bar{D}}{\epsilon}\, dv$$

$\boxed{W_E = \frac{1}{2\epsilon} \int_v D^2\, dv}$ in terms of D.

Conductors dielectrics & capacitance

~~Conductors~~ Current & current density: The current is defined as rate of flow of charge & is measured in Amperes. A current of 1A is said to be flowing across the surface, when a charge of one coulomb is passing across the surface in one second.

i.e. $\boxed{I = \frac{dq}{dt} \text{ Amps}}$

So the current which is existing in the conductor due to drifting of electrons under the influence of the applied voltage is called drift current or conduction current. While in dielectrics there can be flow of charges under the influence of electric field intensity. Such a current is called displacement or convection current. The current flowing across the capacitor through the dielectric seperating its plates is an example of convection current.

The analysis of such currents in field theory is based on defining a current density at a point in the given field.

" The current density is defined as the current passing through the unit surface area, when the surface is held normal to the direction of current. It is a vector quantity. It is denoted by $\bar{J}$ & its unit is A/m^2.

Relation between I & $\bar{J}$:

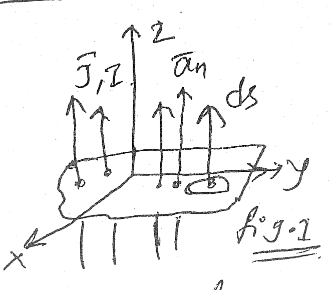


fig.1

As shown in fig ① consider the current density $\bar{J}$, passing normal to the surface ~~area~~ at $z=0$.

$\bar{ds} = ds\, \bar{a}_z$ & $\bar{J} = J\, \bar{a}_z$.

The total current I passing through the elemental surface 'ds' is

$$dI = \bar{J} \cdot \bar{ds}$$

$$\boxed{I = \oint_s \bar{J} \cdot \bar{ds}} \quad //.$$

Relation between J & ρ_v :

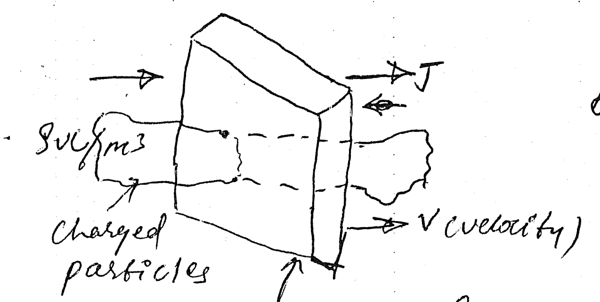


fig 1

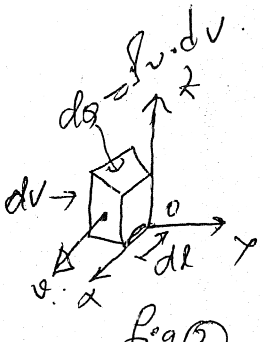


fig ②

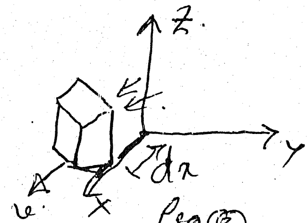


fig ③.

The set of charged particles in a volume gives rise to volume charge density (ρ_v). The current $\bar{J}$ can be related to the velocity with which the volume charge density crosses the surface's at a point as shown in fig ①. The velocity with which the charge is transferred is v m/sec. & is a vector quantity.

Consider a differential volume element dv with charge density of ρ_v C/m3 as shown in fig ②.

Then $dq = \rho_v \cdot dv$.

Let $dl \rightarrow$ Incremental length & $ds \rightarrow$ Incremental surface

Then, $dv = ds \cdot dl$.

$\therefore \; dq = \rho_v \, dv = \rho_v \cdot ds \cdot dl$.

Let the charge is moving with velocity v m/sec along x-axis during finite time interval. dt sec.

$$\therefore \; dI = \frac{dq}{dt}$$

$$= \frac{\rho_v \cdot ds \cdot dl}{dt}$$

$$= \rho_v \cdot ds \cdot \frac{dx}{dt}$$

$$dI = \rho_v \cdot ds \cdot \bar{V} \rightarrow (1)$$

Also $I = \oint_s \bar{J} \cdot \bar{ds}$

$dI = \bar{J} \cdot \bar{ds}$

$= J\, ds \,(\because)$ (when $\bar{J}$ & $\bar{ds}$ are normal to the surface $\bar{J} \uparrow \bar{ds}$)

$\therefore$ equation ① becomes

$dI = J\,ds = \rho_v \cdot ds\, \bar{v}$

$\boxed{\bar{J} = \rho_v \cdot \bar{V}}$ (2)

where $\bar{v}$ is the velocity vector. Such a current density is known as convection current density & the resulting current is known as convection current.

Continuity equation :

The continuity equation of current is based on the principle of conservation of charge. This principle states that charge can ~~be~~ neither be created nor be destroyed.

Consider a closed surface 'S' with a current density $\bar{J}$ A/m², then the total current I crossing the surface is given by $I = \oint_S \bar{J} \cdot \bar{ds} \rightarrow (1)$

The current flows outwards from the closed surface. As the current is flow of positive charges, hence the current I is constituted due to the outward flow of positive charges from closed surface 'S'. According to principle of conservation of charge there must be decrease of an equal amount positive charges inside the closed surface. Hence the outward rate of flow of positive charges gets balanced by the rate of decrease of charge inside the closed surface.

Let Q_i is the charge in the closed surface. Then

$-\frac{dQ_i}{dt}$ = Rate of decrease of charge inside the closed surface

-ve sign indicates ~~that~~ the decrease in charge.

Due to principle of conservation of charge, this rate of decrease of charge is equal to rate of outward flow of charge, which is a current.

$$\therefore \quad I = \oint_S \bar{J} \cdot \bar{ds} = -\frac{dQ_i}{dt} \rightarrow (2)$$

Equation 2 is known as integral form of continuity equation & give the outward flow of current the volume element

If the current is entering

$$\oint_S \bar{J} \cdot \bar{ds} = -I = \frac{dQ_i}{dt} \rightarrow (3)$$

By Divergence theorem,

$$\oint_S \bar{J} \cdot \bar{ds} = \oint_V (\nabla \cdot \bar{J})\, dv \rightarrow (4)$$

From 2 & 4.

$$\oint_V \nabla\cdot\bar{J}\,dv = -\frac{dq_i}{dt}$$

$$= -\frac{d}{dt}\int_v \rho_v\cdot dv \qquad \left[\because q = \int_v \rho_v\cdot dv\right]$$

$$\oint_v (\nabla\cdot\bar{J})dv = -\oint_v \frac{d}{dt}(\rho_v)\,dv$$

For a constant surface, the derivative becomes partial derivative.

$$\therefore \oint_v (\nabla\cdot\bar{J})\,dv = -\oint_v \frac{\partial(\rho_v)}{\partial t}dv$$

The volume integral on both sides is same & cancels.

$$\therefore \boxed{\nabla\cdot\bar{J} = -\frac{\partial\rho_v}{\partial t}}$$ This equation is known as "point form of continuity equation.

For steady currents which are not functions of time, $\frac{\partial\rho_v}{\partial t} = 0$. $\therefore \boxed{\nabla\cdot\bar{J} = 0}$ This equation is known as field equivalent of Kirchoff's current law, which states that the sum of currents at a junction of several conductors is zero.

<u>point form of ohms law</u>: Consider a conductor subjected to an electric field. Under the effect of applied electric field, the available free electrons start moving. The moving electrons strike the adjecent atoms & rebound (recoil) in the random direction. This is known as drifting of electrons. After some time, the electrons obtains the constant average velocity called drift velocity (V_d). The current constituted due to drifting of such electrons in metallic conductor is called drift current. The drift velocity is directly proportional to the applied voltage. i.e. $\bar{V}_d \propto \bar{E}$.

$$V_d = -\mu_e\bar{E} \longrightarrow (1)$$

where μ_e is proportionality constant & is known as mobility of electrons in a given material.

-ve sign indicates that the velocity of electrons is against to the direction of electric field intensity, 'E'.

But we know that,

$$\bar{J} = \rho_v \cdot \bar{V} \rightarrow (2)$$

As the drift velocity is the velocity of free electrons, the above equation can be written as,

$$\bar{J} = \rho_e V_d \rightarrow (2)$$

where, $\rho_e \rightarrow$ charge density due to free electrons & is given by $\rho_e = ne$.

where $n \rightarrow$ No. of free electrons per meter^3

$e \rightarrow$ charge of one electron.

From (1) & (2)

$$\bar{J} = -\rho_e \mu_e \bar{E}$$

$$\boxed{\bar{J} = \sigma \bar{E}} \rightarrow (3)$$

where, $\sigma = -\mu_e \rho_e$ & is known as conductivity of a material.

Equation (3) is known as Ohm's law in point form.

Capacitance: It is a device used to store the charge. It consists of two conducting plates separated by a dielectric medium. 4

~~The cap~~ The capacity of a capacitor ~~capaci~~ is defined in terms of capacitance. The capacitance is defined as the ratio of the magnitude of the total charge accumulated on either of the two conducting plates to the potential difference applied between them. It is denoted by 'C' & its unit is Farad.

Let, Q = charge accumulated in coulombs

V = potential applied b/w the two plates in volts.

Then, $\boxed{C = \frac{Q}{V}}$

As the charge resides only on the surface of the ~~two plates~~, plates.

From Gauss law,

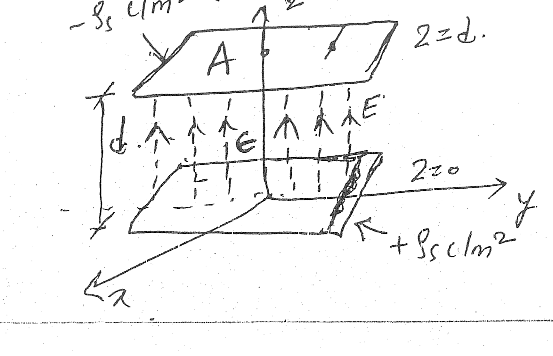
$$Q = \oint_S \bar{D}\cdot d\bar{s}$$

& $$V = -\int \bar{E}\cdot d\bar{l}$$

$$\therefore C = \frac{Q}{V}$$

$$\boxed{C = \frac{\oint_S \bar{D}\cdot d\bar{s}}{-\int \bar{E}\cdot d\bar{l}}}$$

Capacitance of a parallel plate capacitor



Consider a parallel plate capacitor consisting of two conducting plates of area $A\,m^2$ at $z=0$ & $z=d$ mt carrying charge densities $+\rho_s\ c/m^2$ & $-\rho_s\ c/m^2$ respectively.

By definition,

$$c = \frac{Q}{V} \rightarrow (1)$$

But $Q = \rho_s A \rightarrow (2)$

& $V = -\int \bar{E}\cdot\bar{dl} \rightarrow (3)$.

~~Now~~, where, E is the electric field intensity between the plates.

$\therefore \bar{E} =$ Electric field due to plate at $z=0$ + Electric field due to plate at $z=dm$.

$$= \bar{E}_1 + \bar{E}_2$$

$$\cancel{= \frac{\rho_s}{2\epsilon}\bar{a}_z + \left(-\frac{\rho_s}{2\epsilon}\right)}$$

Electric field intensity due to sheet charge is,

$$\bar{E} = \frac{\rho_s}{2\epsilon}\bar{a}_n$$

where $\bar{a}_n \rightarrow$ unit vector indicating the direction of electric field intensity ~~along the~~ which is along the line joining from the sheet charge to the point p where electric field intensity is required.

$$\therefore \bar{E}_1 = \frac{\rho_s}{2\epsilon}(\bar{a}_z)$$

$$\bar{E}_2 = -\frac{\rho_s}{2\epsilon}(-\bar{a}_z)$$

$$= \frac{\rho_s}{2\epsilon}\bar{a}_z.$$

$$\therefore \bar{E} = \frac{\rho_s}{2\epsilon}\bar{a}_z + \frac{\rho_s}{2\epsilon}a_z.$$

$$= \frac{2\rho_s}{2\epsilon}\bar{a}_z$$

$$\bar{E} = \frac{\rho_s}{\epsilon}\bar{a}_z \longrightarrow (4)$$

putting (4) in (3)

$$v = -\int_l \frac{\rho_s}{\epsilon} \bar{a_z} \cdot \bar{dl}$$

$$= -\int_l \frac{\rho_s}{\epsilon} \bar{a_z} \cdot (dx\,\bar{a_x} + dy\,\bar{a_y} + dz\,\bar{a_z})$$ ($\because$ $\bar{dl} = dx\bar{a_x} + dy\bar{a_y} + dz\bar{a_z}$ in Rectangular co-ordinates).

$$= -\int_l \frac{\rho_s}{\epsilon} dz$$

$$= -\int_{z=d}^{0} \frac{\rho_s}{\epsilon} dz.$$

$$= -\frac{\rho_s}{\epsilon} [z]_d^0$$

$$= -\frac{\rho_s}{\epsilon} [0-d]$$

$$v = \frac{\rho_s}{\epsilon} d. \longrightarrow ⑤$$

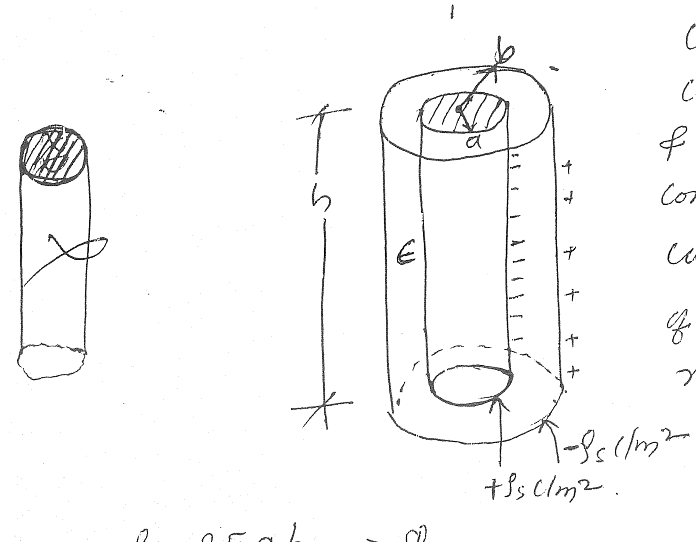
putting 5 ~~of~~ ② in ①

$$C = \frac{\rho_s \cdot A}{\frac{\rho_s \cdot d}{\epsilon}}$$

$$\boxed{C = \frac{\epsilon A}{d}}$$ farads.

Hence the proof.

<u>capacitance of a co-axial cable</u>: consider a co-axial cylindrical capacitor consisting of inner solid conductor of radius 'a' mt & outer ~~solid~~ hollow conductor of radius 'b' mt carrying charge densities of $+\rho_s$ C/m² & $-\rho_s$ C/m² respectively.



$$Q = \rho_s \cdot 2\pi a h. \rightarrow ①$$

$$v = -\oint_l \bar{E} \cdot \bar{dl} \rightarrow ②.$$

Let us compute the electric field intensity between the cable.

consider an imaginary ~~sphere~~ Gaussian surface of radius 'ρ' mt & height 'h' mt.

From Gauss law,

$$\oint_s \bar{D}\cdot\bar{ds} = Q.$$

$$\oint_s D_\rho \bar{a}_\rho \cdot \rho\, d\phi\, dz\, \bar{a}_\rho = Q.$$

$$\oint_s D_\rho\, \rho\, d\phi\, dz = Q$$

$$D_\rho \cdot \rho \int_{z=0}^{h} dz \int_{\phi=0}^{2\pi} d\phi = Q$$

$$\epsilon E_\rho\, \rho\, (z)_0^h (\phi)_0^{2\pi} = Q$$

$$E_\rho = \frac{Q}{2\pi\rho\epsilon h}$$

$$\bar{E} = \frac{Q}{2\pi\epsilon\rho h}\bar{a}_\rho \rightarrow (3)$$

~~But Q~~ putting (1) in (3)

$$\bar{E} = \frac{\rho_s\, 2\pi a h}{2\pi\epsilon\, \rho\, h}\bar{a}_\rho$$

$$\bar{E} = \frac{\rho_s \cdot a}{\epsilon\rho}\bar{a}_\rho \rightarrow (4)$$

we know that,

$d\bar{l} = d\rho\, \bar{a}_\rho + \rho\, d\phi\, \bar{a}_\phi + dz\, \bar{a}_z \rightarrow (5)$ in cylendrical co-ordinate system

putting 4 & 5 in (2)

$$V = -\int_l \frac{\rho_s \cdot a}{\epsilon\rho}\bar{a}_\rho \cdot (d\rho\, \bar{a}_\rho + \rho\, d\phi\, \bar{a}_\phi + dz\, \bar{a}_z)$$

$$= -\int_l \frac{\rho_s \cdot a}{\epsilon\rho}\, d\rho + 0 + 0.$$

$$= -\int_b^a \frac{\rho_s \cdot a}{\epsilon\rho}\, d\rho.$$

$$= -\frac{\rho_s \cdot a}{\epsilon}(\ln\rho)_b^a$$

$$= -\frac{\rho_s \cdot a}{\epsilon}(\ln(a) - \ln(b).)$$

$$= \frac{\rho_s a}{\epsilon} \left(\ln(b) - \ln(a) \right)$$

$$= \frac{\rho_s . a}{\epsilon} \ln(b/a) \longrightarrow (6)$$

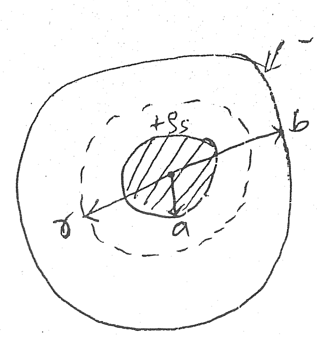
Now, $C = \frac{Q}{V}$

putting (6) & (1) in above eqn,

$$C = \frac{\rho_s . 2\pi\, a h}{\frac{\rho_s . a \ln(b/a)}{\epsilon}}$$

$$\boxed{C = \frac{2\pi\epsilon h}{\ln(b/a)}} \text{ farad.}$$

capacitance of a spherical capacitor :



Consider a spherical capacitor consisting of inner sphere of radius 'a' mt & outer sphere of radius 'b' mt. carrying the charge densities of $+\rho_s \ c/m^2$ & $-\rho_s \ C/m^2$ respectively.

$$C = Q/V \longrightarrow (1)$$

Total charge, $Q = \rho_s 4\pi a^2 \longrightarrow (2)$

$$V = -\int_l \bar{E} \cdot \bar{dl} \longrightarrow 3$$

Let us find the electric field intensity due to in the region a to b.

consider an Gaussian sphere of radius r mt

By Gauss law,

$$\oint_s \bar{D} \cdot \bar{ds} = Q.$$

$$\cancel{\oint \bar{E} \cdot \bar{ds}}$$

$$\oint_s D_r \bar{a_r} \cdot r^2 \sin\theta \, d\theta \, d\phi = Q$$

$$D_r r^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\phi$$

$\cancel{E = \rho_s}$

$D\, r^2 \, (-\sin\theta)_0^{\pi} \, (\phi)_0^{2\pi} \; q$

$D\, r^2 \, (-\sin\pi + \sin 0)\,(2\pi - 0) = q$

$D r^2 \,(2)(2\pi) = \rho_s \cdot 4\pi a^2$ (∵ from ②)

$4\pi r^2 D = \rho_s \cdot 4\pi a^2$

$$D = \frac{\rho_s \cdot a^2}{r^2}$$

$$\bar{E} = \frac{\rho_s \cdot a^2}{\epsilon r^2}\,\bar{a}_r \longrightarrow ④$$

Also $\bar{dl} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\, d\phi\, \bar{a}_\phi$

$$V = -\int \frac{\rho_s \cdot a^2}{\epsilon r^2}\,\bar{a}_r \cdot (dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\, d\phi\,\bar{a}_\phi)$$

$$= -\int \frac{\rho_s \cdot a^2}{\epsilon \cdot r^2}\, dr \cdot + 0 + 0.$$

$$= -\frac{\rho_s \cdot a^2}{\epsilon}\int_b^a \frac{1}{r^2}\, dr$$

$$= -\frac{\rho_s\, a^2}{\epsilon}\left(-\frac{1}{r}\right)_b^a$$

$$= +\frac{\rho_s\, a^2}{\epsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$= \frac{\rho_s \cdot a^{\cancel{2}}}{\epsilon}\,\frac{(b-a)}{\cancel{a}\, b}$$

$$= \frac{\rho_s \cdot a\,(b-a)}{\epsilon\, b} \longrightarrow ⑤$$

putting ⑤ & ② in ①

$$\cancel{V = } \quad C = \frac{\rho_s \cdot 4\pi a^2}{\dfrac{\rho_s \cdot a\,(b-a)}{\epsilon\, b}}$$

$$\boxed{C = \frac{4\pi\epsilon\, ab}{b-a}}$$ ....farads

Capacitance of an isolated sphere: In an isolated sphere, the outer spherical conductor is at ∞. (i.e. $b \to \infty$)

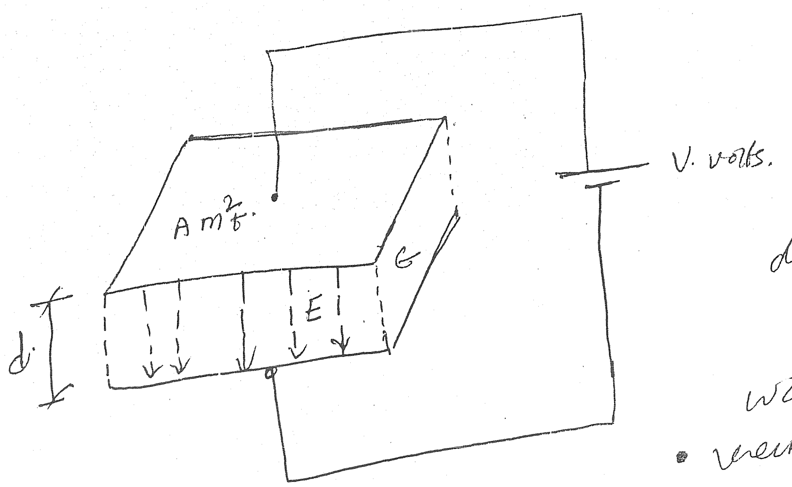
$\therefore$ The capacitance of such an isolated sphere is, (7)

$$C = \lim_{b\to\infty} \frac{4\pi\epsilon\, ab}{b-a}$$

$$= \lim_{b\to\infty} \frac{4\pi\epsilon}{\frac{1}{a}-\frac{1}{b}}$$

$$\boxed{C = 4\pi\epsilon\, a} \text{ farads.}$$

Energy stored in a capacitor: Consider a parallel plate capacitor as shown in the fig.



The electric field intensity at any point in the dielectric medium is ~~given~~

$$\bar{E} = \frac{V}{d}\,\bar{a}_n$$

where, $\bar{a}_n$ is the unit vector with the direction normal to the plates.

$$|E| = \frac{V}{d} \longrightarrow (1)$$

We know that, the energy stored is given by

$$W_E = \frac{1}{2}\int_v \bar{D}\cdot\bar{E}\, dv$$

$$= \frac{1}{2}\int_v \epsilon\bar{E}\cdot\bar{E}\, dv$$

$$= \frac{1}{2}\epsilon\int_v E^2\, dv$$

$$= \frac{1}{2}\epsilon E^2\int_v dv$$

$$= \frac{1}{2}\epsilon\left(\frac{V}{d}\right)^2 A\cdot d$$

$$= \frac{1}{2}\frac{\epsilon A}{d} V^2 \quad (6)$$

$$\boxed{W_E = \frac{1}{2}CV^2} \text{ Joules. } \left(\because C = \frac{\epsilon A}{d}\right)$$

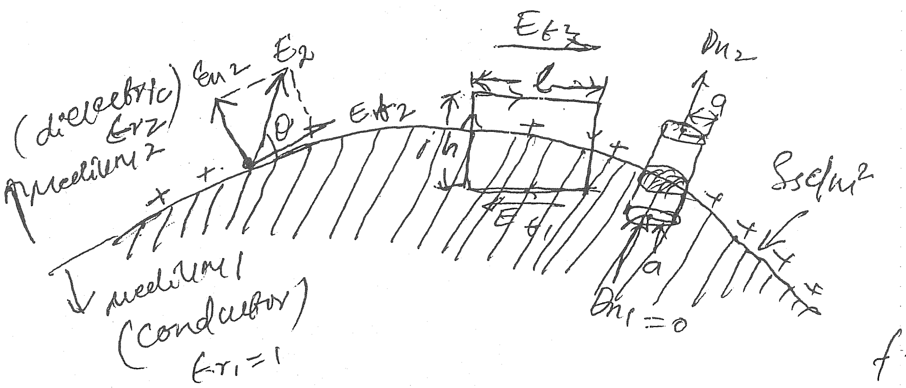
<u>Boundary conditions</u>: when an electric field passes from one medium to another medium, it is important to know the conditions of the field at the boundary between the two medium.

The conditions of the electric field existing at the boundary between the two medium, when it passes from one medium to other are known as <u>boundary conditions</u>

Depending upon the type of medium, there are two situations of boundary conditions.

1. Boundary between conductor & dielectric.
2. Boundary between two dielectric with different properties (ϵ_r)

<u>Boundary conditions between conductor & dielectric</u>



Consider a conductor - dielectric interface with charge density of ρ_s C/m^2 on the interface. The flux density & hence the electric field intensity at the interface can be resolved in to two components as tangential components & normal components to the interface. Now we wish to find the relation between tangential & normal components of field intensities at the interface

a. <u>To obtain tangential components</u>: (D_{tan} & E_t)

A conductor is an equipotential surface. Hence no work is done in moving a point charge on the conductor. So the charge on the conductor doesnot experience any force.

i.e. $\bar{F} = Q \cdot \bar{E}_t = 0.$

$\Rightarrow \boxed{E_t = 0}$ Also $D_t = \epsilon E_t = 0 \Rightarrow \boxed{D_t = 0}$

b. To obtain normal components: Consider a small pill box as shown in the fig.

From, Gauss's law; $\oint_S \bar{D} \cdot \bar{ds} = Q_{enclosed}$

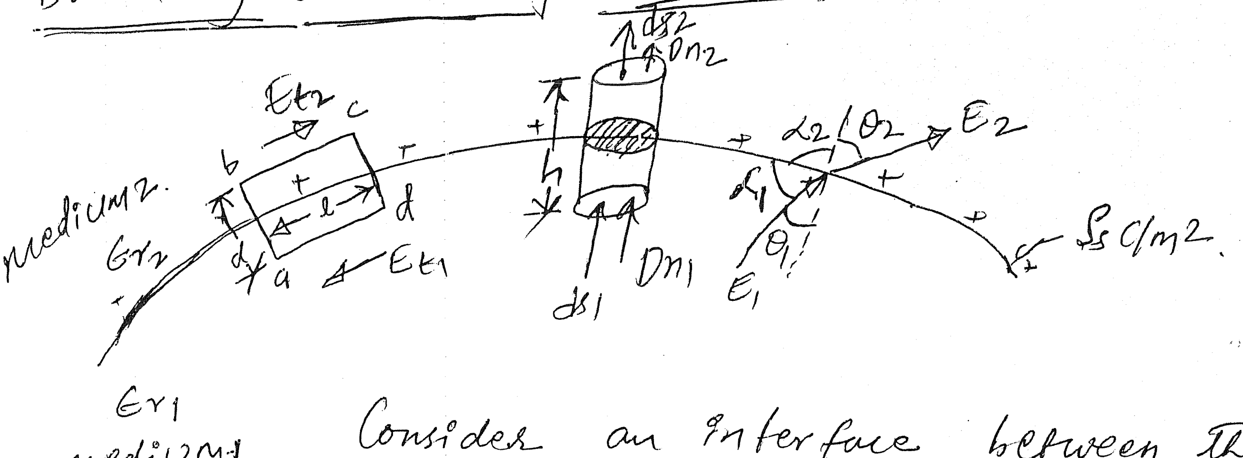
$$\oint_{\text{top surface}} \bar{D} \cdot \bar{ds} + \oint_{\text{Bottom surface}} \bar{D} \cdot \bar{ds} + \oint_{\text{curved surface}} \bar{D} \cdot \bar{ds} = Q_{enclosed}$$

Field in medium 1 = 0 $\therefore \oint_{\text{Bottom surface}} \bar{D} \cdot \bar{ds} = 0$, & $\oint_{\text{curved surface}} \bar{D} \cdot \bar{ds} = 0.$

$\therefore D_n \pi a^2 = \rho_s \pi a^2$ (at the interface, $\rho_s \pi a^2 = \phi$)

$\therefore \boxed{D_n = \rho_s}$ Illy $\boxed{E_n = \frac{D_n}{\epsilon} = \frac{\rho_s}{\epsilon}}$

Boundary conditions for Dielectric - Dielectric Interface



Consider an interface between the two dielectrics with their relative permittivities as shown above.

Let, E_1 = Electric field intensity in medium ①

E_2 = Electric field intensity in medium-2

The electric field intensity & flux density at the interface can be resolved into normal & tangential components. to the ~~the~~ interface.

The con

(9)

a. <u>Expression for tangential components (D_t & E_t)</u>

Consider a closed loop abcd as shown in the fig. Taking $\oint_l \bar{E}\cdot \bar{dl}$ over a closed path abcd,

$$\oint_l \bar{E}\cdot\bar{dl} = \oint_a^b \bar{E}\cdot\bar{dl} + \oint_b^c \bar{E}\cdot\bar{dl} + \oint_c^d \bar{E}\cdot\bar{dl} + \oint_d^a \bar{E}\cdot\bar{dl} = 0 \longrightarrow 1$$

At the interface the length 'd' should be as small as possible. ∴ ab → 0, cd → 0.

∴ $\oint_a^b \bar{E}\cdot\bar{dl} = \oint_c^d \bar{E}\cdot\bar{dl} = 0$, Now the equation 1 becomes,

$$\oint_l \bar{E}\cdot\bar{dl} = \oint_b^c \bar{E}\cdot\bar{dl} + \int_d^a \bar{E}\cdot\bar{dl} = 0.$$

$$E_{t2}\cdot l - E_{t1}\cdot l = 0.$$

$$(E_{t2} - E_{t1})\cdot l = 0$$

$$\therefore E_{t2} - E_{t1} = 0$$

$$\boxed{E_{t2} = E_{t1}}$$

Also $\dfrac{D_{t2}}{\epsilon_2} = \dfrac{D_{t1}}{\epsilon_1}$ $\left(\because \bar{D} = \epsilon\bar{E}, \therefore \bar{E} = \dfrac{\bar{D}}{\epsilon}\right)$

$$\frac{D_{t2}}{\epsilon_0 \epsilon_{r2}} = \frac{D_{t1}}{\epsilon_0 \epsilon_{r1}}$$

$$\boxed{\frac{D_{t2}}{\epsilon_{r2}} = \frac{D_{t1}}{\epsilon_{r1}}}$$

b. <u>Expression for normal components of E & D. (E_n & D_n)</u>

Consider a small pill box of radius 'a' mt & height 'h' mt as shown above in the figure. From Gauss law.

$$\underset{\text{top surface}}{\oint \bar{D}\cdot\bar{ds}} + \underset{\text{curved surface}}{\oint \bar{D}\cdot\bar{ds}} + \underset{\text{Bottom surface}}{\oint \bar{D}\cdot\bar{ds}} = \rho_s \cdot \pi a^2$$

$D_{n2}\pi a^2 + 0 + (-D_{n1})\pi a^2 = \rho_s \cdot \pi a^2$

$(D_{n2} - D_{n1})\cancel{\pi a^2} = \rho_s \cdot \cancel{\pi a^2}$

$\boxed{(D_{n2} - D_{n1}) = \rho_s}$

If the interface is free of charge then, $\rho_s = 0$,

$\boxed{\therefore D_{n2} - D_{n1} = 0}$

$\therefore \boxed{D_{n2} = D_{n1}}$

Also $\epsilon_2 E_{n2} = \epsilon_1 E_{n1}$ $(\because \bar{D} = \epsilon \bar{E})$

$\cancel{\epsilon_0} \epsilon_{r2} E_{n2} = \cancel{\epsilon_0} \epsilon_{r1} E_{n1}$

$\epsilon_{r2} E_{n2} = \epsilon_{r1} E_{n1}$

$\boxed{\dfrac{E_{n1}}{E_{n2}} = \dfrac{\epsilon_{r2}}{\epsilon_{r1}}}$

Let the electric field intensity makes an angle θ_1 & θ_2 with normal to the interface ~~between~~ in medium ① & medium ② having relative permittivity ϵ_{r1} & ϵ_{r2} respectively.

Now, $\left.\begin{aligned} \bar{E}_{t1} &= E_1 \sin\theta_1 \\ \bar{E}_{t2} &= E_2 \sin\theta_2 \end{aligned}\right\} \rightarrow (1)$

At the interface, $\bar{E}_{t1} = \bar{E}_{t2} \rightarrow (2)$

putting ~~$E_1 \sin\theta_1$~~ equation (1) in (2).

$E_1 \sin\theta_1 = E_2 \sin\theta_2 \longrightarrow (3)$

Also, we know that, $D_{n1} = D_{n2}$.

$\therefore \cancel{\epsilon_0} \epsilon_{r1} E_{n1} = \cancel{\epsilon_0} \epsilon_{r2} E_{n2}$

$\epsilon_{r1} E_{n1} = \epsilon_{r2} E_{n2} \longrightarrow (4)$

From the above fig, $E_{n1} = E_1 \cos\theta_1$ & $E_{n2} = E_2 \cos\theta_2 \rightarrow (5)$

$\therefore$ eqn. 4 becomes, $\epsilon_{r1} E_1 \cos\theta_1 = \epsilon_{r2} E_2 \cos\theta_2 \longrightarrow (6)$

Dividing equation ~~(1)~~ (3) by 6.

$$\frac{\cancel{E_1}\sin\theta_1}{\epsilon_{r1}\cancel{E_1}\cos\theta_1} = \frac{\cancel{E_2}\sin\theta_2}{\epsilon_{r2}\cancel{E_2}\cos\theta_2}$$

$$\frac{\tan\theta_1}{\epsilon_{r1}} = \frac{\tan\theta_2}{\epsilon_{r2}}$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}} \longrightarrow (7)$$

//

Let α_1 & α_2 be the angles made between electric field & the dielectric interface, then $\alpha_1 = 90-\theta_1$ & $\alpha_2 = 90-\theta_2 \Rightarrow \theta_1 = 90-\alpha_1$ & $\theta_2 = 90-\alpha_2$.

putting α_1 & α_2 in equation (7)

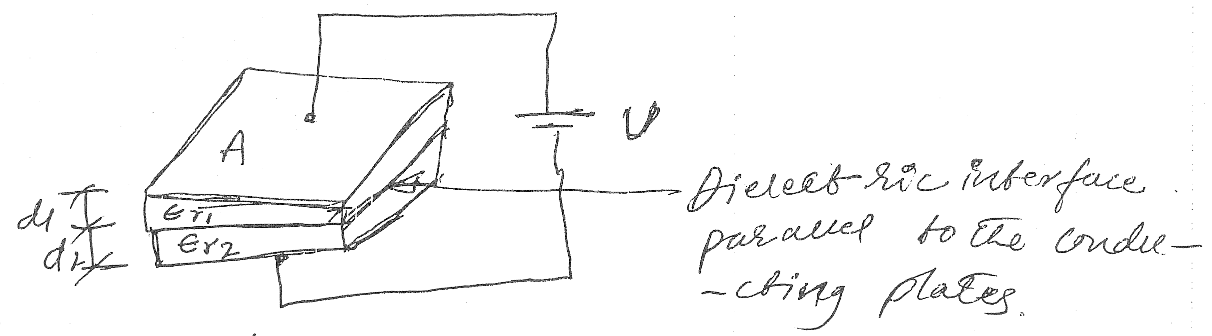
$$\frac{\tan(90-\alpha_1)}{\tan(90-\alpha_2)} = \frac{\epsilon_{r1}}{\epsilon_{r2}}$$

$$\boxed{\frac{\cot\alpha_1}{\cot\alpha_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}} \quad \text{or} \quad \boxed{\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\epsilon_{r1}}{\epsilon_{r2}}}$$

//

Composit parallel plate capacitor: It is a capacitor formed with more than one dielectric. Depending upon the positioning of dielectric, there are two types of composite parallel plate capacitor.

1. Composit parallel plate capacitor with dielectric interface parallel to the conducting plates:



Let $q \rightarrow$ Charge on each plate
$d \rightarrow$ Total thickness of dielectric
$E_1 \rightarrow$ Electric field in region d_1

$E_2 \rightarrow$ Electric field in region d_2.

$V \rightarrow$ applied voltage.

Consider two conducting plates of area $A\, m^2$ & thickness d_1 & d_2 having relative permittivity ϵ_{r1} & ϵ_{r2} respectively are placed one over the other, so that the dielectric interface is parallel to the conducting plates.

by dielectric

Now, the applied voltage is equal to the sum of voltage across each dielectric.

i.e. $V_0 = V_1 + V_2 \longrightarrow 1$

But, $V_1 = E_1 d_1$ & $V_2 = E_2 d_2$

$\therefore$ Eqn. (1) becomes,

$$V = E_1 d_1 + E_2 d_2 \longrightarrow 2.$$

The flux densities at the interface are.

$$D_1 = \epsilon_1 E_1 \,, \quad D_2 = \epsilon_2 E_2 .$$

$$\therefore E_1 = \frac{D_1}{\epsilon_1} \ \& \ E_2 = \frac{D_2}{\epsilon_2} \longrightarrow 3$$

putting E_1 & E_2 in (2)

$$V = \frac{D_1}{\epsilon_1} d_1 + \frac{D_2}{\epsilon_2} d_2 \longrightarrow 4.$$

The magnitudes of surface charged density is same on each plate.

Hence, $\rho_{s} = D_1 = D_2$

$\therefore$ equation (4) becomes,

$$V = \frac{\rho_s d_1}{\epsilon_1} + \frac{\rho_s d_2}{\epsilon_2} = \rho_s \left[\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2} \right]$$

$$V = \frac{Q}{A} \left[\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2} \right] \qquad \left(\because \rho_s = \frac{Q}{A} \right)$$

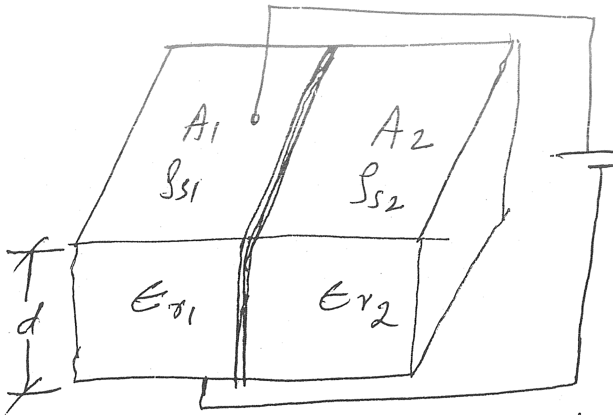
$$\frac{V}{Q} = \frac{d_1}{\epsilon_1 A} + \frac{d_2}{\epsilon_2 A}.$$

$$\frac{V}{Q} = \frac{1}{\epsilon_1 A / d_1} + \frac{1}{\epsilon_2 A / d_2}$$

$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}} \text{ Farads} \qquad \left(C_1 = \frac{\epsilon_1 A}{d_1} \ \& \ C_2 = \frac{\epsilon_2 A}{d_2} \right)$$

In general, $\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots}$ Farads. (11)

2. Composite parallel plate capacitor with dielectric interface perpendicular to the plates:



Consider two conducting plates of area A_1 & A_2 meter² facing the dielectric material of thickness 'd' & relative permittivity ϵ_{r1} & ϵ_{r2} respectively.

Let $V \rightarrow$ applied voltage.

$A_1 \rightarrow$ Area of conducting plate facing dielectric 1 (ϵ_{r1})

$A_2 \rightarrow$ Area of conducting plate facing dielectric 2 (ϵ_{r2})

$d \rightarrow$ Thickness of dielectric 1 (ϵ_{r1}) & dielectric 2 (ϵ_{r2})

As shown in the fig,

$$\left.\begin{aligned} D_1 &= \epsilon_1 E_1 \\ \& \quad D_2 &= \epsilon_2 E_2 \end{aligned}\right\} \rightarrow (1)$$

The electric field intensity is $E = V/d$.

Since the thickness of the dielectric & the potential applied to each dielectric is same,

$$\therefore E_1 = E_2 = V/d \rightarrow (2)$$

putting (2) in (1) $\left. D_1 = \epsilon_1 V/d \ \& \ D_2 = \epsilon_2 \cdot \frac{V}{d} \right\} \rightarrow (3)$

The net charge on the plate is divided in to 2 parts

i.e. $Q = Q_1 + Q_2$

$= \rho_{s1} A_1 + \rho_{s2} A_2$

$= D_1 A_1 + D_2 A_2 \rightarrow (4)$ (Since $\rho_s = D = Q/A$)

putting (3) in (4).

$$Q = \frac{\epsilon_1 V \cdot A_1}{d} + \frac{\epsilon_2 \cdot V \cdot A_2}{d} = V\left[\frac{\epsilon_1 A_1}{d} + \frac{\epsilon_2 A_2}{d}\right]$$

$$\frac{Q}{V} = \frac{\epsilon_1 A_1}{d} + \frac{\epsilon_2 A_2}{d}$$

$\boxed{C = C_1 + C_2}$ where $C_1 = \frac{\epsilon_1 A_1}{d}$ & $C_2 = \frac{\epsilon_2 A_2}{d}$.

In general $\boxed{C = C_1 + C_2 + C_3 + \cdots}$ Farads.

Poisson's & Laplace equations (Unit-III)

When the charge & potential are given at some boundaries of the region, then the potential & the electric field intensity at any region may be obtained using Poisson's & Laplace equation.

Proof: Consider Gauss's law in point form,

$$\nabla \cdot \bar{D} = \rho_v \longrightarrow 1.$$

where $\bar{D} \rightarrow$ flux density.

$\rho_v \rightarrow$ volume charge density.

We know that for homogenous, isotropic & linear medium, $\bar{D}$ & $\bar{E}$ are related as

$$\bar{D} = \epsilon \bar{E}$$

$\therefore$ equation 1. becomes.

$$\nabla \cdot \epsilon \bar{E} = \rho_v$$

$$\nabla \cdot \bar{E} = \frac{\rho_v}{\epsilon}$$

$$\nabla \cdot (-\nabla V) = \frac{\rho_v}{\epsilon} \qquad \left[\because \bar{E} = -\nabla V\right]$$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}} \longrightarrow 2.$$

Equation 2 is known as Poisson's equation.

If in a certain region, volume charge density is zero (i.e. $\rho_v = 0$), which is true for dielectric medium. Now, the Poisson's equation can be written as.

$$\boxed{\nabla^2 V = 0} \longrightarrow 3.$$

Equation 3 is known as Laplace equation.

This equation holds good for charge free region.

The ∇^2 (delsquare) operator is Laplace operator.

The expression for Laplace operator are as follows.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \text{ ---- Rectangular Co-ordinate system.}$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \text{ ... cylendrical " "}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\left(\frac{\partial^2 V}{\partial \phi^2}\right) \text{ .. Spherical " "}$$

Uniqueness theorem: Uniqueness theorem states that, once a method is adopted to solve Laplace or Poisson's equation ~~equation~~ subjected to given boundary conditions, the solutions obtained are unique even if we solve it by any other method.

Proof: Consider Laplace equation, $\nabla^2 V = 0 \longrightarrow 1.$

Let us assume that, we have two solutions of Laplace equation, V_1 & V_2 both general functions of co-ordinates used,

$\therefore \nabla^2 V_1 = 0$ & $\nabla^2 V_2 = 0.$

$\therefore \nabla^2 (V_1 - V_2) = 0. \longrightarrow 2.$

Each solution must also satisfy the boundary conditions & if we represent the given potential values on the boundary by V_b, then value of V_1 on the boundary V_{1b} & the value of V_2 on the boundary V_{2b} must both be identical to V_b

i.e. $V_{1b} = V_{2b} = V_b$ or $V_{1b} - V_{2b} = 0. \Rightarrow V_1 - V_2 = 0 \rightarrow 3.$

We know that, for any scalar 'V' & vector $\bar{D}$, there is a vector identity & is.

$\nabla \cdot V\bar{D} = V(\nabla \cdot \bar{D}) + \bar{D} \cdot \nabla V$

Let us take $(V_1 - V_2)$ as the scalar, & $\nabla(V_1 - V_2)$ as the vector,

$\nabla \cdot ((V_1 - V_2)\nabla(V_1 - V_2)) = (V_1 - V_2)(\nabla \cdot \nabla(V_1 - V_2)) + \nabla(V_1 - V_2) \cdot \nabla(V_1 - V_2)$

Integrating above equation throught the volume enclosed by the boundary surfaces,

$$\int_v \nabla \cdot ((V_1 - V_2)\nabla(V_1 - V_2))\, dv = \int_v (V_1 - V_2)(\nabla \cdot \nabla(V_1 - V_2))\, dv + \int_v \nabla(V_1 - V_2) \cdot \nabla(V_1 - V_2)\, dv \quad 2$$

$$= \int_v (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)]\, dv + \int_v [\nabla(V_1 - V_2)]^2\, dv.$$

By divergence theorem $\int_v \nabla \cdot D\, dv = \oint_s \bar{D} \cdot \bar{ds}$ $\therefore$ above eqn. becomes

$$\oint_s (V_1 - V_2)\nabla(V_1 - V_2) \cdot \bar{ds} = \int_v (V_1 - V_2)\nabla^2(V_1 - V_2)\, dv + \int_v [\nabla(V_1 - V_2)]^2\, dv. \rightarrow (4)$$

putting equating 2 & 3 in (4)

$$0 = 0 + \int_v \nabla(V_1 - V_2)^2\, dv = 0. \Rightarrow (\nabla(V_1 - V_2))^2 = 0.$$

$$\Rightarrow \nabla(V_1 - V_2) = 0. \quad \Rightarrow \boxed{V_1 - V_2 = K = \text{constant}} \rightarrow (5)$$

Equation (5) must hold everywhere, including at the boundary. But by eqn (3) ~~up~~ at the boundary $V_1 - V_2 = 0$. Therefore, the constant in equation must be zero. or $\boxed{V_1 = V_2}$. Thus in a given region, Laplace equation has only one solution that satisfies the boundary condition in the region.

STEADY MAGNETIC FIELDS : UNIT-V

The charges in motion constitutes an electric current. The current carrying conductor produces magnetic field. The dc is a steady flow of current & hence the magnetic field produced by a conductor carrying a dc current is a steady magnetic field. The study of steady magnetic field existing in a given ~~conductor~~ space produced due to the flow of dc current through a conductor is called magnetostatics.

magnetic flux. The effect of magnetic force around the magnet is assumed to be consisting of imaginary magnetic lines of force. The total no. of magnetic ~~β~~ lines of force around the magnet is known as magnetic flux. It is denoted by ϕ & its unit is weber.(wb).

magnetic flux density : The total magnetic flux passing per unit area in a plane normal to the direction of flux is known as magnetic flux density. It is denoted by 'B' & its unit is weber/m².

If B is the normal flux passing through surface area of 'a' mt², then,

$$\boxed{B = \phi/a} \text{ wb/m}^2 \text{ or Tesla.}$$

It is a vector quantity.

magnetic field intensity : The magnetic field intensity at any point in a magnetic field is defined as the force experienced by a unit north pole of one ~~weber~~ weber placed at that point. It is denoted H & its unit is Newton/weber or Ampere Turns/meter.
It's a vector quantity.

Relation between B & H : For a given material the flux density is directly proportional to magnetic field intensity.

$$B \propto H$$

$$\boxed{B = \mu H}$$ where μ is a proportionality constant & is known as permeability of the magnetic medium.

$$\mu = \mu_0 \mu_r$$

where, $\mu_0 = 4\pi \times 10^{-7}$

μ_r = Relative permeability & its value depends upon the type of medium used.

$$\therefore \boxed{\bar{B} = \mu_0 \mu_r \bar{H}}$$

Biot-Savart law : This law states that, the magnetic field intensity at a point 'p' produced by a differential current element is

a) proportional to the product of current I & differential length dl

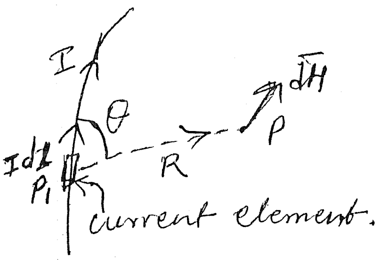
b) the sine of the angle between the element & the line joining point p to the element.

c) & inversely proportional to the square of the distance (R) between the point p & the current element.

i.e. $$d\bar{H} \propto \frac{I\,dl\,\sin\theta}{R^2}$$

$$d\bar{H} = \frac{K\,I\,dl\,\sin\theta}{R^2}$$

where K is proportionality constant. & its value is $K = \frac{1}{4\pi}$.

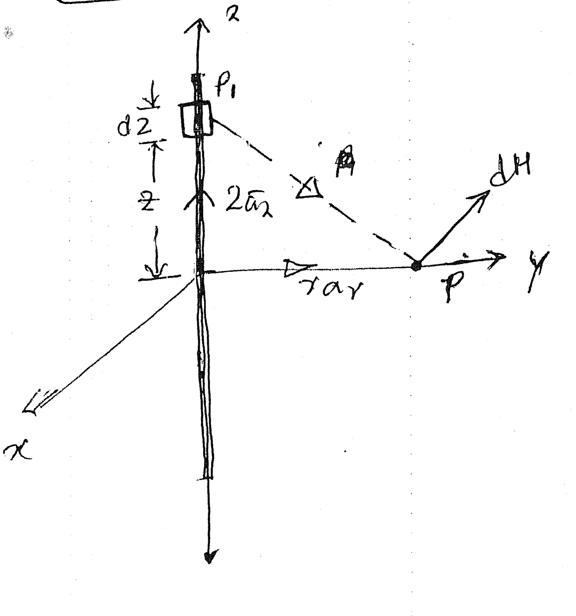


The direction of the magnetic field is normal to the plane containing the differential element & the line joining the element & the point 'p' where H is required.

$$\therefore \boxed{d\bar{H} = \frac{I\,d\bar{l} \times \bar{a}_R}{4\pi R^2}} \quad A/mt$$

where $\bar{a}_R = \frac{\bar{R}}{|R|}$.

Magnetic field due to an infinite length of conductor carrying current :



Consider a conductor of infinite length carrying the current I along z-axis. We wish to find the magnetic field intensity at a point at a distance 'r' from the conductor. Let us obtain the magnetic field intensity $\bar{dH}$ due to a current element

$$I\,d\bar{l} = I\,dz\,\bar{a}_z.$$

$$d\bar{H} = \frac{I\,d\bar{l} \times a_r}{4\pi r^2}$$

$$= \frac{I\,d\bar{l} \times \overline{P_1P}}{4\pi |P_1P|^2} \quad \overline{a}_{P_1P} \rightarrow (1)$$

(2)

From the above fig, $z\bar{a}_z + \overline{P_1P} = r\bar{a}_r \Rightarrow \overline{P_1P} = r\bar{a}_r - z\bar{a}_z$

$\therefore |\overline{P_1P}| = \sqrt{r^2+z^2}$.

$$\bar{a}_{P_1P} = \frac{r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}} \quad \& \quad I\,d\bar{l} = I\,dz\,\bar{a}_z.$$

$$\overline{dH} = \frac{I\,dz\,\bar{a}_z \times (r\bar{a}_r - z\bar{a}_z)}{4\pi(r^2+z^2)^{3/2}}$$

$$= \frac{I\,dz\,(r\bar{a}_\phi - 0)}{4\pi(r^2+z^2)^{3/2}} \qquad \because \bar{a}_z\times\bar{a}_r = \bar{a}_\phi \ \& \ \bar{a}_z\times\bar{a}_z = 0.$$

$$\overline{dH} = \frac{I\,r\,dz}{4\pi(r^2+z^2)^{3/2}}\,\bar{a}_\phi$$

$\therefore$ The magnetic field due to the entire infinite conductor is,

$$\bar{H} = \int_{-\infty}^{+\infty} \frac{I\,r\,dz}{4\pi(r^2+z^2)^{3/2}}\,\bar{a}_\phi = \frac{I\,\bar{a}_\phi}{4\pi}\int_{-\infty}^{\infty}\frac{r\,dz}{(r^2+z^2)^{3/2}}.$$

put, $z = r\tan\theta$.
$dz = r\sec^2\theta\,d\theta$.

when, $z = -\infty$, $\theta_1 = \tan^{-1}(-\infty/r) = -\pi/2$
$z = \infty$, $\theta_2 = \tan^{-1}(\infty/r) = \pi/2$.

$$\bar{H} = \frac{I\,\bar{a}_\phi}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{r\cdot r\sec^2\theta\,d\theta}{(r^2 + r^2\tan^2\theta)^{3/2}}$$

$$= \frac{I\,\bar{a}_\phi}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{r^2\sec^2\theta\,d\theta}{r^3\sec^3\theta} \qquad \because (r^2+r^2\tan^2\theta)^{3/2} = (r^2(1+\tan^2\theta))^{3/2} = (r^2\sec^2\theta)^{3/2} = r^3\sec^3\theta.$$

$$= \frac{I\,\bar{a}_\phi}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{\cos\theta\,d\theta}{r}.$$

$$= \frac{I}{4\pi r}\bar{a}_\phi\,\Big[\sin\theta\Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{I}{4\pi r}\bar{a}_\phi\,(\sin\pi/2 - \sin(-\pi/2))$$

$$= \frac{I}{4\pi r}\bar{a}_\phi\,(1-(-1)).$$

$$= \frac{I}{4\pi r}\bar{a}_\phi \times 2$$

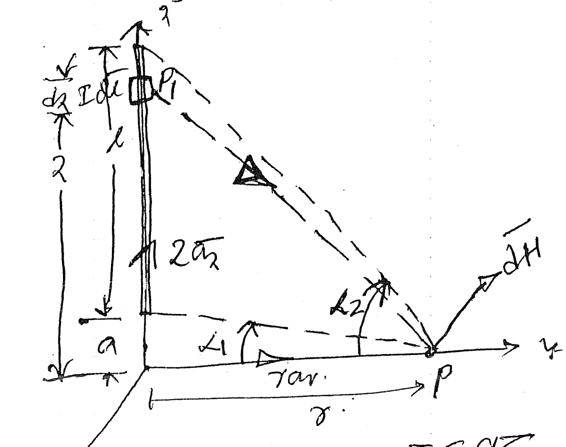
$$\boxed{\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi}$$

where $\bar{a}_\phi = \bar{a}_{Idl} \times \bar{a}_{\perp r}$ from conductor

Also $\bar{B} = \mu\bar{H}$

$$\boxed{\bar{B} = \frac{\mu I}{2\pi r}\bar{a}_\phi} \quad \text{wb/m}^2.$$

Magnetic field due to finite length of conductor carrying current: Consider a conductor of length 'l' carrying current I along the z-axis. We wish to find the magnetic field intensity at a point at a distance 'r' mt from the conductor. We shall first find the magnetic field intensity at P due to a current element $I\,\bar{dl}$ at a distance of 'z' mt.



$$d\bar{H} = \frac{I\,\bar{dl} \times \bar{a}_r}{4\pi r^2} = \frac{I\,\bar{dl} \times \bar{a}_{P_1P}}{4\pi |P_1P|^2}$$

$\therefore$ ~~$\bar{dH} = \frac{I\,\bar{dl} \times \bar{a}_{P_1P}}{4\pi\, P_1P}$~~

$$\bar{dH} = \frac{I\,dz\,\bar{a}_z \times \bar{P_1P}}{4\pi |P_1P|^2\, |P_1P|} = \frac{I\,dz\,\bar{a}_z \times \bar{P_1P}}{4\pi (|P_1P|)^{3/2}}$$

from the above fig, $z\bar{a}_z + \bar{P_1P} = r\bar{a}_r$

$\therefore$ $\bar{P_1P} = r\bar{a}_r - z\bar{a}_z$

$|P_1P| = \sqrt{r^2 + z^2}$

$$\therefore\ \bar{dH} = \frac{I\,dz\,\bar{a}_z \times (r\bar{a}_r - z\bar{a}_z)}{4\pi\,(r^2+z^2)^{3/2}}$$

$$\bar{dH} = \frac{I\,dz\,(r\bar{a}_\phi - 0)}{4\pi\,(r^2+z^2)^{3/2}}$$

$$\bar{dH} = \frac{I\,r\,dz}{4\pi(r^2+z^2)^{3/2}}\,\bar{a}_\phi$$

$$\therefore\ \bar{H} = \int_a^{a+l} \frac{I\,r\,dz}{4\pi\,(r^2+z^2)^{3/2}} = \frac{I r}{4\pi}\,\bar{a}_\phi \int_a^{a+l} \frac{dz}{(r^2+z^2)^{3/2}}$$

put $z = r\tan\theta$

$dz = r\sec^2\theta\, d\theta$

when $z = a$, $\theta = \tan^{-1}(a/r) = \alpha_1$

$z = a+l$, $\theta = \tan^{-1}\left(\frac{a+l}{r}\right) = \alpha_2$

$$\therefore\ \bar{H} = \frac{I r\,\bar{a}_\phi}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{r\sec^2\theta\, d\theta}{(r^2 + r^2\sec^2\theta)^{3/2}} = \frac{I r\,\bar{a}_\phi}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\cancel{r}\sec^2\theta\, d\theta}{r^3 \sec^3\theta}$$

$$= \frac{I\,\cancel{r}}{4\pi r^3} \int_{\alpha_1}^{\alpha_2} \cos\theta\, d\theta = \frac{I}{4\pi r}\,(\sin\theta)_{\alpha_1}^{\alpha_2}$$

$$\bar{H} = \frac{I}{4\pi r}\,\bar{a}_\phi\,(\sin\alpha_2 - \sin\alpha_1)$$

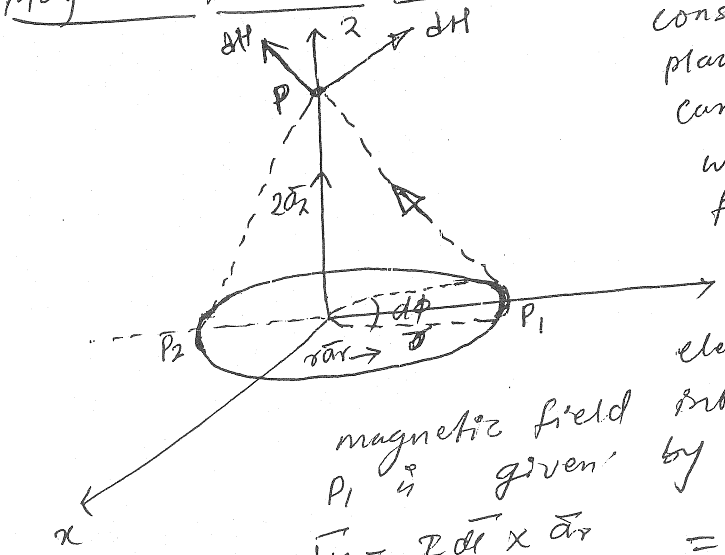
$$\boxed{\bar{H} = \frac{I}{4\pi r}(\sin\alpha_2 - \sin\alpha_1)\,\bar{a}_\phi} \quad A/mt.$$

where α_1 & α_2 are are the angles made by line joi-ning the bottom point & top point of the conductor to the required point where 'H' is asked. with the perpendicular length.

Note : 1. α_1 & α_2 are always measured from $\perp^r$ length & ~~either~~ end of the conductor.

2. α_1 & α_2 are taken to be positive if their direction is same as that of current & negative if their direction is opposit to the current.

<u>magnetic field due to a circular loop current</u> :



consider a current loop in xy plane of radius r meter & carrying a current of I Amps. we wish to find the magnetic field 'H' at a point at a distance z meter on its axis. consider a small elemental length dl at P_1. The magnetic field intensity at P due to $Id\bar{l}$ at P_1 is given by

$$d\bar{H} = \frac{I\,d\bar{l} \times \bar{a}_r}{4\pi r^2} = \frac{I\,d\bar{l} \times \bar{a}_{P_1P}}{4\pi |P_1P|^2}$$

$$I\,d\bar{l} = I\,r\,d\phi\,\bar{a}_\phi ,$$

$$r\bar{a}_r + \overline{P_1P} = z\bar{a}_z$$

$$\overline{P_1P} = z\bar{a}_z - r\bar{a}_r$$

$$|P_1P| = \sqrt{z^2+r^2}$$

$$\bar{a}_{P_1P} = \frac{z\bar{a}_z - r\bar{a}_r}{\sqrt{z^2+r^2}}$$

$$\therefore\ d\bar{H} = \frac{I\,r\,d\phi\,\bar{a}_\phi \times \frac{(z\bar{a}_z - r\bar{a}_r)}{\sqrt{z^2+r^2}}}{4\pi\left(\sqrt{r^2+z^2}\right)^2}$$

$$d\bar{H} = \frac{I\,r\,d\phi\,(z\bar{a}_r + r\bar{a}_z)}{4\pi(r^2+z^2)^{3/2}} \longrightarrow (1).$$

From the above fig. we observe that r-component of the magnetic field due to the two radially opposite elements cancels each other & their z-components add up.

$\therefore$ The magnetic field intensity due to the two current elements at P_1 & P_2 is give by

$$d\bar{H} = \frac{2\, I r\, d\phi\; r\, \bar{a}_z}{4\pi (r^2+z^2)^{3/2}}$$

$$\therefore \bar{H} = \int_{\phi=0}^{2\pi} \frac{I r\, d\phi\; r\, \bar{a}_z}{4\pi (r^2+z^2)^{3/2}}$$

$$\bar{H} = \frac{I r^2}{4\pi (r^2+z^2)^{3/2}} \bar{a}_z \int_0^{2\pi} d\phi.$$

$$= \frac{I r^2}{4\pi (r^2+z^2)^{3/2}} \bar{a}_z \times 2\pi.$$

$$\boxed{\bar{H} = \frac{I r^2}{2 (r^2+z^2)^{3/2}} \bar{a}_z} \quad A/m.$$

Also the flux density at 'p' is given by,

$$\bar{B} = \mu H$$

$$\boxed{\bar{B} = \frac{\mu I r^2}{2 (r^2+z^2)^{3/2}} \bar{a}_z} \quad wb/m^2$$

magnetic field intensity at the centre of the circular loop can be obtained by, putting $z=0$

$$\therefore \bar{H} = \frac{I r^2}{2 (r^2+0)^{3/2}} \bar{a}_z = \frac{I r^2}{2 r^3} \bar{a}_z = \frac{I}{2r} \bar{a}_z \;//$$

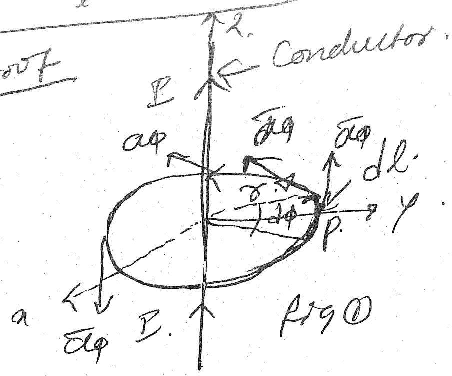
Also $\boxed{\bar{B} = \frac{\mu I}{2r} \bar{a}_z}$ wb/m^2 //

~~From the above fig, we observe that~~

<u>Amper's circuital law</u>: It states that, The line integral of magnetic field intensity H around a closed path is equal to the current enclosed by that path.

$$\text{i.e. } \boxed{\oint_l \bar{H}\cdot d\bar{l} = I}$$

<u>Proof</u>



fig①

Consider a long straight conductor carrying a direct current I Amps placed along z-axis. as shown in fig①. Consider a closed circular path of radius 'r' which encloses the straight conductor carrying current I. The point P is at ⊥er distance r from the conductor. Consider an elemental length $d\bar{l}$ at 'p' which is in $\bar{a}_\phi$ direction & tangential to the circular path at P. $\therefore d\bar{l} = r\,d\phi\,\bar{a}_\phi$.

From Biot-savart law, H at 'p' due to infinite long conductor is, $\bar{H} = \dfrac{I}{2\pi r}\bar{a}_\phi$

$$\therefore \bar{H}\cdot d\bar{l} = \frac{I}{2\pi \cancel{r}}\bar{a}_\phi \cdot \cancel{r}\,d\phi\,\bar{a}_\phi$$

$$= \frac{I}{2\pi}\,d\phi.$$

$$\therefore \oint_l \bar{H}\cdot d\bar{l} = \frac{I}{2\pi}\int_0^{2\pi} d\phi.$$

$$= \frac{I}{2\pi}(\phi)_0^{2\pi}$$

$$= \frac{I}{\cancel{2\pi}} \times \cancel{2\pi}$$

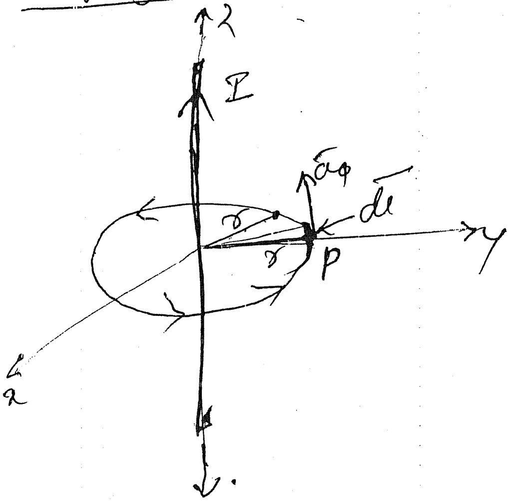
$$\boxed{\oint_l \bar{H}\cdot d\bar{l} = I}$$ Hence the proof.

The closed path choosen around a conductor is known as <u>Amperian path</u>.

<u>Applications of Ampere's law</u>:

1. It is used to find the current enclosed by a path with the knowledge of magnetic field around that closed path.
2. It can be used to find the magnetic field intensity due to symmetric current distributions.

Magnetic field intensity due to an infinite conductor carrying current:



consider an infinite conductor carrying current of I Amps. we wish to find the magnetic field intensity at a point at a distance r from the conductor. The magnitude of H depends upon 'r' & its direction is always tangential to the closed path. so $\bar{H}$ has only $\bar{a}_\phi$ component.

i.e. $\bar{H} = H_\phi \bar{a}_\phi$

$\bar{dl} = r\,d\phi\,\bar{a}_\phi$

From Ampere's circuital law

$$\oint_l \bar{H} \cdot \bar{dl} = I$$

$$\oint_l H_\phi \bar{a}_\phi \cdot r\,d\phi\,\bar{a}_\phi = I \qquad (\because \bar{a}_\phi \cdot \bar{a}_\phi = 1)$$

$$\oint_l H_\phi\, r\,d\phi = I$$

$$H_\phi\, r \int_{\phi=0}^{2\pi} d\phi = I$$

$$H_\phi\, r \cdot (2\pi) = I$$

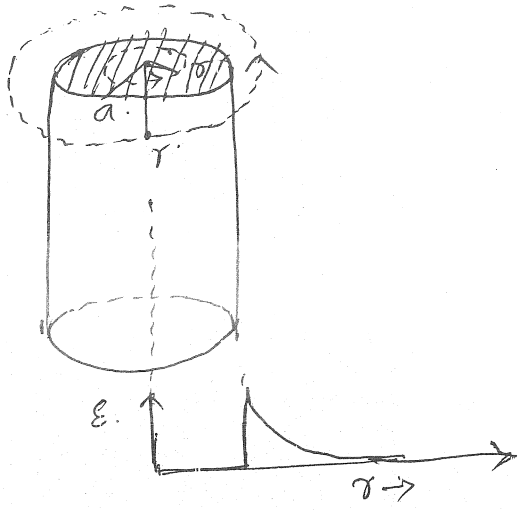
$$H_\phi = \frac{I}{2\pi r}$$

$$\boxed{\bar{H} = \frac{I}{2\pi r}\,\bar{a}_\phi} \quad \text{Ampere/meter.}$$

Also $\bar{B} = \mu H = \frac{\mu I}{2\pi r}\bar{a}_\phi$ wb/m²

Magnetic field intensity due to a thin hollow cylendrical shell carrying current:

Consider a hollow thin cylendrical conductor of radius 'a' carrying current 'I'. we wish to find the magnetic field 'H' at any point outside & inside the conductor.



Case (a): magnetic field H inside the conductor ($r<a$)

Consider a closed path of radius r mt ($r<a$). (5)

From Ampere's Circuital law,

$\bar{H}\cdot\bar{dl} = I \rightarrow (1)$

From the above fig it is clear that the current enclosed by inner closed path of radius r ($r<a$) is zero.

$\therefore$ Equation (1) becomes

$\oint_l \bar{H}\cdot\bar{dl} = 0.$ Hence the field H inside the conductor is 0.

$\Rightarrow \boxed{H = 0}$

Case (b): magnetic field 'H' outside the conductor, ($r>a$).

$\oint_l \bar{H}\cdot\bar{dl} = I$

$\oint_l H_\phi \bar{a}_\phi \cdot r\,d\phi\, a_\phi = I$

$\oint_l H_\phi\, r\,d\phi = I$

$H_\phi\, r \cdot \int_{\phi=0}^{2\pi} d\phi = I$

$H_\phi = \frac{I}{2\pi r}$

$\boxed{\bar{H} = \frac{I}{2\pi r}\,\bar{a}_\phi}$

Also $\bar{B} = \mu\bar{H} = \frac{\mu I}{2\pi r}\,\bar{a}_\phi \ Wb/m^2$

The distribution of electric field is shown above.

Magnetic field intensity due to a solid cylendrical conductor:

Consider a solid cylendrical conductor of radius 'a' meter & carrying a current of I Amps as shown in the fig. we wish to find the magnetic field intensity at any point inside & outside the conductor.

Case (a): magnetic field $\bar{H}$ with in the conductor. ($r<a$).

consider a closed path of radius 'r' meter ($r<a$).

From Ampere's Circuital law,

$$\oint_l \bar{H}\cdot \bar{dl} = I_{enclosed}.$$

$$\oint_l H_\phi \bar{a}_\phi \cdot r d\phi \bar{a}_\phi = J\cdot \pi r^2$$

$$\oint_l H_\phi r d\phi = J\pi r^2$$

$$H_\phi r\int_{\phi=0}^{2\pi} d\phi = J\pi r^2$$

$$H_\phi \cdot r 2\cancel{\pi} = J\cancel{\pi} r^2$$

$$\bar{H} = \frac{J r^2}{2 r} \bar{a}_\phi = \frac{J r}{2} \bar{a}_\phi$$

or $\boxed{\bar{H} = \frac{I r}{2 \cdot \pi a^2} a_\phi}$ A/m.

$J = \frac{\text{current}}{\text{area of conductor}}$

$J = \frac{I}{\pi a^2}$ A/m²

∴ Current in the inner loop = $J \times$ area of loop
$= J \times \pi r^2$ //.

Case (b) : Magnetic field intensity at $(r>a)$ (outside the conductor)

Consider a small closed path of radius r mt $(r>a)$.

By Ampere's law,

$$\oint_l \bar{H}\cdot \bar{dl} = I$$

$$\oint_l H_\phi \bar{a}_\phi \cdot r d\phi \bar{a}_\phi = I$$

$$H_\phi r \cdot \int_0^{2\pi} d\phi = I$$

$$H_\phi r \cdot 2\pi = I$$

$$H_\phi = \frac{I}{2\pi r}$$

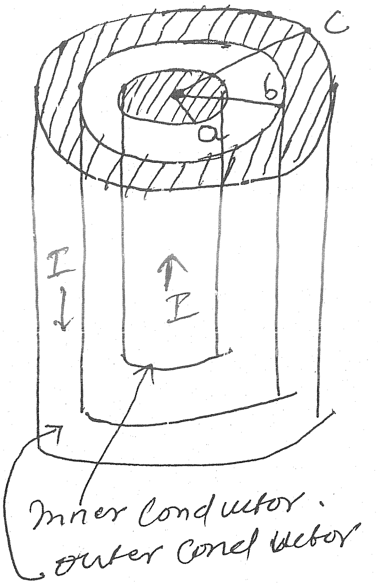
$$\boxed{\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi}$$

Also $\bar{B} = \frac{\mu I}{2\pi r}\bar{a}_\phi$ //.

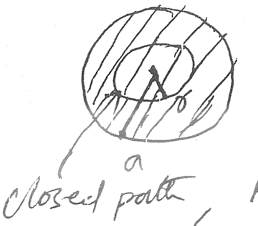
Magnetic field intensity 'H' due to co-axial cable :

Consider a co-axial cable of inner conductor of radius 'a' meter, & outer conductor of inner radius 'b' mt & outer radius 'c' meter.

Let the current I is uniformly distributed in the inner conductor & $-I$ in the outer conductor.



Case I : magnetic field intensity inside the inner conductor. ($r<a$)



Consider a closed Amperian path of radius "r" meter ($r<a$).

From Ampere's circuital law,

$$\oint_l \bar{H}\cdot\bar{dl} = I_{enclosed}$$

$$\oint_l H_\phi \bar{a}_\phi \cdot r\,d\phi\,\bar{a}_\phi = J\cdot \pi r^2$$

$$\oint_l H_\phi\, r\,d\phi = J\,\pi r^2$$

$$H_\phi \cdot r \int_0^{2\pi} d\phi = J\,\pi r^2$$

$$H_\phi\, r\cdot 2\pi = J\pi r^2$$

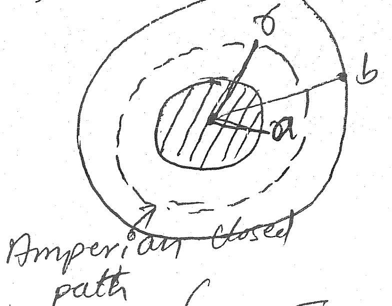
$$\bar{H} = \frac{J\,\pi r^{\cancel{2}}}{2\cancel{\pi} \cancel{r}}\,\bar{a}_\phi$$

$$\bar{H} = \frac{J\,r}{2}\,\bar{a}_\phi$$

$$\boxed{\bar{H} = \frac{I\cdot r}{2\pi a^2}\,\bar{a}_\phi} \text{ A/m} \qquad (\because J = I/\pi a^2)$$

Also $\bar{B} = \mu H = \dfrac{\mu I r}{2\pi a^2}\,\bar{a}_\phi$ wb/m²

Case-II : magnetic field intensity $a<r<b$. (outside the inner conductor)



Consider a Amperian closed path of radius r meter ($a<r<b$) which includes inner conductor.

From Ampere's circuital law,

$$\oint_l \bar{H}\cdot\bar{dl} = I_{enclosed}$$

$$\therefore \int_l H_\phi\,\bar{a}_\phi \cdot r\,d\phi\,\bar{a}_\phi = I$$

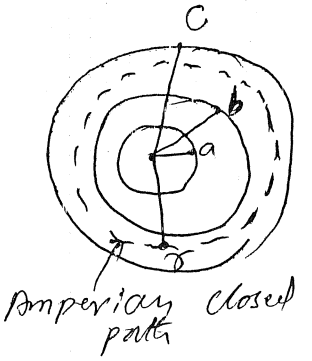
$$H_\phi\, r \int_0^{2\pi} d\phi = I$$

$$H_\phi\, r\,(2\pi) = I$$

$$H_\phi = I/2\pi r$$

$$\boxed{\bar{H} = I/2\pi r\;\bar{a}_\phi} \quad \text{or} \quad \bar{B} = \mu\bar{H} = \frac{\mu I}{2\pi r}\,\bar{a}_\phi \text{ wb/m}^2$$

Case III : magnetic field intensity at $(b < r < c)$
(H inside the outer conductor)



consider a closed path of radius r mt $(b < r < c)$, which encloses part of the current $(-I)$ in the outer conductor & the total current I in the outer conductor.

$$I_{enclosed} = I + J\pi(r^2 - b^2)$$

$$= I + \left(\frac{-I}{\cancel{\pi}(c^2-b^2)}\right) \times \cancel{\pi}(r^2 - b^2)$$

$$= I\left[\frac{c^2 - \cancel{b^2} - r^2 + \cancel{b^2}}{c^2 - b^2}\right]$$

$$= \frac{I}{(c^2-b^2)}(c^2 - r^2).$$

From Amperes circuital law,

$$\oint_l \bar{H} \cdot \bar{dl} = I_{enclosed}.$$

$$\oint_l H_\phi \bar{a}_\phi \cdot r d\phi \bar{a}_\phi = \frac{I(c^2 - r^2)}{(c^2-b^2)}$$

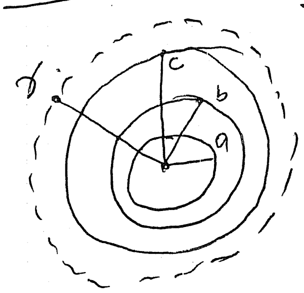
$$H_\phi r \int_0^{2\pi} d\phi = I\left(\frac{c^2-r^2}{c^2-b^2}\right)$$

$$H_\phi r \cdot (2\pi) = I\frac{(c^2-r^2)}{c^2-b^2}$$

$$\boxed{\bar{H} = \frac{I}{2\pi r} \cdot \left[\frac{c^2-r^2}{c^2-b^2}\right] \bar{a}_\phi \text{ A/mt}}$$

$$\bar{B} = \mu \bar{H} = \frac{\mu I}{2\pi r}\left[\frac{c^2-r^2}{c^2-b^2}\right]\bar{a}_\phi \text{ Tesla.}$$

Case IV : H outside the cable $(r > c)$.



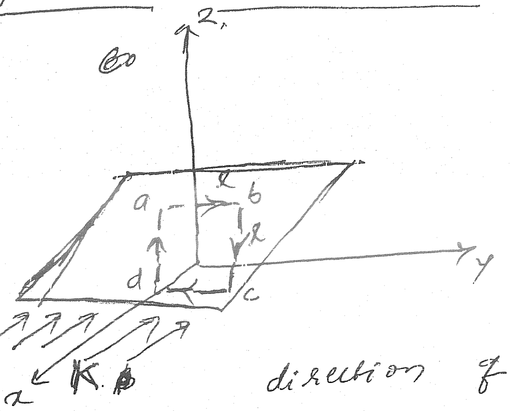
consider an Amperian closed path of radius 'r' meter $(r > c)$, which encloses both the currents $+I$ & $-I$ flowing in the inner & outer conductors respectively.

$$I_{enclosed} = I - I = 0.$$

$$\therefore \oint \bar{H} \cdot \bar{dl} = I = 0$$

$$\Rightarrow \boxed{H = 0} \; //$$

Magnetic field due to infinite sheet of current.



Consider an infinite sheet carrying a current of K Amp/mt length along $(-\bar{a}_x)$ direction. By symmetry & using right hand screw rule, we see that, Magnetic field is uniform & is in the direction of $+\bar{a}_y$ & $-\bar{a}_y$ above & below the sheet respectively. Consider a closed loop abcd extending on both sides of the sheet as shown above.

By Ampere's Circuital law,

$$\oint_{abcd} \bar{H}\cdot\bar{dl} = I_{enclosed}$$

$$\oint_{ab} \bar{H}\cdot\bar{dl} + \oint_{bc} \bar{H}\cdot\bar{dl} + \oint_{cd} \bar{H}\cdot\bar{dl} + \oint_{da} \bar{H}\cdot\bar{dl} = K\cdot l. \rightarrow 1$$

As the thickness of the sheet is negligible sides $bc \rightarrow ad \rightarrow 0$.

$$\therefore \oint_{bc} \bar{H}\cdot\bar{dl} = \oint_{da} \bar{H}\cdot\bar{dl} = 0. \quad \therefore \text{ equation (1) becomes}$$

$$\therefore |H|l + |H|l = Kl.$$

$$2l\cdot H = Kl$$

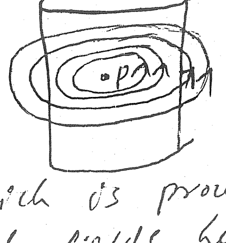
$$\boxed{\bar{H} = \frac{\bar{K}}{2}\times\bar{a}_n}$$

where $\bar{a}_n$ = unit vector perpendicular to the sheet current.

In the above case $\bar{K} = K(-\bar{a}_x)$ & $\bar{a}_n = \bar{a}_z$

$$\therefore \bar{H} = \frac{K}{2}(-\bar{a}_x)\times\bar{a}_z = \frac{K}{2}\bar{a}_y \ A/m.$$

CURL:



Consider the vector field F as shown in the fig. The net outward flux comming out normally through the cylendrical closed surface is zero. Hence the divergence of such a vector field is zero. However there is a source at 'p' which is prouducing this field having zero divergence. Such fields have rotation. So the amount of rotation of the field type should convey the information regarding the source prouducing such fields. Thus the rotation (curl) of a vector field is defined as follows.

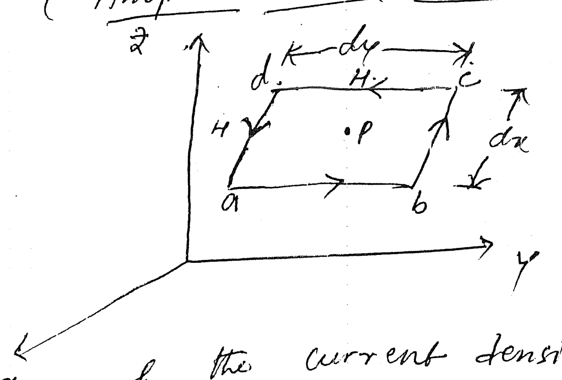
Defn. The curl of a vector field is defined as its net rotation (closed line integral) about a small closed path per unit surface area of the closed path as the path shrinks to zero.

The direction of the curl is normal to the plane containing the path & is given by right hand screw rule.

$$\therefore \boxed{\text{curl } F = \nabla \times \bar{F} = \lim_{ds \to 0} \frac{\oint_s \bar{H} \cdot \bar{dl}}{ds}}$$

where $ds \rightarrow$ Surface area of the enclosed path.

Expression for curl in rectangular co-ordinate system. (Ampere's circuital law in point form)



Consider a differential surface element having the sides dx & dy as shown in the fig. Let the ~~flux~~ magnetic field intensity at 'p' is, $\bar{H} = H_{x0}\bar{a_x} + H_{y0}\bar{a_y} + H_{z0}\bar{a_z} \rightarrow ①$

& the current density be $\bar{J} = J_x\bar{a_x} + J_y\bar{a_y} + J_z\bar{a_z} \rightarrow ②$.

From Amperes circuital law,

$\bar{H} \cdot \bar{dl} = I_{enclosed}$.

Now, $\oint_{ab} \bar{H} \cdot \bar{dl} = H_y\bar{a_y} \cdot dy\bar{a_y} = H_y\, dy$

The ~~int~~ field H_y along ab can be expressed in terms of H_{y0} existing at 'p' & the rate of change of H_y along x-axis with x.

$$\therefore \oint_{ab} \bar{H} \cdot \bar{dl} = \left(H_{y0} + \frac{dx}{2}\frac{\partial H_y}{\partial x}\right) dy \longrightarrow ③$$

$$\text{lly} \quad \oint_{bc} \bar{H} \cdot \bar{dl} = -\left(H_{x0} + \frac{dy}{2} \cdot \frac{\partial H_x}{\partial y}\right) dx \rightarrow ④$$

$$\int_{cd} \bar{H} \cdot \bar{dl} = -[H_y\, dy]$$
$$= -\left(H_{y0} - \frac{dx}{2}\frac{\partial H_y}{\partial x}\right) dy \rightarrow 5$$

$$\oint_{da} \bar{H} \cdot \bar{dl} = +H_x\, dx = \left[H_{x0} - \frac{dy}{2}\frac{\partial H_x}{\partial x}\right] dx \rightarrow ⑥$$

$$\therefore \oint_{abcd} \bar{H} \cdot \bar{dl} = \oint_{ab} \bar{H} \cdot \bar{dl} + \oint_{bc} \bar{H} \cdot \bar{dl} + \oint_{cd} \bar{H} \cdot \bar{dl} + \oint_{da} \bar{H} \cdot \bar{dl}$$

$$= \cancel{H_{y0}}\, dy + \frac{dx\,dy}{2}\frac{\partial H_y}{\partial x} - H_{x0}\cancel{dx} - \frac{dx\,dy}{2}\frac{\partial H_x}{\partial y} - \cancel{H_{y0}\,dy}$$
$$+ \frac{dx\,dy}{2}\frac{\partial H_y}{\partial x} + \cancel{H_{x0}}\,dx - \frac{dx\,dy}{2}\frac{\partial H_x}{\partial y}$$

$$= dx\,dy\frac{\partial H_y}{\partial x} - dx\,dy\frac{\partial H_x}{\partial y}$$

$$= \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) dx\,dy.$$

divide $dx\,dy$ on both sides

& Taking $\lim_{dx\,dy \to 0}$ on both sides

$$\lim_{dx\,dy\to 0} \frac{\oint_{abcd} \bar{H}\cdot \bar{dl}}{dx\,dy} = \lim_{dx\,dy\to 0} \frac{\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \cancel{dx\,dy}}{\cancel{dx\,dy}}$$

$$= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \rightarrow 8 \qquad \left(\because \frac{\oint_{abcd} \bar{H}\cdot\bar{dl}}{dx\,dy} = \frac{I}{dx\,dy}\right.$$

$\left.\because \text{current density normal to the surface } dx\,dy = J_z\right)$

Considering an elementary closed path in the YZ plane, we get ~~current~~ current density normal to this plane i.e. (J_x)

$$\therefore \lim_{dy\,dz\to 0} \frac{\oint_{abcd} \bar{H}\cdot\bar{dl}}{dy\,dz} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \rightarrow 9.$$

Illy Considering the closed path in the zx plane.

$$\lim_{dx\,dz\to 0} \frac{\oint_{abcd} \bar{H}\cdot\bar{dl}}{dx\,dz} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \longrightarrow 10$$

putting 8, 9, 10 in (8)

$$J = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\bar{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\bar{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\bar{a}_z$$

$\boxed{\bar{J} = \nabla \times \bar{H}}$ → Ampere's law in pointform.

where,
$$\nabla\times\bar{H} = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{bmatrix}$$ → Rectangular Co-ordinate system.

$$\nabla\times\bar{H} = \frac{1}{r}\begin{bmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ H_r & rH_\phi & H_z \end{bmatrix}$$ → cylendrical co-ordinate system.

$$= \frac{1}{r^2\sin\theta}\begin{bmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ H_r & rH_\theta & r\sin\theta\, H_\phi \end{bmatrix}$$ → spherical co-ordinate system

physical significance of curl.

1. Curl of a vector field results in to a vector field.
2. The curl of a vector field at a point gives the amount of rotation of the field at that point.
3. The magnitude of the curl gives magnitude of the source producing this field.

5. If the curl of a vector field exists, then the field is called rotational.

6. For irrotational vector fields (fields which donot form closed path), the curl is zero.

<u>Stokes theorem</u>: It states that the line integral of a magnetic field intensity over the closed path is equal to the integral of curl of magnetic field intensity over the surface enclosed by the closed path.

$$\oint_l \bar{H}\cdot d\bar{l} = \oint_s (\nabla\times\bar{H})\cdot d\bar{s}$$

<u>Proof</u>. From Ampere's circuital law in integral form

$$\oint_l \bar{H}\cdot d\bar{l} = I$$

Also $\cancel{I} = \oint_s \bar{J}\cdot d\bar{s}$

$$\therefore \oint_l \bar{H}\cdot d\bar{l} = \oint_s \bar{J}\cdot d\bar{s}$$

From Amperes circuital law in point form

$$\nabla\times\bar{H} = \bar{J}$$

$$\boxed{\therefore \oint_l \bar{H}\cdot d\bar{l} = \int_s (\nabla\times\bar{H})\cdot d\bar{s}}$$ Hence the proof.

properties of curl:

1. $\nabla \times \nabla V = 0$
2. $\nabla \cdot (\nabla \times \bar{A}) = 0$
3. $\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 A$.
4. $\nabla \times (\bar{A} + \bar{B}) = \nabla \times \bar{A} + \nabla \times \bar{B}$.
5. $A \times \nabla(\phi) = -\nabla \times (\phi(A)) + \phi(\nabla \times \bar{A})$.

(1) "

flux density:

$\bar{B} = \phi/s$ wb/m2

$\bar{B} = \frac{d\phi}{ds}$

$d\phi = \bar{B} \cdot \bar{ds}$

$$\boxed{\phi = \int_s \bar{B} \cdot \bar{ds}}$$

$\bar{B} = \mu \bar{H}$

$\phi = \int_s \mu \cdot \bar{H} \cdot \bar{ds}$

$$\therefore \boxed{\phi = \mu \int_s \bar{H} \cdot \bar{ds}}$$

Scalar magnetic potential:

WKT in electrostatics,

$\bar{E} = -\nabla V_m$.

In magnetostatics, magnetic field intensity is related with magnetic potential as

$\bar{H} = -\nabla V_m$.

Taking curl on both sides

$\nabla \times \bar{H} = -\nabla \times \nabla V_m$

$\nabla \times \bar{H} = 0$ $\quad (\because \nabla \times \nabla V = 0)$.

Thus scalar magnetic potential V_m can be defined for source free region. This is applicable to permanent magnets, where $J=0$.

$\bar{H} = -\nabla V_m$ when $J=0$.

$\therefore$ Ily $V_{m_{ab}} = -\int_b^a \bar{H} \cdot \bar{dl}$ for specified path.

Laplace equation for scalar magnetic potential:

W.KT. $\phi = \oint_s \bar{B} \cdot \bar{ds}$

where $ds \rightarrow$ open surface through which the magnetic flux is passing.

Consider a closed surface which is defining a certain volume. The magnetic flux lines always exists in the form of closed loop. Thus for a closed surface the no. of magnet flux lines entering must be equal to the no of flux lines leaving. The single magnetic pole cannot exists like a single isolated electric charge. No magnetic flux can reside in a closed surface. Hence the integral $\bar{B}\cdot d\bar{s}$ evaluated over a closed surface is always zero.

$$\therefore \boxed{\int_s \bar{B}\cdot d\bar{s} = 0.}$$

This law is known as Gauss law in integral form.

Taking divergence on both sides

$$\oint_s \bar{B}\cdot d\bar{s} = \int_{vol} (\nabla\cdot\bar{B})\, dv = 0.$$

$dv \rightarrow$ volume enclosed by the closed surface.

$$\therefore \boxed{\nabla\cdot\bar{B} = 0.}$$

This equation is known as point form of Gauss law in magnetostatics.

$$\nabla\cdot \bar{B} = 0 \qquad (\bar{B} = \mu\bar{H})$$

$$\nabla\cdot \mu\bar{H} = 0$$

$$\mu\,\nabla\cdot\bar{H} = 0.$$

$$\nabla\cdot\bar{H} = 0 \qquad (\bar{H} = -\nabla V_m)$$

$$\nabla\cdot -\nabla V_m = 0$$

$$\boxed{\nabla^2 V_m = 0} \quad \text{for } J = 0.$$

This equation is known as Laplace equation of scalar magnetic potential.

<u>Vector magnetic potential</u>: Vector magnetic potential '$\bar{A}$' is a quantity which satisfies the following conditions.

1) $\nabla\times\bar{A} = \bar{B}$

2) $\nabla\cdot\bar{A} = 0$

where $\bar{B} = \mu\bar{H}$ is the magnetic flux density.

expression for ~~vector mo~~

Expression for vector magnetic potential:

W.K.T from Biot-Savart law,

$$d\bar{H} = \frac{I\,\bar{dl} \times \bar{a}_r}{4\pi r^2}$$

$$= \frac{\bar{J}\,ds\,\bar{dl} \times \bar{a}_r}{4\pi r^2} \qquad \because \left(J = \frac{I}{ds}\right)$$

$$= \frac{\bar{J}\,dv \times \bar{a}_r}{4\pi r^2} \qquad (\because dv = ds \cdot dl) \quad (\because \text{volume} = \text{area} \times \text{length})$$

$$= \frac{\bar{J} \times \bar{a}_r \cdot dv}{4\pi r^2}$$

$$\bar{H} = \oint_v \frac{\bar{J} \times \bar{a}_r}{4\pi r^2}\,dv$$

$$= \oint_v \frac{\bar{J} \times \frac{\bar{a}_r}{r^2}}{4\pi}\,dv$$

$$= \int_v \frac{\bar{J} \times (-\nabla(\frac{1}{r}))}{4\pi}\,dv \qquad \left[\because \frac{\bar{a}_r}{r^2} = -\nabla\left(\frac{1}{r}\right)\right.$$

$$\cancel{\nabla(\tfrac{1}{r})} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\right) a_r + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\right)\bar{a} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\right)\bar{a}$$

$$= \frac{1}{r^2} a_r + 0 + 0 = \frac{\bar{a}_r}{r^2}$$

$$= \oint_v \frac{-\bar{J} \times \nabla(\frac{1}{r})}{4\pi}\,dv \rightarrow ①$$

W.K.T from the vector identity,

$$\bar{A} \times \nabla\phi = -\nabla \times (\phi\bar{A}) + \phi(\nabla \times \bar{A})$$

$$\therefore \bar{J} \times \nabla\left(\frac{1}{r}\right) = -\nabla \times \left(\frac{\bar{J}}{r}\right) + \frac{1}{r}(\nabla \times \bar{J})$$

For constant current density. $\nabla \times J = 0$.

$$\therefore \bar{J} \times \nabla\left(\frac{1}{r}\right) = -\nabla \times \left(\frac{\bar{J}}{r}\right)$$

$\therefore$ eqn ① becomes

$$\bar{H} = -\int_v \frac{-\nabla \times (\bar{J}/r)}{4\pi}\,dv$$

$$= \nabla \times \oint_v \frac{\bar{J}/r}{4\pi}\,dv$$

Now, $\bar{B} = \mu\bar{H} = \nabla \times \frac{\mu}{4\pi} \oint_v \left(\frac{\bar{J}}{r}\right) dv \rightarrow ②$

$$\boxed{\bar{B} = \nabla \times \bar{A}}$$

where $\bar{A}$ is a vector magnetic potential, & is given by

$$\boxed{\bar{A} = \frac{\mu}{4\pi} \oint_v \frac{\bar{J}}{r}\,dv}$$

~~W.K.T~~ ~~B = μH~~

=

Poisson's equation for vector magnetic potential : ($\nabla^2 A = -\mu J$)

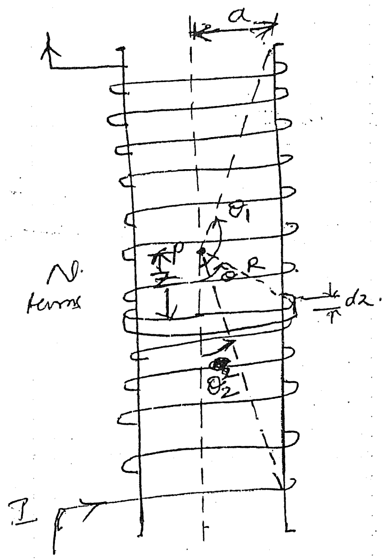
W.K.T. $\bar{B} = \nabla \times \bar{A}$

Taking curl on both sides

$$\nabla \times \bar{B} = \nabla \times \nabla \times \bar{A}$$
$$= \nabla(\nabla \cdot \bar{A}) - \nabla^2 A \qquad (\because \nabla \times \nabla \times \bar{H} = \nabla(\nabla\cdot\bar{A}) - \nabla^2 A)$$
$$= 0 - \nabla^2 A \qquad (\nabla \cdot \bar{A} = 0).$$
$$\nabla^2 A = -\nabla \times \bar{B}$$
$$= -\nabla \times \mu\bar{H}$$
$$= -\mu(\nabla \times \bar{H})$$
$$= -\mu \bar{J} \qquad (\nabla \times \bar{H} = J \text{ from Ampere's circuit law})$$

$\boxed{\nabla^2 \bar{A} = -\mu \bar{J}}$ is known as 'poisson's eqn. for 1. ~~vector~~ magnetic fields.

Expression for $\bar{H}$ & $\bar{B}$ at a point on the axis of solenoid of infinite length :



A solenoid is a coil wound uniformly around a non-magnetic material.

Let $a \to$ radius of coil
$N \to$ No of turns of the coil
$I \to$ current through the coil.

Let the axis of the coil is along z-axis.

Consider the differential thickness dz of the solenoid. Total no. of turns on this thickness are

$$n = \frac{N}{l} dz$$

Let P be a point at a distance of 'z' from dz.

θ_1 & θ_2 are the angles made by z-axis with top & bottom end of the coil respectively.

$\theta \to$ angle made by the axis of coil & the line joining the point 'p' & dz.

WKT dH due to a circular loop at 'p' at a distance 'z' on the loop axis is

$$d\bar{H} = \int_{\phi=0}^{2\pi} \frac{I \cdot a^2 \, d\phi}{2\pi (z^2+a^2)^{3/2}} \bar{a}_z$$

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$$|dH| = \frac{I a^2 (\frac{N}{l}) dz}{2(a^2+z^2)^{3/2}}$$ ($\because$ the coil contains $(\frac{N}{l} dz)$ turns).

$$|\bar{H}| = \int_{\theta_1}^{\theta_2} |d\bar{H}|$$

$$= \int_{\theta_1}^{\theta_2} \frac{I a^2 (\frac{N}{l}) dz}{2(z^2+a^2)^{3/2}}$$

from the fig $\tan\theta = a/z$, $\frac{z}{a} = \cot\theta \Rightarrow z = a\cot\theta$

$\therefore dz = -a\,\mathrm{cosec}^2\theta\, d\theta$

$$= \int_{\theta_1}^{\theta_2} \frac{I a^2 (\frac{N}{l})(-a\,\mathrm{cosec}^2\theta)\, d\theta}{2(a^2 + a^2\cot^2\theta)^{3/2}}$$

$$= \int_{\theta_1}^{\theta_2} \frac{-I a^3 (\frac{N}{l})\,\mathrm{cosec}^2\theta\, d\theta}{2\cdot a^3\,\mathrm{cosec}^3\theta}$$ ($(1+\cot^2\theta)^{3/2} = (\mathrm{cosec}^2\theta)^{3/2} = \mathrm{cosec}^3\theta$)

$$= -\frac{IN}{2l}\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta$$

$$= -\frac{IN}{2l}\left(-\cos\theta\right)_{\theta_1}^{\theta_2}$$

$$|\bar{H}| = \frac{IN}{2l}(\cos\theta_2 - \cos\theta_1) \quad \text{AT/mt.}$$

$$|\bar{B}| = \mu|\bar{H}| = \frac{\mu I N}{2l}(\cos\theta_2 - \cos\theta_1) \quad \text{wb/mt}^2$$

If the solenoid is having large length,

$\theta_1 = 180°$ & $\theta_2 = 0°$

$$\therefore |\bar{H}| = \frac{NI}{2l}(\cos 0 - \cos 180°)$$

Expression for H at the centre of long solenoid.

$$= \frac{NI}{2l}\cdot 2$$

$$\boxed{\bar{H} = \frac{NI}{l}} \quad \text{AT/mt}$$

Expression for H at the end of solenoid

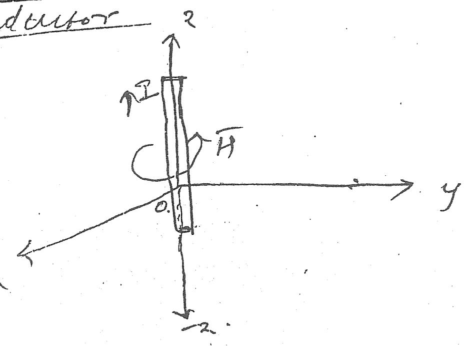
At the end of solenoid, $\theta_1 = 90°$ & $\theta_2 = 0$

$$\therefore |\bar{H}| = \frac{NI}{2l}(\cos\theta_2 - \cos\theta_1)$$

$$= \frac{NI}{2l}(\cos 0 - \cos 90)$$

$$\boxed{|\bar{H}| = \frac{NI}{2l}} = \frac{\bar{H}_{\text{at the centre}}}{2}$$

Expression for ~~scalar~~ vector magnetic potential in a region surrounding an infinitely long straight conductor



Consider an infinitely long conductor carrying direct current of I amps placed along z-axis. as shown in the fig.

WKT H, due to infinit conductor is given by.

$$\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi$$

$$\bar{B} = \mu_0 \bar{H} = \frac{\mu_0 I}{2\pi r}\,\bar{a}_\phi \longrightarrow ①$$

WKT, $\bar{B} = \nabla \times \bar{A} \longrightarrow ②$

Assuming cylindrical co-ordinates

$$\bar{B} = \nabla \times \bar{A} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ A_r & rA_\phi & A_z \end{vmatrix}$$

from ①

$$\frac{\mu_0 I}{2\pi r}\,\bar{a}_\phi = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$= \frac{1}{r}\left\{\left[\frac{\partial A_z}{\partial\phi} - \frac{\partial}{\partial z}(rA_\phi)\right]\bar{a}_r - r\bar{a}_\phi\left[\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right] + \bar{a}_z\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial}{\partial\phi}A_r\right)\right\}$$

Equating the coefficients of $\bar{a}_\phi$

$$\frac{\mu_0 I}{2\pi r} = -\frac{1}{r}r\left[\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right] = \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]$$

from ① B is a function of r, so, $\frac{\partial A_r}{\partial z} = 0$.

$$\therefore -\frac{\partial A_z}{\partial r} = +\frac{\mu_0 I}{2\pi r}$$

$$\partial A_z = -\frac{\mu_0 I}{2\pi r}\,\partial r.$$

Integrating on b.s.

$$\int \partial A_z = \int -\frac{\mu_0 I}{2\pi r}\,\partial r$$

$$A_z = -\frac{\mu_0 I}{2\pi}\ln r + C_1$$

Assume at $r = r_0$, $A_z = 0$

$$0 = -\frac{\mu_0 I}{2\pi}\ln r_0 + C_1 \Rightarrow C_1 = \frac{\mu_0 I}{2\pi}\ln(r_0)$$

$$\therefore A_z = -\frac{\mu_0 I}{2\pi}\ln r + \frac{\mu_0 I}{2\pi}\ln(r_0)$$

$$= \frac{\mu_0 I}{2\pi}\{\ln r_0 - \ln r\}$$

$$A_z = \frac{\mu_0 I}{2\pi}\ln\left(\frac{r_0}{r}\right)$$

$$\boxed{\bar{A} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{r_0}{r}\right)\bar{a}_z}$$

Wb/m.

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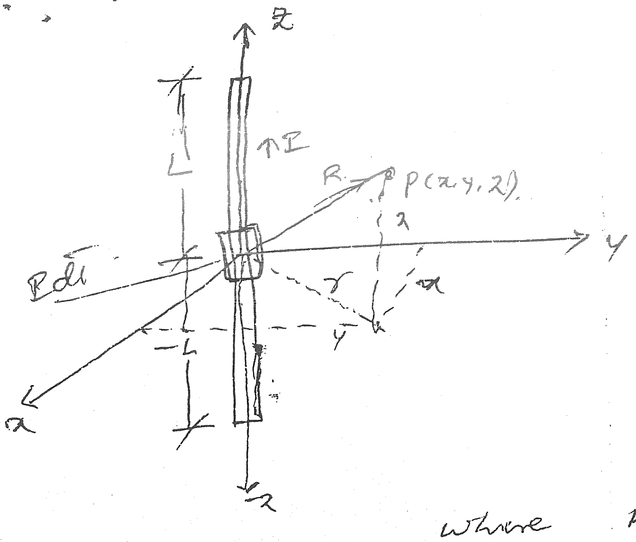
Expression for vector magnetic potential due to finite conductor :



Consider a straight conductor of length 2L carrying a current of I amps.

Consider a differential current element,

$I\bar{dl} = I\,dz\,\bar{a_z}$.

The vector magnetic potential $\bar{A}$ due to the differential current element is given by,

$$\bar{A} = \int \frac{\mu_0 I \bar{dl}}{4\pi R}$$

where $R = \sqrt{x^2+y^2+z^2}$
$= \sqrt{r^2+z^2}$ where $r^2 = x^2+y^2$.

$$\therefore \bar{A} = \int_{-L}^{L} \frac{\mu_0 I\, dz}{4\pi\sqrt{r^2+z^2}}\,\bar{a_z}$$

$$= 2\int_0^L \frac{\mu_0 I\, dz}{4\pi\sqrt{r^2+z^2}}\,\bar{a_z}$$

$$= \frac{\mu_0 I}{2\pi}\int_0^L \frac{dz}{\sqrt{r^2+z^2}}\,\bar{a_z}$$

Now $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)$

$$\therefore A = \frac{\mu_0 I}{2\pi}\left[\ln\left(z+\sqrt{r^2+z^2}\right)\right]_0^L \bar{a_z}$$

$$= \frac{\mu_0 I}{2\pi}\left[\ln\left(L+\sqrt{r^2+z^2}\right) - \ln\left(0+\sqrt{r^2+0}\right)\right]\bar{a_z}$$

$$= \frac{\mu_0 I}{2\pi}\left[\ln\left(L+\sqrt{r^2+z^2}\right) - \ln r\right]\bar{a_z}$$

$$\boxed{\bar{A} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{L+\sqrt{r^2+z^2}}{r}\right)\bar{a_z}} \quad wb/m.$$

For long length of the conductor $L \gg r$,

$L^2+r^2 \simeq z^2$.

$$\bar{A} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{L+\sqrt{z^2}}{r}\right) a_z.$$

$$\boxed{\bar{A} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{2z}{r}\right)}\ \bar{a_z}\ wb/m.$$

<u>To find $\bar{H}$ from $\bar{A}$</u> WKT $\bar{A} = \frac{\mu_0 I}{2\pi} \ln\left(\frac{2L}{r}\right) \bar{a}_z$

$\bar{B} = \nabla \times \bar{A}$

$$= \frac{1}{r} \begin{bmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \partial/\partial r & \partial/\partial \phi & \partial/\partial z \\ 0 & 0 & \frac{\mu_0 I}{2\pi} \ln\left(\frac{2L}{r}\right) \end{bmatrix}$$

$$= \frac{1}{r} \left[\bar{a}_r \cdot (0) - r\,\bar{a}_\phi \left\{ \frac{\mu_0 I}{2\pi} \frac{\partial}{\partial r}\left(\ln \frac{2L}{r}\right) - 0 \right\} - \bar{a}_z (0 - 0) \right]$$

$$= \frac{1}{r} \left[-\cancel{r} \frac{\mu_0 I}{\cancel{2}\pi} \bar{a}_\phi \frac{1}{\frac{2L}{r}} \cdot \left(-\frac{2L}{r^2}\right) \right]$$

$$= \frac{1}{r} \left[\frac{\mu_0 I}{2\pi} \bar{a}_\phi \right]$$

$$\bar{B} = \frac{\mu_0 I}{2\pi r} \bar{a}_\phi$$

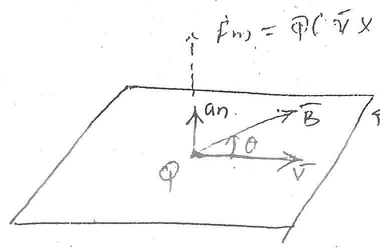
$$\bar{H} = \frac{\bar{B}}{\mu_0} = \frac{I}{2\pi r} \bar{a}_\phi$$

$$\boxed{\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi}$$

Magnetic force material & Inductance ①

Force on a moving charge:

$F_m = Q(\bar{v} \times B)$



← plane containing $\bar{v}$ & B

$\bar{a}_n \to$ Normal to the plane

W.K.T the static electric field $\bar{E}$ exerts a force on a charge Q & is given by

$$F = Q\bar{E} \longrightarrow ①$$

The direction of force is same as the direction of field.

Now if a moving charge is placed in a magnetic field of flux density $\bar{B}$, it experiences a force. This force is a function of charge Q, velocity $\bar{v}$, & the angle between $\bar{v}$ & $\bar{B}$.

$$\therefore F = Q v B \sin\theta$$

$$\bar{F} = Q(\bar{v} \times \bar{B}) \longrightarrow ②$$

[∵ By the defn. of cross product $\bar{A} \times \bar{B} = A B \sin\theta$]

The direction of this force is $\perp^{lr}$ to direction of $\bar{v}$ & $\bar{B}$.

Now the total force on a moving charge due to electric field $\bar{E}$ & magnetic flux density $\bar{B}$ is

$$\bar{F} = Q\bar{E} + Q(\bar{v} \times \bar{B})$$

$$\boxed{\bar{F} = Q(\bar{E} + \bar{v} \times \bar{B})}$$

This eqn. is known as Lorentz eqn.

The direction of magnetic force is independent $\perp^{lr}$ to the direction of velocity vector of a particle, hence kinetic energy of a particle remains unchanged. & magnetic field is incapable of transferring energy to the moving charge. But in case of electric field, direction of force is ~~independent~~ independent of velocity vector. Thus electric force performs work on the charge.

Force on a differential current element:

The force exerted on a differential element of charge dQ moving in a steady magnetic field is given by,

$$d\bar{F} = dQ(\bar{v} \times \bar{B}) \longrightarrow ①$$

The current density is given by,

$$\bar{J} = I/ds$$

$$= \frac{dQ}{dt} \cdot \frac{1}{ds} \qquad \left(\because I = \frac{dQ}{dt}\right)$$

~~$d\bar{F} = dq$~~

$$\bar{J} = \frac{dq}{dt} \cdot \frac{1}{\frac{dv}{dl}}$$

(dv = differential volume
$dv = ds \cdot dl$
$\therefore ds = dv/dl$)

$$= \frac{dq}{dv} \cdot \frac{dl}{dt}$$

$$\bar{J} = \rho_v \cdot \bar{v} \rightarrow (2)$$

[ρ_v = volume charge density
$= dq/dv$
$\bar{v}$ = velocity of moving charge
$= \frac{dl}{dt}$ or $\frac{dx}{dt}$]

Also, $\rho_v = \frac{dq}{dv}$

$$dq = \rho_v \cdot dv \rightarrow (3)$$

putting (3) in (1)

$$d\bar{F} = \rho_v \cdot dv \; \bar{v} \times \bar{B}$$

$$= dv \cdot \rho_v \bar{v} \times \bar{B}$$

$$= dv(\bar{J} \times \bar{B}) \rightarrow (4) \text{ (from eqn (2))}$$

But WKT,

$$\bar{J}\,dv = \bar{K}\,ds = I\,d\bar{l}$$

$\therefore$ The force exerted on a current density $(\bar{K}\,ds)$ is given by,

$$d\bar{F} = (\bar{K} \times \bar{B})\,ds \rightarrow (5)$$

Ily the force exerted on a differential current element $I d\bar{l}$ is given by,

$$d\bar{F} = I\,d\bar{l} \times \bar{B} \rightarrow (6)$$

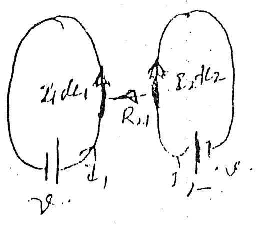
The total force is obtained by integrating eqn (4, 5 & 6).

$$\bar{F} = \int_v (\bar{J} \times \bar{B})\,dv \rightarrow (6A)$$

$$\bar{F} = \int_s (\bar{K} \times \bar{B})\,ds \rightarrow (7)$$

$$\bar{F} = \int_l (I\,d\bar{l} \times \bar{B}) \rightarrow (8) \; = I L B \sin\theta$$

Force between differential current elements:



When a current flows through one of the conductor, the magnetic field is produced around the conductor & if another conductor which carries a current of I_2 is placed in this magnetic field, then the force is exerted on the second current element. If the directions of both the currents are same, then the conductors experiences force of attraction. & if the directions of both the currents are opposite to each other, then the conductors experience a force of repulsion. (2)

consider the two currents $I_1 dl_1$ & $I_2 dl_2$ carrying the currents I_1 & I_2 respectively in the same direction.

The force exerted on element $I_1 dl_1$ due to the magnetic field $\bar{B}_2$ produced by other element $I_2 dl_2$ is given by,

$$d(d\bar{F}_1) = I_1 d\bar{L}_1 \times d\bar{B}_2 \rightarrow (1)$$

From Biot-savart law, the magnetic field produced by current element $I_2 d\bar{L}_2$ is given by,

$$d\bar{B}_2 = \mu_0 \, d\bar{H}_2$$

$$= \mu_0 \left[\frac{I_2 d\bar{L}_2 \times \bar{a}_{R21}}{4\pi R_{21}^2} \right] \rightarrow (2)$$

putting (2) in (1)

$$d(dF_1) = I_1 d\bar{L}_1 \times \mu_0 \left[\frac{I_2 d\bar{L}_2 \times \bar{a}_{R21}}{4\pi R_{21}^2} \right]$$

$$= \frac{\mu_0 I_1 I_2}{4\pi} \left[\frac{d\bar{L}_1 \times d\bar{L}_2 \times \bar{a}_{R21}}{|R_{21}^2|} \right]$$

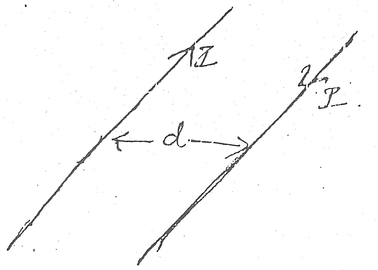
$\therefore$ The net force is given by,

$$\bar{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{L}_1 \times (d\bar{L}_2 \times \bar{a}_{R21})}{|R_{21}^2|} \; //$$

IIIy the force F_2 exerted on the current element 2 due to the magnetic field B_1 produced by the current element 1 is,

$$\bar{F}_2 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{L}_2 \times (d\bar{L}_1 \times \bar{a}_{R12})}{R_{12}^2} .$$

Expression for force between two parallel conductors:



consider two infinite parallel conductor carrying a current of I amp & seperated by distance of d m as shown in the fig.

According to Biot-savart law, the magnetic field intensity at any one conductor due to other conductor is given by,

$$\bar{H} = \int \frac{I d\bar{l}_1 \times \bar{a}_R}{4\pi R^2} \ A/m$$

w.k.t. for an infinite conductor, the $\bar{H}$ at a distance 'd' is given by,

$$|\bar{H}| = \frac{I}{2\pi d} \rightarrow ①$$

Also the magnitude of force F_1 is given by

$$F = BIL \sin\theta \rightarrow ②$$

Now, $B = \mu_0 H$.

$$B = \mu_0 \frac{I}{2\pi d} \rightarrow ③ \text{ (from ①)}$$

putting ③ in ②

$$F = \frac{\mu_0 I}{2\pi d} I L \sin\theta$$

$$= \frac{\mu_0 I^2}{2\pi d} L \sin\theta$$

For parallel conductors, $\theta = 90°$,

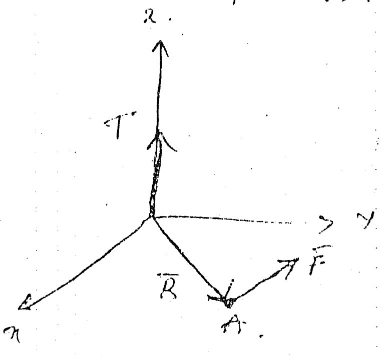
$$F = \frac{\mu_0 I^2}{2\pi d} L \cdot 1$$

$$\boxed{F = \frac{\mu_0 I^2 L}{2\pi d}}$$

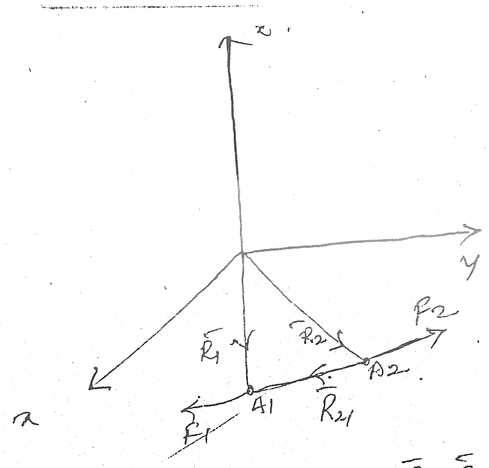
Force/unit length $= \frac{\mu_0 I^2}{2\pi d}$ N/m.

Magnetic torque: The torque about a specified point is defined as the product of the moment arm $\bar{R}$ & the force $\bar{F}$. Torque is a vector.

$$\bar{T} = \bar{R} \times \bar{F} \ \ Nm.$$



Then the torque $\bar{T}$ about the origin is equal to the vector product of $\bar{R}$ & $\bar{F}$. The magnitude of torque is equal to the product of magnitudes of $\bar{R}$ & $\bar{F}$ & the sine of angle between $\bar{R}$ & $\bar{F}$. The direction of $\bar{T}$ is normal to both $\bar{R}$ & $\bar{F}$.

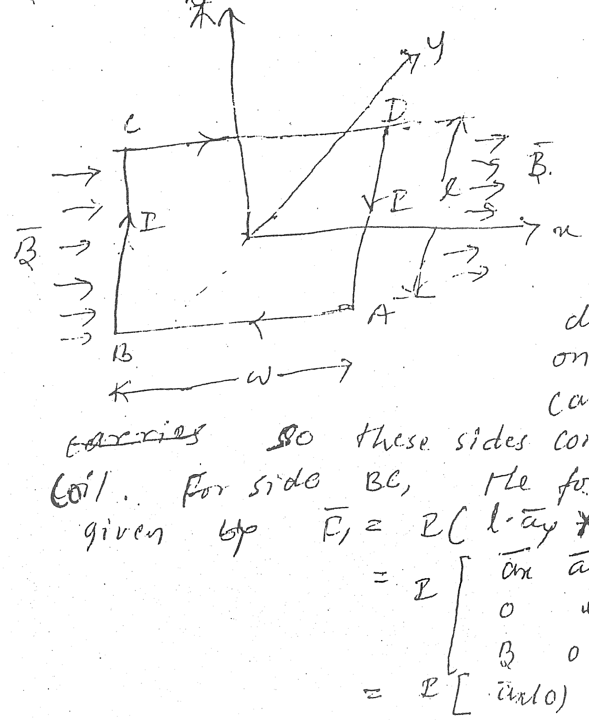


Consider the two forces F_1 & F_2 are applied at points A_1 & A_2 respectively. The arms for the two forces drawn from the origin be R_1 & R_2 respectively as shown above. Assume that $\bar{F}_2 = -\bar{F}_1$. Then the total torque T about the origin due to the two forces is

given by,
$$T = \bar{R}_1 \times \bar{F}_1 + \bar{R}_2 \times \bar{F}_2$$
$$= \bar{R}_1 \times \bar{F}_1 + \bar{R}_2 \times (-\bar{F}_1)$$
$$= (\bar{R}_1 - \bar{R}_2) \times \bar{F}_1$$
$$= \bar{R}_{21} \times \bar{F}_1$$

where $\bar{R}_{21} = \bar{R}_1 - \bar{R}_2$ is a vector joining A_2 to A_1

Expression for torque in a planar coil carrying current I.



Consider a rectangular planar coil of length 'l' along y-axis & width 'w' along x-axis. The coil is placed in the uniform magnetic field $\bar{B}$ along +ve x-direction.

As shown in the fig the sides AB & CD are parallel to the direction of $\bar{B}$, no force will be exerted on these sides. The coil sides BC & ~~DA~~ carries current in +y & -y directions ~~carries~~ so these sides contribute in force exerted on a planar coil. For side BC, the force exerted on a planar coil is given by
$$\bar{F}_1 = I(l\,\bar{a}_y \times B\bar{a}_x)$$
$$= I\begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & l & 0 \\ B & 0 & 0 \end{bmatrix}$$
$$= I[\bar{a}_x(0) - \bar{a}_y(0) + \bar{a}_z(-Bl)].$$
$$\bar{F}_1 = -BIl\,\bar{a}_z$$

Ily. for side DA, the force exerted is given by
$$\bar{F}_2 = I(-l\,\bar{a}_y \times B\bar{a}_x)$$
$$= I\begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & -l & 0 \\ B & 0 & 0 \end{bmatrix}$$
$$= I(\bar{a}_x(0) - \bar{a}_y(0) + \bar{a}_z(+Bl))$$
$$= BIl\,\bar{a}_z \ \ N.$$

For the current element along side BC, the moment arm is, $\bar{R}_1 = -\frac{w}{2}\bar{a}_x$

IIly for the current element along side DA, the moment arm is given by, $\bar{R}_2 = \frac{w}{2}\bar{a}_x$

Thus the total torque eqn. about y-axis is given by

$$\bar{T} = \bar{T}_1 + \bar{T}_2$$

$$= \bar{R}_1 \times \bar{F}_1 + \bar{R}_2 \times \bar{F}_2$$

$$= \left(-\frac{w}{2}\bar{a}_x \times -BIl\,\bar{a}_z\right) + \frac{w}{2}\bar{a}_x \times (BIl\,\bar{a}_z)$$

$$= -\frac{w}{2}BIl\,\bar{a}_y - \frac{w}{2}BIl\,\bar{a}_y$$

$$= -wBIl\,\bar{a}_y$$

$$= -(w \cdot l)\,BI\,\bar{a}_y$$

$$\boxed{T = -ABI\,\bar{a}_y}\ \text{Nm} \quad (A = w \times l)$$

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

$$M^2 = K^2 L_1 L_2$$

Magnetic Boundary conditions

The conditions of the magnetic field existing at the boundary of the two media, when the magnetic field passes from one medium to other are known as magnetic boundary conditions. At the boundary $\bar{B}$ & $\bar{H}$ vectors are resolved in to two components.

a) Tangential to boundary ($\bar{B}_{tan}$)
b) Normal to boundary ($\bar{B}_n$)

Boundary ~~component~~ conditions for normal components:

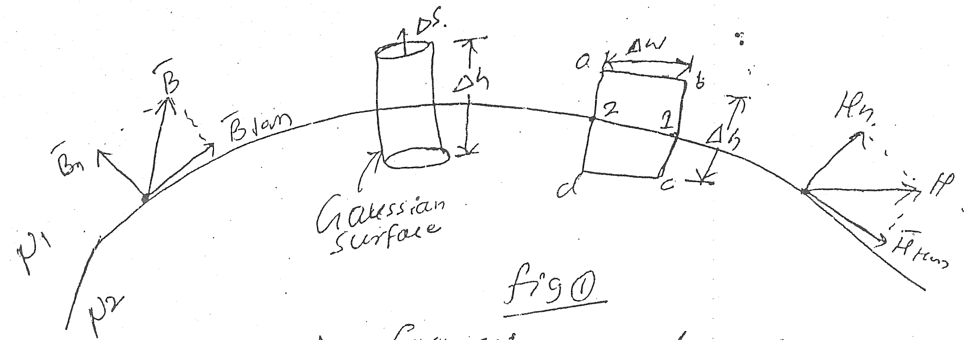


fig ①

consider a closed Gaussian ~~cylinderical~~ surface in the form of right circular cylinder.
Applying ~~Ampere's~~ ~~Circuital~~ ~~law~~ Gauss law

$$\oint \bar{B}\cdot \bar{ds} = \phi = 0 \quad (\because \text{flux within the closed surface is zero})$$

$$\therefore \oint_{\text{top surface}} \bar{B}\cdot\bar{ds} + \oint_{\text{curved surface}} \bar{B}\cdot\bar{ds} + \oint_{\text{Bottom surface}} \bar{B}\cdot\bar{ds} = 0 \longrightarrow ①$$

Let B_{N1} & B_{N2} are the normal components of $\bar{B}$, in medium1 & medium2 respectively.

$$B_{N1}\Delta s + 0 + (-B_{N2})\Delta s = 0$$

(-ve sign $\because$ the flux is entering the surface in medium2)

$$B_{N1}\Delta s - B_{N2}\Delta s = 0$$

$$B_{N1}\Delta s = B_{N2}\Delta s$$

$$\boxed{B_{N1} = B_{N2}} \quad — ②$$

$$\mu H_{N1} = \mu H_{N2}$$

$$\mu_0\mu_{r1} H_{N1} = \mu_0\mu_{r2} H_{N2}$$

$$\boxed{\frac{H_{N1}}{H_{N2}} = \frac{\mu_{r2}}{\mu_{r1}}}$$

//

Boundary conditions for tangential components

From Ampere's circuital law,

$$\oint \bar{H}\cdot\bar{dl} = I \quad \rightarrow (1)$$

consider a rectangular closed path abcd as shown in in fig (1). ∴ eqn (1) becomes

$$\oint_{a}^{b} \bar{H}\cdot\bar{dl} + \int_{b}^{1} \bar{H}\cdot\bar{dl} + \int_{1}^{c} \bar{H}\cdot\bar{dl} + \int_{c}^{d} \bar{H}\cdot\bar{dl} + \int_{d}^{2} \bar{H}\cdot\bar{dl} + \int_{2}^{a} \bar{H}\cdot\bar{dl} = I \rightarrow (2)$$

Let us assume that over a small width Δw, $\bar{H}$ can be assumed constant say H_{tan1} in medium1 & H_{tan2} in medium2. Similarly along Δh, $\bar{H}$ can be assumed constant say H_{N1} & H_{N2} in medium1 (μ_1) & medium 2 (μ_2) respectively.

consider a surface current of K A/m normal to the path, eqn (2) becomes,

$$K\Delta w = H_{tan1}\Delta w + H_{N1}\frac{\Delta h}{2} + H_{N2}\frac{\Delta h}{2} - H_{tan2}\Delta w - H_{N2}\frac{\Delta h}{2} - H_{N1}\frac{\Delta h}{2} = K\Delta w.$$

To get the conditions at the boundary, $\Delta h = 0$.

$$\therefore K\Delta w = H_{tan1}\Delta w - H_{tan2}\Delta w$$

$$K = H_{tan1} - H_{tan2}$$

$$H_{tan1} - H_{tan2} = \bar{K} \times \bar{a}_{N12}$$

where $\bar{a}_{N12}$ is the unit vector normal to the boundary from medium1 to medium2.

When the boundary condition is free from current, K=0

$$H_{tan1} = H_{tan2}$$

$$\frac{B_{tan1}}{\mu_1} = \frac{B_{tan2}}{\mu_2}$$

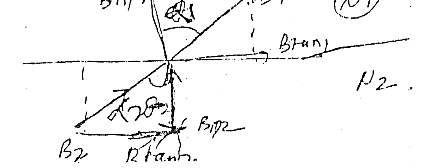
$$\frac{B_{tan1}}{B_{tan2}} = \frac{\mu_1}{\mu_2} = \frac{\mu_0\mu_{r1}}{\mu_0\mu_{r2}} = \frac{\mu_{r1}}{\mu_{r2}}$$

$$\boxed{\frac{B_{tan1}}{B_{tan2}} = \frac{\mu_{r1}}{\mu_{r2}}}$$

Ex. P.T. the magnetic field making an angle α_1 & α_2 with normal to the interface between the ~~dielectrics~~ magnetic medium, μ_{r1} & μ_{r2} is

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{B_{tan1}}{B_{tan2}} = \frac{\mu_{r1}}{\mu_{r2}}.$$

Proof.



From fig,

$$\tan\alpha_1 = \frac{B_{tan1}}{B_{N1}}$$

$$\tan\alpha_2 = \frac{B_{tan2}}{B_{N2}}$$

(2)

$$\therefore \frac{\tan\alpha_1}{\tan\alpha_2} = \frac{B_{tan1}}{B_{N1}} \times \frac{B_{N2}}{B_{tan2}}$$

$$= \frac{B_{tan1}}{\cancel{B_{N1}}} \times \frac{\cancel{B_{N2}}}{B_{tan2}} \quad (B_{N1} = B_{N2})$$

$$= \frac{B_{tan1}}{B_{tan2}}$$

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\mu_{r1}}{\mu_{r2}} \; //$$

magnetic circuits :

mmf = magneto motive force

$= NI$

$= N \cdot \oint \bar{H} \cdot d\bar{l}$

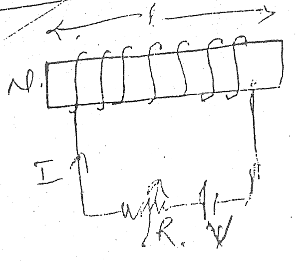
Reluctance, $S = \frac{mmf}{\phi}$

$$= \frac{N \oint \bar{H} \cdot d\bar{l}}{\oint_s \bar{B} \cdot d\bar{s}}$$

Self inductance : (L) Self inductance of a coil is defined as the no of flux linkages produced in one coil for every one Ampere of current flowing through the same coil

$$\therefore \boxed{L = \frac{N\phi}{I}}$$

Inductance of a solenoid



W.K.T. ; $H = \frac{NI}{l} \rightarrow (1)$

Also, $\phi = B \cdot A$

$= \mu H A \quad ; (B = \mu H)$

$= \frac{\mu N I A}{l} \rightarrow (2) \; (\text{from } (1))$

But inductance of a coil

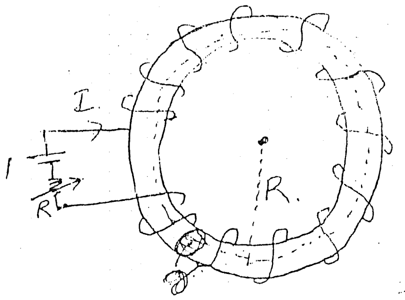
by defn. is $L = N\phi / I \rightarrow (3)$

putting 3 in (2)

$$L = \frac{N}{I} \cdot \frac{\mu N I A}{l}$$

$$\boxed{L = \frac{N^2 \mu A}{l}} \; //$$

Inductance of a toroid :



Consider a toroid ring with N-turns and & carrying current I-amps.

Let, R → Radius of toroid

r → radius of cross-section of ring.

The magnetic flux density inside a toroidal ring is given by,

$$B = \frac{\mu N I}{2\pi R},$$

But flux linkage,

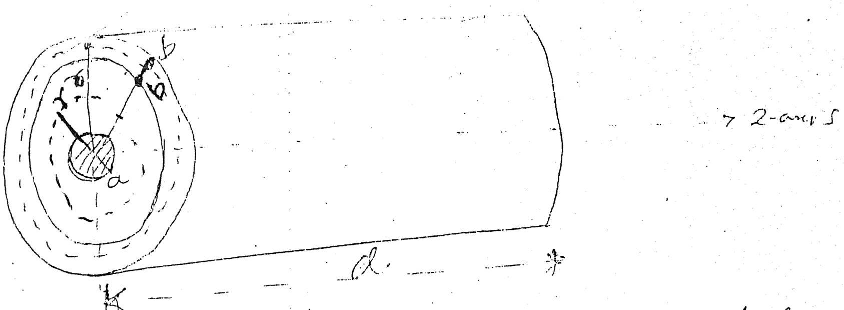
$$N\phi = NBA = N \cdot \frac{\mu N I A}{2\pi R} = \frac{\mu N^2 I A}{2\pi R}$$

But $L = \frac{N\phi}{I}$

$$\therefore L = \frac{\mu N^2 I A}{2\pi R \cdot I}$$

$$\boxed{L = \frac{\mu N^2 A}{2\pi R}}$$ Henry. where $A = \pi r^2$.

Inductance of a co-axial cable :



Consider a co-axial cable with inner conductor of radius a & outer conductor of radius 'b'.

Field intensity at any point between inner & outer conductor is,

$$\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi$$

$$\bar{B} = \mu \bar{H}$$

$$= \mu \frac{I}{2\pi r}\bar{a}_\phi$$

$$\therefore \phi = \int_s \bar{B} \cdot \bar{ds}$$

$$= \int_s \frac{\mu I}{2\pi r}\bar{a}_\phi \cdot dr\,dz\,\bar{a}_\phi = \int_s \frac{\mu I}{2\pi r}\,dr\,dz.$$

③

$$\phi = \int_{z=0}^{d} \int_{r=a}^{b} \left(\frac{\mu I}{2\pi r}\right) dr\,dz$$

$$= \frac{\mu I}{2\pi} \int_0^d dz \cdot \int_a^b \frac{1}{r}\, dr$$

$$= \frac{\mu I}{2\pi} (z)_0^d (\ln r)_a^b$$

$$= \frac{\mu I}{2\pi}\, d \cdot \ln(b) - \ln(a)$$

$$= \frac{\mu I}{2\pi}\, d \ln(b/a)$$

$$= \frac{\mu I d}{2\pi} \ln(b/a) \rightarrow ?$$

Now, $L = \frac{N\phi}{I}$

$\therefore L = \frac{\phi}{I} \quad (N=1)$

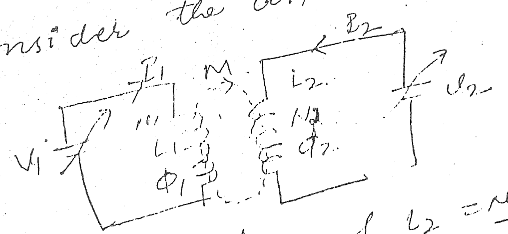
$\therefore = \frac{\mu \cancel{I} d}{2\pi \cdot \cancel{I}} \ln(b/a)$

$$\boxed{L = \frac{\mu d \ln(b/a)}{2\pi}} \text{ Henry}$$

//.

mutual inductance: mutual inductance between the two coils is defined as the flux linkages in the coil due to the current produced by the other coil.

consider the coil shown below,



Then $L_1 = \frac{N_1\phi_1}{I_1}$ & $L_2 = \frac{N_2\phi_2}{I_2}$.

mutual inductance between the two coils is,

$M = \frac{N_1\phi_2}{I_2}$ or $M = \frac{N_2\phi_1}{I_1}$ // is Henry..

Unit of mutual inductance

consider $M = \frac{N_2\phi_1}{I_1} \rightarrow ①$

WKT $H_1 = \frac{N_1 I_1}{l_1}$

$\therefore B_1 = \mu H_1$

$= \frac{\mu N_1 I_1}{l_1}$

~~WKT $H_1 = N$~~

$\phi_1 = B_1 A_1$

$= \frac{\mu N_1 I_1}{l_1} A_1 \rightarrow ②$

putting ② in ①

$M = \frac{\mu N_1 N_2 \cancel{I_1} A_1}{\cancel{I_1} l_1}$

$$\boxed{M = \frac{\mu N_1 N_2 A_1}{l_1}}$$

or $$\boxed{M = \frac{\mu N_1 N_2 A_2}{l_2}}$$

Energy stored in a magnetic field :

Consider a coil of inductance L through which a current is increasing from a value 0 to a maximum value I.

WKT, $P = eI$

$= L\frac{dI}{dt} I$

$P = LI\frac{dI}{dt}$

Also $P =$ work done per unit time

$= W/t$

$W = P \cdot t$

$dW = P \cdot dt$

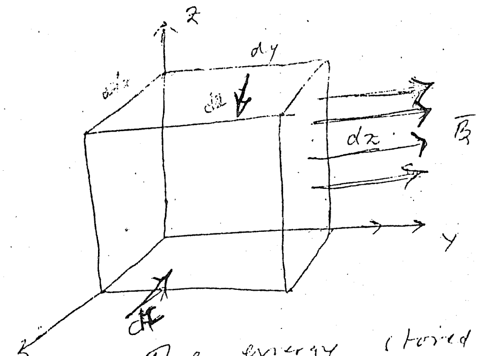
$\therefore dW = LI\frac{dI}{dt} dt$

$= LI\, dI$

$W = \int_0^I LI\, dI$

$$\boxed{W = \frac{LI^2}{2}}$$

Energy density in a magnetic field :



The energy stored in a inductor is given by

$W = \frac{1}{2} LI^2$

Consider a differential volume element, $dv = dx\, dy\, dz$. in a magnetic field $\bar{B}$ as shown above. Consider that at the top & bottom surfaces of a differential volume, conducting sheets with current dI are present.

The inductance of the sheet is,

$dL = \frac{N d\phi}{dI} = \frac{d\phi}{dI}$ $(N=1)$

$dL = \frac{B\, ds}{dI} = \frac{B\, dx\, dz}{dI}$ $(ds = dx\, dz)$

$dL = \frac{\mu H\, dx\, dz}{dI} \rightarrow (1)$ $(B = \mu H)$

From Ampere's circuital law,

$dz = H\,dy$. —②

The energy stored in a differential volume is given by, ④

$dW_m = \frac{1}{2}\, dL\, dI^2$ —③

putting ①&② in ③

$$dW_m = \frac{1}{2}\left[\frac{\mu H\, dx\, dz}{H\, dy}\right]\left[H\, dy\right]^2$$

$$= \frac{1}{2}\mu H^2\, dx\, dy\, dz.$$

$$= \frac{1}{2}\mu H^2\, dv. \qquad (dv = dx\, dy\, dz)$$

The energy density function is defined as,

$$w_m = \lim_{dv \to 0} \frac{dW_m}{dv} = \frac{1}{2}\mu H^2 \quad J/m^3$$

$$w_m = \frac{1}{2}\mu H \cdot H = \frac{1}{2} B H$$

$$w_m = \frac{1}{2}\mu\left(\frac{B}{\mu}\right)^2 = \frac{1}{2}\frac{B^2}{\mu}$$

Also the energy density can be found out as,

$$W_m = \int dW_m\, dv$$

$$= \frac{1}{2}\int_v \mu H^2\, dv$$

$$= \frac{1}{2}\int_v BH\, dv$$

$$= \frac{1}{2}\int_v \frac{B^2}{\mu}\, dv$$

TIME VARYING MAGNETIC FIELDS - [UNIT-VI]

The magnetic fields which vary with respect to time are called as time varying magnetic fields. These ~~fie~~ magnetic fields are produced by time varying currents (i.e. Alternating currents).

<u>Faraday's laws</u>: <u>First law</u>: Whenever the magnetic flux linking with a conductor changes an emf is induced in the conductor.

<u>Second law</u>: It states that, the magnitude of emf induced in a conductor is directly proportional to the rate of chang of flux linking with the conductor.

Consider a coil having N turns is linking with a flux of $d\phi$ wb in dt seconds, then

$$\boxed{e = N\frac{d\phi}{dt}}$$ where $e \rightarrow$ emf induced.

$$e = \frac{d\phi}{dt}$$ $N = 1$ turn i.e for one conductor.

<u>Lenz's law</u>: It states that, the direction of electro-magnetically induced emf is such as to oppose the cause producing it.

$$\therefore e = -N\frac{d\phi}{dt}$$

$$e = -\frac{d\phi}{dt} \quad (N=1)$$

But $\phi = \int_S \bar{B}\cdot\bar{ds}$

$$\therefore e = -\frac{d}{dt}\int_S \bar{B}\cdot\bar{ds} \longrightarrow (1)$$

Also the potential developed at a point in an electric field 'E' is given by

$$e = \oint_l \bar{E}\cdot\bar{dl} \longrightarrow (2)$$

From (1) & (2)

$$\boxed{\oint_l \bar{E}\cdot\bar{dl} = -\frac{d}{dt}\int_S \bar{B}\cdot\bar{ds}} \longrightarrow (3)$$

Eqn. 3 is known as Faraday's law in integral or differential form.

Conditions for the induced emf (Types of emf):

mainly there are two conditions for emf induced depending upon whether the conductor is subjected to time changing magnetic field or the conductor is moving in the constant magnetic field.

1) ~~The~~ condition 1: Conductor is stationary & the magnetic field is changing w.r.t time (statically induced emf) :→ The emf induced in a conductor due to time varying flux linking with it is known as statically induced emf. or ~~transformer emf~~.

$$\therefore \ e = -\frac{d\phi}{dt}$$

$$e = -\frac{d}{dt}\int_s \bar{B}\cdot\bar{ds}$$

$$\therefore \ e = \oint_l \bar{E}\cdot\bar{dl} = -\frac{d}{dt}\int_s \bar{B}\cdot\bar{ds} \rightarrow ①$$

From Storkes theorem, ~~$\int \bar{E}\cdot dl$~~

$$\oint_l \bar{H}\cdot\bar{dl} = \int_s (\nabla\times\bar{H})\cdot\bar{ds}$$

$$\text{Ily} \quad \oint_l \bar{E}\cdot\bar{dl} = \int_s (\nabla\times\bar{E})\cdot\bar{ds} \rightarrow ②$$

putting ② in ①

$$\int_s (\nabla\times\bar{E})\cdot\bar{ds} = -\frac{d}{dt}\int_s \bar{B}\cdot\bar{ds}$$

Surface integral on both sides cancels.

$$\therefore \ \nabla\times\bar{E} = -\frac{d\bar{B}}{dt}$$

$$\text{or} \quad \boxed{\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}} \rightarrow ③$$

Eqn. ③ is known as point form of Faraday's law.

Types of statically induced emf;

a) self induced emf: The emf induced in a conductor due to changing ~~flux~~ flux linking with the same conductor is known as self induced emf.

B) Mutually induced emf : The emf induced in one coil (or Conductor) due to changing flux linking with it, & this changing flux is prouduced by ~~the~~ changing current flowing in the other coil is called as mutually induced emf.

The principle of statically induced emf is used in transformers.

II) Condition-2 : The magnetic field is stationary (i does not vary w.r.t time) & the conductor is moving with velocity. or (Dynamically induced emf)

The emf induced due to relative motion between the conductor moving with velocity & stationary magnetic field is known as dynamically induced emf or motional emf or rotational emf.

Consider a charge 'Q' moving in a magnetic field of fluxdensity 'B' wb/m2. The force experienced on this charge is

$$\bar{F} = Q \cdot (\bar{V} \times \bar{B})$$

But $\bar{E} = \frac{\bar{F}}{Q}$

$$\bar{E} = \frac{Q(\bar{V} \times \bar{B})}{Q}$$

$$\bar{E} = \bar{V} \times \bar{B} \text{ V/m.} \rightarrow (1)$$

The expression for dynamically induced emf is

$$e = \oint_{l} \bar{E} \cdot \bar{dl}$$

$$\boxed{e = \int_{l} (\bar{V} \times \bar{B}) \cdot \bar{dl}} \text{ volts.}$$

The principle of dynamically induced emf is used in generator.

MAXWELL's 2nd equation in point form: (Modified form of Amper's circuit law: (Displacement current)) 4

From Amperes circuital law,

$\nabla \times \bar{H} = \bar{J}$

Taking ~~curl~~ divergence on both sides

$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} \rightarrow (1)$

By the property of curl,

$\nabla \cdot (\nabla \times \bar{H}) = 0 \rightarrow (1A)$, ∴ eqn (1) becomes

$\nabla \cdot \bar{J} = 0 \rightarrow (2)$

But by continuity equation,

$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \rightarrow (3)$

From eqn (2) & (3) it is clear that, for the time varying fields, Amper's circuital law becomes inconsistent with continuity equation.

∴ The Amper's circuital law need to be modified.

Let $\nabla \times \bar{H} = \bar{J} + \phi \rightarrow (4)$

Taking divergence on both sides.

$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \phi.$

$0 = \nabla \cdot \bar{J} + \nabla \cdot \phi$ (from (1A) →).

$\therefore \nabla \cdot \phi = -\nabla \cdot \bar{J}$

$= -\left(-\frac{\partial \rho_v}{\partial t}\right)$ (By continuity eqn, from eqn (3))

$= \frac{\partial \rho_v}{\partial t}$

(By point form of Gauss law, $\nabla \cdot \bar{D} = \rho_v$)

$\nabla \cdot \phi = \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$

$\nabla \cdot \phi = \nabla \cdot \frac{\partial \bar{D}}{\partial t}$

$\therefore \phi = \frac{\partial \bar{D}}{\partial t} \rightarrow (5)$

putting (5) in (4)

$$\boxed{\nabla \times \bar{H} = \bar{J} + \frac{\partial D}{\partial t}}$$

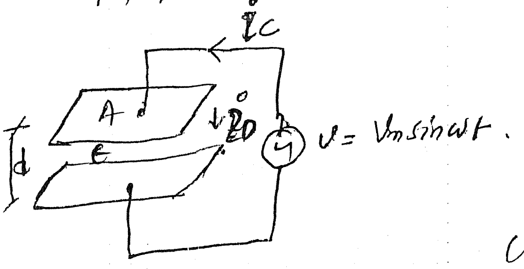
$\nabla \times \bar{H} = \bar{J}_c + \bar{J}_D$

$J_c \rightarrow$ conduction current density. This current is due to flow charges in the conductor.

$\bar{J}_D$ is called the displacement current density & this is obtained due to the time varying electric flux density. 5.

<u>Relation between conduction current in the connecting leads & displacement current in the dielectric in a parallel plate capacitor, when time varying ~~ϕ~~ voltage is applied :</u>

or Q: Show that for a parallel plate capacitor subjected time varying fields, the displacement current in the dielectric equals the conduction current in the wire.



Consider a parallel plate capacitor connected across an alternating voltage source. The electrons move up to the conducting plates constituting conduction current i_c.

& is given by

$$i_c = C \cdot \frac{dv}{dt}$$

We know that

$$C = Q/v.$$

$$\therefore Q = C \cdot V$$

Differentiating w.r.t time

$$\frac{dq}{dt} = C \cdot \frac{dv}{dt}.$$

$$\boxed{i_c = C \cdot \frac{dv}{dt}} \rightarrow ① \quad \left(\because \frac{dq}{dt} = i \right)$$

Now the electric field between the two plates varies w.r.t. time & this rate of variation is called displacement current density & is given by

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t}$$

$$= \frac{\partial}{\partial t}(\epsilon \bar{E})$$

$$= \epsilon \frac{\partial}{\partial t}(|E|)$$

$$= \epsilon \frac{\partial}{\partial t}\left(\frac{v}{d}\right) \qquad \left(\because |\bar{E}| = v/d \right)$$

where $v \rightarrow$ potential applied

$d \rightarrow$ thickness of dielectric slab.

$$\frac{i_D}{A} = \frac{\epsilon}{d}\frac{dv}{dt}$$ ($J_D = \frac{i_D}{A}$ current / ~~per~~ unit area = current density)

$$i_D = \frac{\epsilon A}{d}\frac{dv}{dt}$$

But for parallel plate capacitor

$$C = \frac{\epsilon A}{d}$$

$$\therefore \boxed{i_D = C\cdot\frac{dv}{dt}} \longrightarrow (2)$$

from (1) & (2) $i_c = i_D = C\cdot\frac{dv}{dt}$. Hence the proof.

Maxwell's Equations: Maxwell's equations are nothing but, a set of 4 equations derived from ~~Ampere's circuit law,~~ Gauss law, Ampers circuital law, & Faraday's law for electric & magnetic fields. These equations can be expressed in point form & integral form.

Equation I: Statement: (Gauss's law) The total normal electric flux passing from a closed surface is equal to the charge enclosed by the surface.

i.e. $\boxed{\oint_s \bar{D}\cdot\bar{ds} = Q_{enclosed}.} \rightarrow (1)$ This is known as maxwell's 1st equation in ~~point form~~ integral form.

Now $\oint_s \bar{D}\cdot\bar{ds} = Q_{enclosed}$

$$\oint_s \bar{D}\cdot\bar{ds} = \int_v \rho_v\, dv \rightarrow (A) \quad (\because \rho_v = dq/dv)$$

By divergence theorem,

$$\oint_s \bar{D}\cdot\bar{ds} = \oint_v (\nabla\cdot\bar{D})\, dv \rightarrow (B).$$

$\therefore$ Comparing eqn (A) & (B)

$$\boxed{\nabla\cdot\bar{D} = \rho_v} \longrightarrow (2)$$

Eqn (2) is known as point form of maxwell's 1st equation.

Maxwell's 2nd equation: From Ampere's circuital law.

$$\oint_l \bar{H}\cdot\bar{dl} = I_{enclosed}$$

$$\oint_l \bar{H}\cdot\bar{dl} = \oint_s \bar{J}\cdot\bar{ds} \longrightarrow$$

The modified form of Ampere's circuital law, giving the relation between the displacement current density & the conduction current density is,

$$\boxed{\oint_l \bar{H}\cdot d\bar{l} = \oint_s \bar{J}\cdot d\bar{s} + \oint_s \frac{\partial \bar{D}}{\partial t}\cdot d\bar{s}} \longrightarrow 3.$$

This eqn. is known as integral form of Amper's law or integral form of maxwell's 2nd equation.

Statement: The line integral of magnetic field intensity around any closed path in a magnetic field is equal to the sum of conduction current & the rate of change of displacement current density through any closed surface bounded by the closed path.

Now $\oint_l \bar{H}\cdot d\bar{l} = \oint_s \bar{J}\cdot d\bar{s} + \oint_s \frac{\partial \bar{D}}{\partial t}\cdot d\bar{s} = \oint_s \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)\cdot d\bar{s}$

By Stokes theorem, $\oint_l \bar{H}\cdot d\bar{l} = \oint_s (\nabla\times\bar{H})\cdot d\bar{s}$

above eqn. becomes,

$\therefore \cancel{\oint_l \bar{H}\cdot d} \oint_s (\nabla\times\bar{H})\cdot d\bar{s} = \oint_s \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)\cdot d\bar{s}$

$$\Rightarrow \boxed{\nabla\times\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}} \longrightarrow 4.$$

Eqn 4 is known as maxwell's 2nd equation in point form or Ampere's circuital in point form.

<u>Maxwell's 3rd equation : (Faraday's law)</u> :

From Faraday's law,

$$e = -N\frac{d\phi}{dt}$$

$$= -N\frac{d}{dt}\left(\int_s \bar{B}\cdot d\bar{s}\right) \qquad \left(\because \phi = \int_s \bar{B}\cdot d\bar{s}\right)$$

$$= -N\frac{\partial}{\partial t}\left(\int_s \bar{B}\cdot d\bar{s}\right)$$

But $e = \oint_l \bar{E}\cdot d\bar{l}$

$$\therefore \oint_l \bar{E}\cdot d\bar{l} = -\frac{\partial}{\partial t}\left(\int_s \bar{B}\cdot d\bar{s}\right) \qquad (\text{for } N=1)$$

$$\boxed{\oint_l \bar{E}\cdot d\bar{l} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}} \longrightarrow 5.$$

This eqn is known as ~~differential~~ integral form of maxwell's 3rd equation

From Stokes' theorem,

$$\oint_l \bar{H}\cdot d\bar{l} = \oint_S (\nabla\times\bar{H})\cdot d\bar{s}$$

$\|{}^{ly}$ $$\int_l \bar{E}\cdot d\bar{l} = \int_S (\nabla\times\bar{E})\cdot d\bar{s}$$

$\therefore$ from eqn ⑤ $$\int_S (\nabla\times\bar{E})\cdot d\bar{s} = -\int_S \frac{\partial\bar{B}}{\partial t}\cdot d\bar{s}$$

$$\Rightarrow \boxed{\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}} \longrightarrow 6.$$

This eqn is known as point form (differential form) of Maxwell's 3rd equation or point form Faraday's law.

Statement: The emf around any closed path in an electric field is equal to the negative rate of change of magnetic flux density through any closed surface bounded by the closed path.

Maxwell's 4th equation (Gauss law in magnetostatics): From Gauss's law in integral form for magnetic field,

$$\oint_S \bar{B}\cdot d\bar{s} = \phi = 0$$

$$\therefore \boxed{\oint_S \bar{B}\cdot d\bar{s} = 0} \longrightarrow ⑦$$

maxwell's 4th eqn in differential form.

From divergence theorem,

$$\oint_S \bar{B}\cdot d\bar{s} = \oint_v (\nabla\cdot\bar{B})\, dv.$$

$$\therefore \oint_v (\nabla\cdot\bar{B})dv = 0 \quad \text{(from eqn ⑦)}.$$

As $dv \neq 0$.

$$\therefore \boxed{\nabla\cdot\bar{B} = 0.} \longrightarrow 8.$$ Maxwell's 4th eqn in point form.

Statement: The net magnetic flux over any closed surface in a magnetic field is zero. ie, no isolated (single) magnetic pole exists.

Maxwell's equations for any medium, General set:

	Integ form	Integral form.	Law from which derived.
1	$\nabla\cdot\bar{D} = \rho_v$	$\oint_s \bar{D}\cdot\bar{ds} = Q_e = \int_v \rho_v dv$	Gauss's law (Electric field)
2.	~~$\oint_l \bar{H}\cdot\bar{dl}$~~ $\nabla\times\bar{H} = \bar{J}_c + \frac{\partial\bar{D}}{\partial t}$	$\oint_l \bar{H}\cdot\bar{dl} = \oint_s \bar{J}\cdot\bar{ds} + \oint_s \frac{\partial\bar{D}}{\partial t}\cdot\bar{ds}$	Ampere's circuital law
3.	$\nabla\times E = -\frac{\partial\bar{B}}{\partial t}$	$\oint_l \bar{E}\cdot\bar{dl} = \int_s -\frac{\partial\bar{B}}{\partial t}\cdot\bar{ds}$	Faraday's law.
4.	$\nabla\cdot\bar{B} = 0$.	$\oint_s \bar{B}\cdot\bar{ds} = 0$.	Gauss's law (magnetic field) (Non existance of monopole)

Maxwell's equations for free space conditions:

For free space condition, $\rho_v = 0$ & $J_c = 0$.

	Point form	Integral form	Law from which derived.
1	$\nabla\cdot\bar{D} = \rho_v = 0$.	$\oint_s \bar{D}\cdot\bar{ds} = 0$	Gauss's law.
2.	$\nabla\times\bar{H} = \frac{\partial\bar{D}}{\partial t}$	$\oint_l \bar{H}\cdot\bar{dl} = \oint_s \frac{\partial\bar{D}}{\partial t}\cdot\bar{ds}$	Ampere's law.
3	$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$	$\oint_l \bar{E}\cdot\bar{dl} = -\oint_s \frac{\partial\bar{B}}{\partial t}\cdot\bar{ds}$	Faraday's law.
4.	$\nabla\cdot\bar{B} = 0$	$\oint_s \bar{B}\cdot\bar{ds} = 0$.	Gauss's law (magnetic field)

Maxwell's equations for static conditions:

For static conditions., $\partial\bar{D}/\partial t = \partial\bar{B}/\partial t = 0$.

	Point form	Integral form	Law from which derived.
1.	$\nabla\cdot\bar{D} = \rho_v$	$\oint_s \bar{D}\cdot\bar{ds} = \int_v \rho_v dv$	Gauss's law.
2.	$\nabla\times\bar{H} = \bar{J}$	$\oint_l \bar{H}\cdot\bar{dl} = \oint_s \bar{J}\cdot\bar{ds}$	Ampere's law.
3.	$\nabla\times\bar{E} = 0$	$\oint_l \bar{E}\cdot\bar{dl} = 0$	Faraday's law.
4	$\nabla\cdot\bar{B} = 0$	$\oint_s \bar{B}\cdot\bar{ds} = 0$.	Gauss's law.

Maxwell's equations for good conductor

For good conductor, $\rho_v = 0$, $J_c >> \partial \bar{D}/\partial t$

S	Point form	Integral form	Law from which derived
1.	$\nabla \cdot \bar{D} = 0$	$\oint_s \bar{D} \cdot \bar{ds} = 0$	Gauss's law.
2.	$\nabla \times \bar{H} = \bar{J}$	$\oint_l \bar{H} \cdot \bar{dl} = \oint_s \bar{J_c} \cdot \bar{ds}$	Ampere's law.
3.	$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint_l \bar{E} \cdot \bar{dl} = -\oint_s \frac{\partial \bar{B}}{\partial t} \cdot \bar{ds}$	Faraday's law.
4	$\nabla \cdot \bar{B} = 0$	$\oint_s \bar{B} \cdot \bar{ds} = 0.$	Gauss law (magnetic)

Maxwell's equations for harmonically varying fields.

Let the fields E & B are varying harmonically with time,

then, $\bar{D} = D_0 e^{j\omega t}$ & $\bar{B} = B_0 e^{j\omega t}$.

$$\therefore \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t}(D_0 e^{j\omega t}) = D_0 e^{j\omega t} \cdot j\omega = j\omega (D_0 e^{j\omega t}) = j\omega \bar{D}$$

Illy $\frac{\partial \bar{B}}{\partial t} = j\omega \bar{B}$.

$\therefore$ The maxwell's equations are as given below,

I) $\nabla \cdot \bar{D} = \rho_v$ & $\oint_s \bar{D} \cdot \bar{ds} = \int_v \rho_v \, dv$

II) $\nabla \times \bar{H} = \bar{J_c} + \frac{\partial \bar{D}}{\partial t} = \bar{J} + j\omega \bar{D}$

$\nabla \times \bar{H} = \sigma \bar{E} + j\omega \epsilon \bar{E}$ $\quad (\because \bar{D} = \epsilon \bar{E}, \ \bar{J_c} = \sigma \bar{E})$

$\nabla \times \bar{H} = (\sigma + j\omega\epsilon) \bar{E}$ //

$$\oint_l \bar{H} \cdot \bar{dl} = + \int_s \frac{\partial \bar{D}}{\partial t} \cdot \bar{ds} + \oint_s \bar{J_c} \cdot \bar{ds}$$

$$= + \int_s \cancel{\tfrac{\partial}{\partial t}} \, j\omega \bar{D} \cdot \bar{ds} + \oint_s \sigma \bar{E} \cdot \bar{ds}$$

$$= \cancel{\sigma} \int_s j\omega\epsilon \, \bar{E} \cdot \bar{ds} + \oint_s \sigma \bar{E} \cdot \bar{ds}$$

$$= (j\omega\epsilon + \sigma) \int_s \bar{E} \cdot \bar{ds} \ //$$

III) $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -j\omega B = -j\omega \mu \bar{H}$ // $\quad (\because \bar{B} = \mu \bar{H})$

$$\oint_l \bar{E} \cdot \bar{dl} = -\int_s \frac{\partial \bar{B}}{\partial t} \cdot \bar{ds} = -\int_s j\omega \bar{B} \cdot \bar{ds} = -j\omega\mu \int_s \bar{H} \cdot \bar{ds}.$$

IV) $\nabla \cdot \bar{B} = 0$ // & $\int_s \bar{B} \cdot \bar{ds} = 0$ //.

	Point form (Differential form)	Integral form	Law from which derived
1.	$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot \bar{ds} = \int_v \rho_v dv$	Gauss's law.
2.	$\nabla \times \bar{H} = (\sigma + jw\epsilon)\bar{E}$	$\oint_l \bar{H} \cdot \bar{dl} = (\sigma + jw\epsilon)\int_S \bar{E} \cdot \bar{ds}$	Ampere's law.
3.	$\nabla \times \bar{E} = -jw\bar{B}$ $= -jw\mu\bar{H}$	$\oint_l \bar{E} \cdot \bar{dl} = -\int_S jw\bar{B} \cdot \bar{ds}$ $= -jw\mu\int_S \bar{H} \cdot \bar{ds}$	Faraday's law.
4.	$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot \bar{ds} = 0$	Gauss's law.

Relation b/w conduction current density (J_c) & displacement current density (J_D):

or with usual notation prove that, $|\bar{J}_c / \bar{J}_D| = \sigma / w\epsilon$

From, ohm's law in point form, $\bar{J}_c = \sigma \bar{E} \rightarrow (1)$

Let $\bar{D} = D_0 e^{jwt}$

& $\bar{E} = E_0 e^{jwt}$.

$\therefore \bar{J}_c = \sigma E_0 e^{jwt} \rightarrow (2)$

Now, displacement current density,

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t}(\epsilon E)$$

$$= \frac{\partial}{\partial t}(\epsilon E_0 e^{jwt})$$

$$= \epsilon E_0 jw e^{jwt}$$

$$\bar{J}_D = \epsilon jw \bar{E}_0 e^{jwt} \rightarrow (3)$$

Now, $$\frac{\bar{J}_c}{\bar{J}_D} = \frac{\sigma \cancel{E_0} \cancel{e^{jwt}}}{\epsilon jw \cancel{E_0} \cancel{e^{jwt}}}$$

$$= \frac{\sigma}{jw\epsilon}$$

$$\boxed{\left|\frac{\bar{J}_c}{\bar{J}_D}\right| = \frac{\sigma}{w\epsilon}}$$ Hence the proof.

Comparison between Electric & magnetic circuit

Electric circuit	Magnetic circuit
1. The circuit which forms closed path for the flow of electric current is called electric circuit	1 The circuit which forms closed path ~~fo~~ to provduce the flux is known as magnetic circuit.
2. Emf is the driving force for the flow of current & measured in volts.	3. mmf is the driving force to proudcce flux & is measured in AT/mt.
3. Resistance apposes the flow of electric current	3. Reluctance apposes the flux path.
4. ohms law : $I = V/R$	4. Ohms law : $\phi = \frac{NI}{S} = \frac{mmf}{Reluctance}$ ~~where~~
5. ~~E~~ Electric field intensity (E)	5. Magnetic field intensity (H)
6. ~~curre~~ Flux density : $D = \phi/A$ c/m²	6. flux density $\bar{B} = \phi/A$ – wb/m²,
7. Reciprocal of resistance is conductance	7 Reciprocal of reluctance is permeance.
8. In electric circuit, the current actually flows, ~~in~~ i.e. there is a movement of electron.	8. Due to mmf the flux gets established & does not flow in the sense in which current flows.

%

not closed ~~lines start~~ from +ve charge & ~~ending~~ -ve charge

~~magnetic lines of flux are closed lines~~

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Retarded Potential :-

Vector magnetic potentials & scalar electric potentials are given by

$$\nabla \times \bar{A} = \bar{B} \longrightarrow (1)$$

$$\bar{E} = -\nabla V \longrightarrow (2)$$

Taking curl to equ^n (2)

$$\nabla \times \bar{E} = -\nabla \times \nabla V$$

But curl of gradient is zero.

$$\nabla \times E = 0 \longrightarrow (3)$$

From maxwells equ^n for time varying fields

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

∴ The maxwells equ^n is inconsistant with potential equ^n

∴ This equ^n need to be modified

$$\bar{E} = -\nabla V + N \longrightarrow (5)$$

Taking curl on Both sides

$$\nabla \times \bar{E} = -\nabla \times \nabla V + \nabla \times N$$

$$\nabla \times \bar{E} = 0 + \nabla \times N. \qquad (\nabla \times \nabla \phi = 0)$$

$$\nabla \times N = \nabla \times \bar{E}$$

$$= -\frac{\partial \bar{B}}{\partial t} \qquad (\because \nabla \times E = -\frac{\partial \bar{B}}{\partial t})$$

$$= -\frac{\partial}{\partial t}(\nabla \times \bar{A}) \qquad (\because \bar{B} = \nabla \times \bar{A})$$

$$\nabla \times N = -\nabla \times \frac{\partial \bar{A}}{\partial t}$$

$$N = -\frac{\partial A}{\partial t} \longrightarrow (6)$$

Putting equ^n (6) in (5)

$$\bar{E} = -\nabla V - \frac{\partial A}{\partial t} \longrightarrow (7)$$

From gauss law in point form

$$\nabla \cdot \bar{D} = \rho_V$$

$$\nabla \cdot \varepsilon \bar{E} = \rho_V \qquad (\because \bar{D} = \varepsilon \bar{E})$$

$$\varepsilon \nabla \cdot \bar{E} = \rho_V$$

$$\nabla \cdot \bar{E} = \frac{\rho_V}{\epsilon} \longrightarrow (8)$$

putting equn (7) in (8)

$$\nabla \cdot \left(-\nabla V - \frac{\partial A}{\partial t}\right) = \frac{\rho_V}{\varepsilon}$$

$$-\nabla^2 V - \frac{\partial}{\partial t} \nabla \cdot \bar{A} = \frac{\rho_V}{\varepsilon}$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \bar{A}) = -\frac{\rho_V}{\epsilon} \longrightarrow (9)$$

consider Maxwells equn.

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

$$\nabla \times \frac{\bar{B}}{\mu} = \bar{J} + \epsilon \frac{\partial \bar{E}}{\partial t} \qquad \left(\begin{array}{l} \bar{B} = \mu \bar{H} \\ \bar{D} = \epsilon \bar{E} \end{array}\right)$$

$$\frac{1}{\mu}(\nabla \times \nabla \times \bar{A}) = \bar{J} + \epsilon \frac{\partial \bar{E}}{\partial t} \qquad (\bar{B} = \nabla \times \bar{A})$$

$$\frac{1}{\mu}(\nabla(\nabla \cdot \bar{A}) - \nabla^2 A) = \bar{J} + \epsilon \frac{\partial \bar{E}}{\partial t} \qquad (\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A} - \nabla^2 A))$$

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 A = \mu \bar{J} + \mu \epsilon \frac{\partial \bar{E}}{\partial t}$$

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 A = \mu \bar{J} + \mu \epsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \bar{A}}{\partial t}\right) \quad (\text{from } 7)$$

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 A = \mu \bar{J} - \mu \epsilon \left(\nabla \cdot \frac{\partial V}{\partial t} + \frac{\partial^2 A}{\partial t^2}\right) \longrightarrow (10)$$

From Lorentz condition for potential

$$\nabla \cdot \bar{A} = -\mu \epsilon \frac{\partial V}{\partial t}$$

substituting this equn in (9) & (10),

$$\nabla^2 V + \frac{\partial}{\partial t}\left(-\mu \epsilon \frac{\partial V}{\partial t}\right) = -\frac{\rho_V}{\epsilon} \longrightarrow (11)$$

$$-\nabla\left(\cancel{\mu \epsilon \frac{\partial V}{\partial t}}\right) - \nabla^2 A = \mu J - \cancel{\mu \epsilon \nabla \frac{\partial V}{\partial t}} - \mu \epsilon \frac{\partial^2 A}{\partial t^2}$$

$$-\nabla^2 A = \cancel{+\mu J} - \mu\epsilon \frac{\partial^2 A}{\partial t^2}$$

(14)

$$\nabla^2 A = -\mu J + \mu\epsilon \frac{\partial^2 A}{\partial t^2} \longrightarrow (12)$$

During time varying fields the potential at a point at given time 't' is due to the state of charge distribution (ρ_v) or current (I) that was existing at an earlier time t'. If R is the distance between the source to the point at which potential is obtained the t & t' are retarded as

$$t' = t - \frac{R}{V}$$

where $v \longrightarrow$ velocity of light $= 3\times10^8$ m/s.

Thus t' the earlier time at which the state of the source which corresponds to the potential at the given instant A.

UNIFORM PLANE WAVES [UNIT-VII]

The waves are the means of transporting energy or information from source to destination. The waves consisting of electric & magnetic fields ~~&~~ are known as electromagnetic waves. These waves are functions of time & space.

Ex: Radio waves, light rays, radar beams, television signals etc.

Initially maxwell predicted the existance of electromagnetic waves & later on it was stated by prof Henrich Hertz.

Uniform plane wave: It is defined as a wave whose value remains constant throught a plane which is transversed to the direction of propogation of the wave. For example, if the wave is ~~travalling~~ in $\bar{a}_z$ direction, then the plane z=constant will be perpendicular to the direction of propogation. In the plane wave with z=const, the variables are x & Y. Hence, for this uniform plane wave, E & H are independent of x & y co-ordinates and are functions of 'z' co-ordinates only.

General wave equation in electric field:

The maxwell's 2nd & 3rd equations which are functions of both time & space is given by,

$$\nabla\times\bar{H} = \bar{J}+\frac{\partial\bar{D}}{\partial t} \rightarrow (1)$$

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} = -\mu\frac{\partial\bar{H}}{\partial t} \rightarrow (2)$$

Taking curl on both sides of eqn (2).

$$\nabla\times\nabla\times\bar{E} = \nabla\times -\mu\frac{\partial\bar{H}}{\partial t}$$

$$= -\mu\frac{\partial}{\partial t}(\nabla\times\bar{H}) \rightarrow (3)$$

putting (1) in (3)

$$\nabla\times\nabla\times\bar{E} = \cancel{\nabla\times}\left(-\mu\frac{\partial}{\partial t}\left\{\bar{J}+\frac{\partial\bar{D}}{\partial t}\right\}\right.$$

From the property of curl,

$$\left.\begin{aligned}\nabla\times\nabla\times\bar{E} &= \nabla(\nabla\cdot\bar{E})-\nabla^2\bar{E}\\ \&\ \bar{J} = \sigma\bar{E},\ \ &\bar{D} = \epsilon\bar{E}\end{aligned}\right\} (4)$$

putting (4) in (3)

$$\nabla(\nabla\cdot\bar{E})-\nabla^2\bar{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\bar{E}+\epsilon\frac{\partial\bar{E}}{\partial t}\right)$$

$$\therefore \boxed{\nabla^2 E - \mu\sigma \frac{\partial \bar{E}}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2} = \nabla\left(\frac{\rho_v}{\epsilon}\right)} \rightarrow ⑤ \quad \because \nabla\cdot\bar{D} = \rho_v, \quad \nabla\cdot D = \rho_v/\epsilon.$$

$$\text{or} \quad \boxed{\nabla^2(\bar{D}/\epsilon) - \frac{\mu\sigma}{\epsilon}\frac{\partial \bar{D}}{\partial t} - \mu \frac{\partial^2 \bar{D}}{\partial t^2} = \nabla\left(\frac{\rho_v}{\epsilon}\right)} \rightarrow ⑥$$

For free space conditions, $\sigma = 0$, $\rho_v = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$.

Eqn 5 becomes.

$$\boxed{\nabla^2 \bar{E} - \mu_0 \epsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2} = 0}$$

$$\text{or} \quad \frac{\nabla^2 \bar{D}}{\epsilon_0} - \mu_0 \frac{\partial^2 \bar{D}}{\partial t^2} = 0 //$$

· General wave-equation in magnetic field :

The maxwell's 2nd & 3rd are given by.

$$\nabla\times\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}.$$

$$\text{or} \quad \nabla\times\bar{H} = \sigma\bar{E} + \epsilon\frac{\partial E}{\partial t} \rightarrow ① \quad (\because \bar{J} = \sigma\bar{E}, \ \bar{D} = \epsilon\bar{E})$$

$$\nabla\times\bar{E} = \frac{-\partial \bar{B}}{\partial t}$$

$$\nabla\times\hat{E} = -\mu\frac{\partial \bar{H}}{\partial t} \rightarrow ② \qquad (\because \bar{B} = \mu\bar{H})$$

Taking curl on both sides of eqn ①

$$\nabla\times\nabla\times\bar{H} = \sigma\,\nabla\times\bar{E} + \epsilon\frac{\partial}{\partial t}\nabla\times\bar{E} \longrightarrow ③$$

putting ② in ③

$$\nabla\times\nabla\times\bar{H} = \sigma\left(-\mu\frac{\partial \bar{H}}{\partial t}\right) + \epsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial H}{\partial t}\right)$$

$$= -\mu\sigma\frac{\partial \bar{H}}{\partial t} + \mu\epsilon\frac{\partial^2 H}{\partial t^2} \longrightarrow ④$$

$$\text{But} \quad \nabla\times\nabla\times\bar{H} = \nabla(\nabla\cdot\bar{H}) - \nabla^2\bar{H}$$

$$\therefore \nabla(\nabla\cdot\bar{H}) - \nabla^2\bar{H} = -\mu\sigma\frac{\partial \bar{H}}{\partial t} - \mu\epsilon\frac{\partial^2 H}{\partial t^2}$$

$$0 - \nabla^2\bar{H} = -\mu\sigma\frac{\partial \bar{H}}{\partial t} - \mu\epsilon\frac{\partial^2 H}{\partial t^2} \qquad \left(\because \begin{array}{l}\nabla\cdot B = 0\\ \mu\nabla\cdot H = 0\end{array}\right)$$

$$\therefore \boxed{\nabla^2\bar{H} - \mu\sigma\frac{\partial \bar{H}}{\partial t} - \mu\epsilon\frac{\partial^2 \bar{H}}{\partial t^2} = 0.}$$

for free space conditions, $\sigma = 0$, $\mu = \mu_0$, $\epsilon = \epsilon_0$.

$$\therefore \boxed{\nabla^2 H - \mu_0\epsilon_0\frac{\partial^2 H}{\partial t^2} = 0} \longrightarrow ⑤$$

$$\text{or} \quad \frac{\nabla^2\bar{B}}{\mu_0} - \epsilon_0\frac{\partial^2 B}{\partial t^2} = 0 \qquad (\because \bar{H} = \bar{B}/\mu)$$

Consider a non conducting medium with, ϵ & μ.

then equation ⑤ becomes.

$$\therefore \nabla^2 \bar{H} - \mu\epsilon \frac{\partial^2 \bar{H}}{\partial t^2} = 0$$

$$\text{or } \nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = 0. \quad \longrightarrow (6)$$

In general, a classical wave travelling with velocity v m/sec. is given by.

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \longrightarrow (7)$$

comparing (7) with (6)

$$\frac{1}{v^2} = \mu\epsilon$$

$$\boxed{v = \frac{1}{\sqrt{\mu\epsilon}}} \quad v \text{ is called wave velocity.}$$

for free space, $\mu = \mu_0$, $\epsilon = \epsilon_0$.

$$\therefore v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3\times10^8 \text{ m/sec.}$$

i.e. in free space the wave travels at a velocity of 3×10^8 m/s which is equal to the speed of light.

Transverse nature of Uniform plane wave :

Consider a uniform plane wave propogating in x-direction in a medium which is free of charge, & let the field be, $\bar{E} = E_x \bar{a}_x + E_y \bar{a}_y + E_z \bar{a}_z$ V/m

Now, $\nabla\cdot\bar{D} = \rho_v$

$\nabla\cdot\epsilon\bar{E} = 0 \quad (\because \rho_v = 0.)$

$\therefore \nabla\cdot\bar{E} = 0. \quad (\epsilon \neq 0)$

$$\left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right)\cdot\left(E_x\bar{a}_x + E_y\bar{a}_y + E_z\bar{a}_z\right) = 0$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0. \longrightarrow (1)$$

If the uniform plane wave is travalling in x direction then E, is a function of x & t. i.e. E(x,t).

then $E_y = E_z = \text{constant}$.

$$\therefore \frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z} = 0.$$

$\therefore$ eqn (1) becomes,

$$\frac{\partial E_x}{\partial x} = 0 \longrightarrow (2)$$

Eqn. (2) is true when $E_x = \text{constant}$ or $E_x = 0$.

Now consider the wave equation, with $\sigma = 0$.

$$\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = 0 \longrightarrow 3.$$

In terms of x, y & z components of E, the above eqn becomes,

$$\frac{\partial^2 E_x}{\partial x^2} - \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} = 0, \quad \frac{\partial^2 E_y}{\partial y^2} - \mu\epsilon \frac{\partial^2 E_y}{\partial t^2} = 0, \quad \frac{\partial^2 E_z}{\partial z^2} - \mu\epsilon \frac{\partial^2 E_z}{\partial t^2} = 0. \longrightarrow 4$$

putting 2 in 4

$$0 - \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} = 0$$

$$\therefore \frac{\partial^2 E_x}{\partial t^2} = 0 \longrightarrow 5 \quad (\because \mu\epsilon \neq 0)$$

so $\frac{\partial E_x}{\partial t}$ = Constant, or $\frac{\partial E_x}{\partial t} = 0$.

Equation 3 holds good, when
i) $E_x = 0$ ii) E_x is constant w.r.t. time iii) E_x is uniformly increasing with time.

But for a wave motion E_x will not be constant in time or E_x will not be increasing uniformly with time. Thus we can conclude that, for a wave motion, along x-axis, $\boxed{E_x = 0.}$

It means that x-component of E is zero.

Thus we can conclude that, uniform plane electromagnetic waves do not have the components of in the direction of propogation (i.e $E_x = H_x = 0$). These waves have the components of E & H in the direction perpendicular (Transverse) to the direction of propogation. Thus uniform plane electromagnetic waves are transverse in nature.

Intrinsic impedence of a perfect Dielectric:

(Relation between E & H in ~~free~~ perfect dielectric)

The ratio of electric field intensity to the magnetic field intensity in a given medium is known as intrinsic impedence. It is denoted by η.

$$\therefore \boxed{\eta = \frac{|E|}{|H|}}$$

We know that $\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t}$

$$\begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{bmatrix} = -\mu \frac{\partial}{\partial t}\left(H_x \bar{a}_x + H_y \bar{a}_y + H_z \bar{a}_z\right) \rightarrow (1)$$

Consider an uniform plane wave propogating along x-direction, then $E_x = H_x = 0$. Also there will be no variation of field components along y & z axis by transverse property of E.M. waves. ie $\partial/\partial y = \partial/\partial z = 0$.

$\therefore$ eqn. (1) becomes

$$\begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & 0 & 0 \\ 0 & E_y & E_z \end{bmatrix} = -\mu \frac{\partial}{\partial t}\left(H_y \bar{a}_y + H_z \bar{a}_z\right)$$

$$\bar{a}_x(0-0) - \bar{a}_y\left(\frac{\partial E_z}{\partial x} - 0\right) + \bar{a}_z\left(\frac{\partial E_y}{\partial x}\right) = -\mu \frac{\partial H_y}{\partial t}\bar{a}_y - \mu\frac{\partial H_z}{\partial t}\bar{a}_z$$

$$-\frac{\partial E_z}{\partial x}\bar{a}_y + \frac{\partial E_y}{\partial x}\bar{a}_z = -\mu\frac{\partial H_y}{\partial t}\bar{a}_y - \mu\frac{\partial H_z}{\partial t}\bar{a}_z$$

Equating y & z components

$$\frac{\partial E_z}{\partial x} = \mu\frac{\partial H_y}{\partial t} \rightarrow (2), \quad \frac{\partial E_y}{\partial x} = -\mu\frac{\partial H_z}{\partial t} \rightarrow (3).$$

Let $\bar{E}_y = f_1(x - vt)$ where $v = 1/\sqrt{\mu\epsilon}$ = velocity of wave

From eqn. (3), $\frac{\partial(f_1(x-vt))}{\partial x} = -\mu\frac{\partial H_z}{\partial t} \rightarrow (4)$

Let $(x - vt) = u \Rightarrow \frac{\partial u}{\partial x} = 1$, & $\frac{\partial u}{\partial t} = -v \Rightarrow \partial t = -\frac{\partial u}{\partial v}$

$\therefore$ Eqn. (4) can be written as.

$$\frac{\partial}{\partial x} f_1(u) = -\mu\frac{\partial H_z}{\partial t}$$

$$\frac{\partial f_1(u)}{\partial u}\cdot\frac{\partial u}{\partial x} = -\mu\frac{\partial H_z}{\partial t} \quad (\because \text{By Chain Rule})$$

$$\frac{\partial f_1(u)}{\partial u}\cdot 1 = -\mu\frac{\partial H_z}{\partial t}$$

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu}\cdot\frac{\partial f_1(u)}{\partial u}$$

$$\frac{\partial H_z}{\cancel{\partial t}} = -\frac{1}{\mu}\frac{\partial f_1(u)}{\partial u}\cdot\partial t \qquad \left(\partial t = -\frac{\partial u}{\partial v}\right)$$

$$= -\frac{1}{\mu}\frac{\partial f_1(u)}{\partial \cancel{u}}\cdot\left(-\frac{\partial u}{v}\right)$$

$$\partial H_z = +\frac{1}{\mu v}\,\partial f_1(u)$$

On integrating on both sides ~~w.r.t~~

$$\int \partial \bar{H}_z = \frac{1}{\mu v}\int \partial f_1(u)$$

$$\bar{H}_z = \frac{1}{\mu v} f_1(u) = \frac{1}{\mu v} f_1(x-vt) = \frac{1}{\mu v}\bar{E}_y$$

$$\therefore \frac{\bar{E}_y}{\bar{H}_z} = \mu \cdot v = \frac{\mu \cdot 1}{\sqrt{\mu \epsilon}} = \sqrt{\frac{\mu}{\epsilon}} \longrightarrow (5) \Rightarrow \bar{E}_y = \sqrt{\mu/\epsilon}\,\bar{H}_z \rightarrow (6)$$

Ily from equation ~~5~~ (8), we can write

$$\frac{\bar{E}_z}{\bar{H}_y} = -\sqrt{\mu/\epsilon} \Rightarrow \bar{E}_z = -\sqrt{\mu/\epsilon}\,\bar{H}_y \rightarrow (7)$$

Now, $\eta = \frac{|E|}{|H|} = \frac{\sqrt{E_y^2 + E_z^2}}{\sqrt{H_y^2 + H_z^2}}$

$$\eta = \frac{\sqrt{(\sqrt{\mu/\epsilon}\,H_z)^2 + (-\sqrt{\mu/\epsilon}\cdot H_y)^2}}{\sqrt{H_z^2 + H_y^2}}$$

$$= \sqrt{\mu/\epsilon}\left(\frac{\cancel{\sqrt{H_z^2 + H_y^2}}}{\cancel{\sqrt{H_z^2 + H_y^2}}}\right)$$

$$\boxed{\eta = \sqrt{\mu/\epsilon}}\ \Omega.$$

For free space, $\epsilon = \epsilon_0$, $\mu = \mu_0$.

$\therefore \eta = \sqrt{\mu_0/\epsilon_0} = 377\,\Omega$

~~mutual~~ ~~pro~~ orthogonal property of electric & magnetic field.

Consider an electromagnetic wave propogating along x-axis, Then, $\bar{E} = E_y \bar{a}_y + E_z \bar{a}_z$ & $\bar{H} = H_y \bar{a}_y + H_z \bar{a}_z$.

Now, $\bar{E}\cdot\bar{H} = (E_y \bar{a}_y + E_z \bar{a}_z)\cdot(H_y \bar{a}_y + H_z \bar{a}_z)$

$$= E_y H_y + E_z H_z \longrightarrow (1)$$

But. W.K.T, $\frac{E_y}{H_z} = \sqrt{\frac{\mu}{\epsilon}}$ & $\frac{E_z}{H_y} = -\sqrt{\mu/\epsilon}$

putting above in eqn (1)

$$\bar{E}\cdot\bar{H} = \sqrt{\mu/\epsilon}\,H_z H_y + (-\sqrt{\mu/\epsilon})\,H_y H_z.$$

$$= \sqrt{\frac{\mu}{\epsilon}}\,(\cancel{H_z H_y} - \cancel{H_z H_y})$$

$$\boxed{\bar{E}\cdot\bar{H} = 0}$$

$\Rightarrow \bar{E} \perp^{r} \bar{H}$

Hence ~~H~~ E & H are orthogonal to each other.

* Q No1: For an EM wave show that E & H are mutually perpendicular to each other. i.e. $E \perp \bar{H}$

Equation of uniform plane wave for sinusoidal excitation or (wave equation in phasor form) or (Uniform plane wave in any medium: (4) Consider an electromagnetic wave propogating along x-axis, then.

$$\frac{\partial^2 E_y}{\partial x^2} - \mu\sigma\frac{\partial E_y}{\partial t} - \mu\epsilon\frac{\partial^2 E_y}{\partial t^2} = 0 \longrightarrow (1)$$

Let $E_y = E_{ym}\, e^{j\omega t} \longrightarrow (2)$

Then, $\frac{\partial E_y}{\partial t} = E_{ym}\, e^{j\omega t} \cdot j\omega$

$= j\omega\, E_{ym} e^{j\omega t}$

$= j\omega\, E_y$ (from (2))

|||ly $\frac{\partial^2 E_y}{\partial t^2} = (j\omega)^2 E_y$

putting above derivatives in (1)

$$\frac{\partial^2 E_y}{\partial x^2} - \mu\sigma\, j\omega E_y - \mu\epsilon (j\omega)^2 E_y = 0$$

$$\frac{\partial^2 E_y}{\partial x^2} = \mu\sigma j\omega E_y + \mu\epsilon j^2\omega^2 E_y$$

$= (-\mu\epsilon\omega^2 + j\mu\sigma\omega) E_y$

$= (j\omega\mu\sigma - \omega^2\mu\epsilon) E_y$

$\frac{\partial^2 E_y}{\partial x^2} = \gamma^2 E_y \rightarrow (3)$ where, $\gamma = \sqrt{j\omega\mu\sigma - \omega^2\mu\epsilon}$ & is known as propogation constant.

|||ly $\therefore \nabla^2 \bar{E} = \gamma^2 \bar{E}$

|||ly $\nabla^2 H = \gamma^2 H$

putting (2) in (3)

$$\frac{\partial^2}{\partial x^2}(E_{ym}\, e^{j\omega t}) = \gamma^2 E_{ym}\, e^{j\omega t}$$

$$\cancel{e^{j\omega t}}\frac{\partial^2}{\partial x^2} E_{ym} = \gamma^2 E_{ym}\, \cancel{e^{j\omega t}}$$

$$\frac{\partial^2 E_{ym}}{\partial x^2} = \gamma^2 E_{ym} \longrightarrow (4)$$

Let the solution of this wave be,

$E_{ym} = C_1 e^{-\gamma x} + C_2 e^{\gamma x} \rightarrow (5)$ putting 5 in (2)

$\bar{E}_y = (C_1 e^{-\gamma x} + C_2 e^{\gamma x})\, e^{j\omega t} \longrightarrow (6)$

Let $\gamma = \sqrt{j\omega\mu\sigma - \omega^2\mu\epsilon} = \alpha + j\beta$

$\therefore E_y = (C_1 e^{-(\alpha+j\beta)x} + C_2 e^{(\alpha+j\beta)x})\, e^{j\omega t}$

$= C_1 e^{-\alpha x + j\beta x + j\omega t} + C_2 e^{\alpha x + j\beta x + j\omega t}$

$$E_y = c_1 e^{-\alpha x} \cdot e^{-j\beta(x-\frac{w}{\beta}t)} + c_2 e^{\alpha x} \cdot e^{j\beta(x+\frac{w}{\beta}t)} \rightarrow (5)$$

The real part of above equation represents attenuation of the wave along the direction of propogation. When the wave propogates through any medium, it gets attenuate in any medium. That means the amplitude of the wave reduces ~~along the~~. Hence α is known as attenuation constant & its unit is neper/meter.

Also, when a wave propogates through the medium, phase change occurs such a phase change is expressed by an imaginary part of ~~propogation constant~~ eqn (5) or propogation constant ($\gamma = \sqrt{j\omega\mu\sigma - \omega^2\mu\epsilon}$). It is called phase constant (β). It is measured in rad/mt.

The vector addition of attenuation constant & phase constant is known as propogation constant.

i.e $\gamma = \alpha + j\beta = \sqrt{j\omega\mu\sigma - \omega^2\mu\epsilon}$

Expression for α, β, η in any general medium

Attenuation constant α: We know that

$$\gamma = \sqrt{-\omega^2\mu\epsilon + j\omega\mu\sigma} = \alpha + j\beta \rightarrow (1)$$

squaring on both sides

$$(\alpha+j\beta)^2 = (-\omega^2\mu\epsilon + j\omega\mu\sigma)$$

$$\alpha^2 + j^2\beta^2 + j2\alpha\beta = -\omega^2\mu\epsilon + j\omega\mu\sigma$$

$$\alpha^2 - \beta^2 + j2\alpha\beta = -\omega^2\mu\epsilon - j\omega\mu\sigma \rightarrow (2) \quad (j^2=-1)$$

Equating the real & imaginary parts.

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon \rightarrow (2) \qquad 2\alpha\beta = -\omega\mu\sigma \rightarrow (3)$$

~~$2\alpha\beta = j\omega\mu\sigma$~~

squaring the above equation on both sides.

$$4\alpha^2\beta^2 = \omega^2\mu^2\sigma^2$$

$$4\alpha^2(\alpha^2 + \omega^2\mu\epsilon) = \omega^2\mu^2\sigma^2 \quad (\text{from (2) } \beta^2 = \alpha^2 + \omega^2\mu\epsilon)$$

$$4\alpha^4 + 4\alpha^2\omega^2\mu\epsilon = \omega^2\mu^2\sigma^2$$

$$4\alpha^4 + 4\alpha^2\omega^2\mu\epsilon - \omega^2\mu^2\sigma^2 = 0.$$

$$\alpha^2 = \frac{-4\omega^2\mu\epsilon \pm \sqrt{16\omega^4\mu^2\epsilon^2 + 16\omega^2\mu^2\sigma^2}}{2 \times 4}$$

Taking only the positive term, $\because$ α^2 cannot be negative, then.

if $ax^4 + bx^2 + c = 0$

Then $x^2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$x = \sqrt{\frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

$$\therefore \alpha^2 = \frac{-4\omega^2\mu\epsilon + 4\sqrt{\omega^4\mu^2\epsilon^2 + \omega^2\mu^2\sigma^2}}{8}$$

$$= -\frac{\omega^2\mu\epsilon}{2} + \frac{1}{2}\sqrt{\omega^4\mu^2\epsilon^2\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)}$$

$$= -\frac{\omega^2\mu\epsilon}{2} + \frac{\omega^2\mu\epsilon}{2}\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}$$

$$\therefore \alpha = \sqrt{-\frac{\omega^2\mu\epsilon}{2} + \frac{\omega^2\mu\epsilon}{2}\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}}$$

$$= \sqrt{\frac{\omega^2\mu\epsilon}{2}\left(-1 + \sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}\right)}$$

$$\boxed{\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right)}} \rightarrow (4)$$

Neper/meter.

Phase constant: From eqn (2)

$$\beta^2 = \alpha^2 + \omega^2\mu\epsilon$$

putting (4) in the above eqn.

$$\beta^2 = \left(\omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right)}\right)^2 + \omega^2\mu\epsilon$$

$$= \frac{\omega^2\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right) + \omega^2\mu\epsilon$$

$$= \frac{\omega^2\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} - 1 + 2\right)$$

$$= \frac{\omega^2\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right)$$

$$\boxed{\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right)} \text{ rad/mt}} \rightarrow (5)$$

Intrinsic impedence of a general medium:

~~we~~ The maxwells 3rd eqn is

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t}$$

for a wave propogating along x-axis, $H_x = E_x = 0$

& $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$.

$$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = -\mu j\omega(H_x\bar{a}_x + H_y\bar{a}_y + H_z\bar{a}_z)$$

$$\left(j\omega = \frac{\partial}{\partial t}\right)$$

$$\begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & 0 & 0 \\ 0 & E_y & E_z \end{bmatrix} = -j\omega\mu(H_y\bar{a}_y + H_z\bar{a}_z)$$

$$\bar{a}_x(0) - \bar{a}_y\left(\frac{\partial E_z}{\partial x} - 0\right) + \bar{a}_z\left(\frac{\partial E_y}{\partial x}\right) = -j\omega\mu(H_y\bar{a}_y + H_z\bar{a}_z)$$

$$-\frac{\partial E_z}{\partial x}\bar{a}_y + \frac{\partial E_y}{\partial x}\bar{a}_z = -j\omega\mu H_y\bar{a}_y - j\omega\mu H_z\bar{a}_z$$

Equating y & z-components

$$\frac{\partial E_z}{\partial x} = j\omega\mu H_y \rightarrow (1) \quad \& \quad \frac{\partial E_y}{\partial x} = -j\omega\mu H_z \rightarrow (2)$$

Let $E_y = C_1 e^{-\gamma x}$ ∴ from (2)

$$\frac{\partial}{\partial x}(C_1 e^{-\gamma x}) = -j\omega\mu H_z$$

$$-\gamma C_1 e^{-\gamma x} = -j\omega\mu H_z$$

$$-\gamma E_y = -j\omega\mu H_z$$

$$-\gamma E_y = -j\omega\mu H_z$$

$$\frac{E_y}{H_z} = \frac{j\omega\mu}{\gamma} \rightarrow (3)$$

Ily $\frac{E_z}{H_y} = \frac{-j\omega\mu}{\gamma} \rightarrow (4)$

$$\eta = \left|\frac{E}{H}\right| = \frac{\sqrt{E_y^2 + E_z^2}}{\sqrt{H_y^2 + H_z^2}}$$

$$= \frac{\sqrt{\left(\frac{j\omega\mu}{\gamma}H_z\right)^2 + \left(-\frac{j\omega\mu}{\gamma}H_y\right)^2}}{\sqrt{H_y^2 + H_z^2}} \quad (\text{from } (3) \,\&\, (4))$$

$$= \frac{j\omega\mu}{\gamma}\frac{\cancel{\sqrt{H_z^2 + H_y^2}}}{\cancel{\sqrt{H_y^2 + H_z^2}}}$$

$$\eta = \frac{j\omega\mu}{\gamma}$$

$$= \frac{j\omega\mu}{\sqrt{j\omega\mu\sigma - \omega^2\mu\epsilon}} \qquad \left(\because \gamma = \sqrt{j\omega\mu\sigma - \omega^2\mu\epsilon}\right)$$

$$= \frac{j\omega\mu}{\sqrt{j\omega\mu\sigma + j^2\omega^2\mu\epsilon}} \qquad (\because j^2 = -1)$$

$$\eta = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}}$$

$$= \frac{j\omega\mu}{\sqrt{j\omega\mu}\sqrt{\sigma + j\omega\epsilon}}$$

$$= \frac{\sqrt{j\omega\mu}}{\sqrt{\sigma + j\omega\epsilon}}$$

$$\boxed{\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}}\ \Omega.$$

For ~~good~~ perfect dielectric, where $\sigma = 0$

$$\therefore \eta = \sqrt{\frac{j\omega\mu}{j\omega\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}\ //$$

for free space conditions $\epsilon_r = \mu_r = 1$.

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\ \Omega$$

~~Wave propogation in good conductor :~~
~~(Expression for α, β, η for good conducting medium.)~~

Wave length ; The distance travelled by the wave in which its phase angle changes by 2π radians. It is denoted by λ & its unit is mtr.

$$\therefore \boxed{\lambda = \frac{2\pi}{\beta}\ \text{mt}}$$

Wave propogation in good conductor :
(Expression for $\alpha, \beta, \eta, \lambda$ & v for good conducting medium)

we know that $\bar{J}_c = \sigma\bar{E}$ & $\bar{J}_D = \frac{\partial \bar{D}}{\partial t}$

for good conductor, $J_c >> \bar{J}_D$

$$|\sigma\bar{E}| >> |j\omega\bar{D}|$$

$$|\sigma\bar{E}| >> |j\omega\epsilon\bar{E}|$$

$$\sigma >> |j\omega\epsilon|$$

$$\boxed{\frac{\sigma}{\omega\epsilon} >> 1}$$

i) attenuation constant; $\alpha = \sqrt{\frac{\omega^2\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]}$

As $\frac{\sigma}{\omega\epsilon} >> 1$ $\therefore 1 + \frac{\sigma^2}{(\omega\epsilon)^2} \simeq \frac{\sigma^2}{\omega^2\epsilon^2}$.

$$\therefore \alpha = w\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{\frac{\sigma^2}{w^2\epsilon^2}}-1\right)}$$

$$= w\sqrt{\frac{\mu\epsilon}{2}\left(\frac{\sigma}{w\epsilon}-1\right)}$$

$$= w\sqrt{\frac{\mu\epsilon}{2}\times\frac{\sigma}{w\epsilon}} \qquad \left(\frac{\sigma}{w\epsilon} >> 1 \therefore \frac{\sigma}{w\epsilon}-1 \simeq \frac{\sigma}{w\epsilon}\right)$$

$$= \sqrt{\frac{w^2\mu\sigma}{2w}}$$

$$\boxed{\alpha = \sqrt{\frac{w\mu\sigma}{2}}} \text{ neper/mt.}$$

ii) phase Constant :

$$\beta = w\sqrt{\frac{\mu\epsilon}{2}\sqrt{\frac{\sigma^2}{w^2\epsilon^2}+1})+1}$$

$$\frac{\sigma}{w\epsilon} >> 1 \therefore \frac{\sigma^2}{w^2\epsilon^2}+1 \simeq \frac{\sigma^2}{w^2\epsilon^2}$$

$$\beta = w\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{\frac{\sigma^2}{w^2\epsilon^2}}+1\right)} = w\sqrt{\frac{\mu\epsilon}{2}\left(\frac{\sigma}{w\epsilon}+1\right)}$$

$$= w\sqrt{\frac{\mu\epsilon}{2}\cdot\frac{\sigma}{w\epsilon}} \qquad \left(\because \frac{\sigma}{w\epsilon}+1 \simeq \frac{\sigma}{w\epsilon}\right)$$

$$\boxed{\beta = \sqrt{\frac{w\mu\sigma}{2}}} //$$

III) Intrinsic impedence :

$$\eta = \frac{jw\mu}{\gamma}$$

$$= \frac{jw\mu}{\alpha + j\beta}$$

$$= \frac{jw\mu}{\sqrt{\frac{w\mu\sigma}{2}}+j\sqrt{\frac{w\mu\sigma}{2}}}$$

$$= \frac{jw\mu}{\sqrt{\frac{w\mu\sigma}{2}}(1+j)}$$

$$= \sqrt{\frac{w^2\mu^2}{\frac{w\mu\sigma}{2}}}\cdot\frac{j}{(1+j)}$$

$$\eta = \sqrt{\frac{2w\mu}{\sigma}}\,\frac{j}{1+j}\times\frac{1-j}{1-j}$$

$$= \sqrt{\frac{2w\mu}{\sigma}}\,\frac{j(1-j)}{1+j^2}$$

$$= \sqrt{\frac{2w\mu}{\sigma}}\,\frac{(j+1)}{2} \qquad j^2=-1$$

$$\boxed{\eta = (1+j)\sqrt{\frac{w\mu}{2\sigma}}}\ \Omega$$

Velocity :

$$v = \omega/\beta = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \sqrt{\frac{2\omega^2}{\omega\mu\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}} \text{ m/sec}$$

Wave length :

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \sqrt{\frac{4\pi^2}{\omega\mu\sigma/2}} = \sqrt{\frac{8\pi^2}{\omega\mu\sigma}} \text{ mt.}$$

Wave propogation in good dielectric (lossy dielectric)

(expression for $\alpha, \beta, v, \lambda, \theta$ for good (lossy) dielectric)

For lossy dielectric or good dielectric

$$|J_c| << \bar{J}_D \leq c\left|\frac{\partial D}{\partial t}\right|$$

$$|\sigma E| << |j\omega\epsilon E| \Rightarrow \sigma \cancel{E} << \omega\epsilon \cancel{E}.$$

$$\therefore \boxed{\frac{\sigma}{\omega\epsilon} << 1.}$$

i) attenuation constant :

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)}$$

By Binomial theorem, $(1+a)^n = (1+na)$, neglecting higher order terms, $\therefore \left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/2} = \left(1+\frac{1}{2}\frac{\sigma^2}{\omega^2\epsilon^2}\right)$

$$\therefore \alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(1+\frac{\sigma^2}{2\omega^2\epsilon^2}-1\right)} = \omega\sqrt{\frac{\mu\cancel{\epsilon}}{2}\times\frac{\sigma^2}{2\omega^2\epsilon^{\cancel{2}}}}$$

$$= \sqrt{\frac{\cancel{\omega^2}\sigma^2\mu}{4\,\epsilon\,\cancel{\omega^2}}} =$$

$$\boxed{\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}} \text{ neper/mt.}$$

ii) phase constant ;

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}+1\right)}$$

$$= \omega\sqrt{\frac{\mu\epsilon}{2}\left(1+\frac{\sigma^2}{2\omega^2\epsilon^2}+1\right)}$$

$(1+a)^n = (1+na)$ By Binomial theorem

$$= \omega\sqrt{\frac{\mu\epsilon}{2}\left(2+\frac{\sigma^2}{2\omega^2\epsilon^2}\right)} = \omega\sqrt{\frac{\mu\epsilon}{\cancel{2}}\times\cancel{2}\left(1+\frac{\sigma^2}{4\omega^2\epsilon^2}\right)}$$

$$\beta = w\sqrt{\mu\epsilon}\cdot\left(1+\frac{\sigma^2}{8w^2\epsilon^2}\right) \qquad (1+a)^n = (1+na)$$

$$\boxed{\beta = w\sqrt{\mu\epsilon}}\ \text{rad/mt.} \quad \left(\because \frac{\sigma}{w\epsilon} << 1 \quad 1+\frac{\sigma^2}{8w^2\epsilon^2} \simeq 1\right)$$

III) Intrinsic Impedence :

$$\eta = \frac{jw\mu}{\gamma} = \frac{jw\mu}{\sqrt{jw\mu\sigma - w^2\mu\epsilon}} = \frac{jw\mu}{\sqrt{-w^2\mu\epsilon\left(1-\frac{j\sigma}{w\epsilon}\right)}}$$

$$= \frac{jw\mu}{\sqrt{j^2w^2\mu\epsilon}\cdot\sqrt{1-\frac{j\sigma}{w\epsilon}}} \qquad (\because j^2=-1)$$

$$= \frac{jw\mu}{jw\sqrt{\mu\epsilon}\left(\sqrt{1-\frac{j\sigma}{w\epsilon}}\right)}$$

$$= \sqrt{\frac{\mu}{\epsilon}}\cdot\left(1-\frac{j\sigma}{w\epsilon}\right)^{-\frac{1}{2}} \quad \because \cancel{\frac{\sigma}{w\epsilon} << 1} \quad \begin{matrix}(1+a)^n = (1+na) \\ \left(1-\frac{j\sigma}{w\epsilon}\right)^{-\frac{1}{2}} = \left(1-\left(-\frac{1}{2}\right)\frac{j\sigma}{w\epsilon}\right) \\ = 1+\frac{j\sigma}{2w\epsilon}\end{matrix}$$

$$\boxed{\eta = \sqrt{\frac{\mu}{\epsilon}}\left(1+\frac{j\sigma}{2w\epsilon}\right)}\ \Omega.$$

IV) wave length : $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{w\sqrt{\mu\epsilon}}$ mt.

V) wave Velocity : $v = \frac{w}{\beta} = \frac{w}{w\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}}$ m/sec.

wave propogation in perfect dielectric :
(expression for $\alpha, \beta, \gamma, v, \lambda$ for perfect dielectric medium)

for perfect dielectric $\sigma \simeq 0$.

$$\therefore \left|\frac{J_c}{J_D}\right| = \frac{\sigma}{w\epsilon} \Rightarrow 0.$$

i) attenuation constant :

$$\alpha = w\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{w\epsilon}\right)^2}-1\right)}$$

$$= w\sqrt{\frac{\mu\epsilon}{2}(1+0-1)} \qquad \frac{\sigma}{w\epsilon} \simeq 0.$$

$$\boxed{\alpha = 0}$$

ii) **phase constant :**

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}+1\right)}$$

$$= \omega\sqrt{\frac{\mu\epsilon}{2}(1+0+1)} \qquad \frac{\sigma}{\omega\epsilon} \to 0.$$

$$= \omega\sqrt{\frac{\mu\epsilon}{\cancel{2}}\times\cancel{2}}$$

$$\boxed{\beta = \omega\sqrt{\mu\epsilon}} \text{ rad/mt.}$$

iii) **Intrinsic impedence :**

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\epsilon}} = \sqrt{\frac{j\omega\mu}{\omega\epsilon\left(j+\frac{\sigma}{\omega\epsilon}\right)}}$$

$$= \sqrt{\frac{j\omega\mu}{\omega\epsilon j(1+0)}}$$

$$\boxed{\eta = \sqrt{\frac{\mu}{\epsilon}}}\ \Omega.$$

iv) **phase velocity** : $v = \frac{\omega}{\beta} = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}}$ m/s.

V) **wave length** : $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\epsilon}}$ mt.

conditions to identify the given medium :

1. For good conductor : $\left|\frac{J_c}{J_D}\right| >> 1$
2. " good dielectric (lossy dielectric : $\left(\frac{J_c}{J_D}\right) << 1$
3. " perfect dielectric (lossless dielectric) : $\left|\frac{J_c}{J_D}\right| << 1$ it is of the order of 10^{-3} or less.
4. " free space : $\left|\frac{J_c}{J_D}\right| = 0.$

Important Equations

1. General wave eqn in E & H

$$\nabla^2 E - \mu\sigma \frac{\partial E}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2} = \nabla\left(\frac{\rho_v}{\epsilon}\right)$$

$$\nabla^2 H - \mu\sigma \frac{\partial H}{\partial t} - \mu\epsilon \frac{\partial^2 H}{\partial t^2} = 0.$$

2. General wave eqns in freespace ($\rho_v = \sigma = 0$)

$$\nabla^2 E - \mu\epsilon \frac{\partial^2 E}{\partial t^2} = 0$$

$$\nabla^2 H - \mu\epsilon\, \partial^2 H/\partial t^2 = 0.$$

3. $v = 1/\sqrt{\mu\epsilon}$

4. $\eta = \sqrt{\mu/\epsilon}$ for perfect dielectric.

5. General equations for α, β, η.

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right)} \text{ neper/mt.}$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right)} \text{ Rad/mt.}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

$$\lambda = 2\pi/\beta$$

$$v = \omega/\beta.$$

6. Wave propogation in good conductor.
(value of $\alpha, \beta, \eta, \lambda$ & v)

a. For good conductor $\frac{\sigma}{\omega\epsilon} \gg 1$.
($J_c \gg J_D$)

b. $\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$

c. $\beta = \sqrt{\frac{\omega\mu\sigma}{2}}$

d. $\eta = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$.

e. $v = \frac{2\pi}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}$

f. $\lambda = 2\pi/\beta = 2\pi/\sqrt{\omega\mu\sigma/2}$.

7. Wave propogation in good dielectric

a. $|J_c| \ll |J_D| \Rightarrow \frac{\sigma}{\omega\epsilon} \ll 1$.

b. $\alpha = \frac{\sigma}{2}\sqrt{\mu/\epsilon}$.

c. $\beta = \omega\sqrt{\mu\epsilon}$

d. $\eta = \sqrt{\frac{\mu}{\epsilon}}\left[1 + \frac{j\sigma}{2\omega\epsilon}\right]$

e. $\lambda = 2\pi/\beta$

f. $v = \omega/\beta$.

8. wave propogation in perfect dielectric.

a. for perfect dielectric ($J_c/J_D = 0$), $\Rightarrow \frac{\sigma}{\omega\epsilon} = 0$

b. $\alpha = 0$

c. $\beta = \omega\sqrt{\mu\epsilon}$

d. $\eta = \sqrt{\mu/\epsilon}$

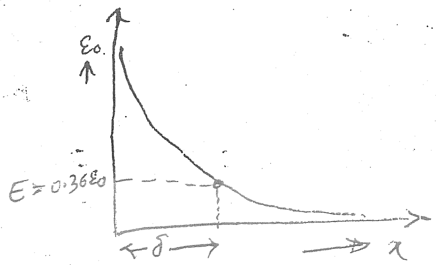
e $v = \omega/\beta$

f $\lambda = 2\pi/\beta$.

9. δ = skin depth

$$\delta = \sqrt{\frac{1}{\pi f \mu\sigma}} = \sqrt{\frac{2}{\omega\mu\sigma}}.$$

Skin effect & skin depth :



when an electromagnetic wave enters in to a conducting medium, its amplitude decreases exponentially & becomes practically zero after penetrating in to a small thickness as shown in the fig. As a result the current induced by the wave exists only near the surface of the conductor. This effect is called <u>skin effect.</u>

If x is the distance travelled in a medium, then electric field intensity ~~and~~ any distance 'x' is given by,

$$E = E_0 e^{-\alpha x} \longrightarrow (1)$$

where, $E_0 \rightarrow$ amplitude of the wave at the time of penetration ($x=0$)

$\alpha \rightarrow$ attenuation constant.

$\therefore$ ~~$E = \frac{1}{e}$~~ Let $x = \delta$ & $x = 1/\alpha = \delta$

$$\therefore E = E_0 e^{-\alpha \cdot 1/\alpha} = E_0 e^{-1}$$

$$\therefore \boxed{E = 0.368 E_0}$$

where δ is called skin depth or depth of penetration.

~~It is d~~ So, the depth of a conductor at which the amplitude of an incident wave reduces to 0.368 times its value of amplitude at the time of incidence is called depth of penetration or skin depth.

$$\therefore \delta = 1/\alpha$$

For good conductor,

$$\alpha = \sqrt{\frac{\omega \mu \sigma}{2}}$$

$$\therefore \delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\frac{\omega\mu\sigma}{2}}}$$

$$= \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$= \sqrt{\frac{2}{2\pi f \mu \sigma}}$$

$$\boxed{\therefore \delta = \sqrt{\frac{1}{\pi f \mu \sigma}}} \text{ mt.}$$

Poynting vector & poynting theorem :

This theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease of energy stored with in the volume minus the ohmic power dissipated. The power radiated is given by, $\boxed{\bar{P} = \bar{E} \times \bar{H}}$

where P is called poynting vector.

Consider $\bar{E} = E_m \cos(\omega t - \beta z)\, \bar{a}_x$ in free space

Now, $\eta = E_m / H_m$

$$\therefore \bar{H} = H_m = \frac{E_m}{\eta} \cos(\omega t - \beta z)\, \bar{a}_y$$

$$\therefore \bar{P} = \bar{E} \times \bar{H}$$

$$= E_m \cos(\omega t - \beta z)\, \bar{a}_x \times \frac{E_m}{\eta} (\cos \omega t - \beta z)\, \bar{a}_y$$

$$\bar{P} = \underline{\underline{\frac{E_m^2}{\eta} \cos^2(\omega t - \beta z)\, \bar{a}_z}} \quad \text{watts}/m^2$$

The average power density is given by,

$$P_{avg} = \frac{1}{T} \int_0^T \frac{E_m^2}{\eta} \cos^2(\omega t - \beta z)\, \bar{a}_z\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{E_m^2}{\eta} \left(\frac{1 + \cos 2(\omega t - \beta z)}{2} \right) \bar{a}_z\, dt \qquad \left(\because \cos^2\theta = \frac{1 + \cos 2\theta}{2} \right)$$

$$= \frac{E_m^2}{2\pi \eta \times 2} \left(t + \frac{\sin (2(\omega t - \beta z)}{2\omega} \right)_0^{2\pi}$$

$$= \frac{E_m^2}{4\pi\eta} \left[\left(2\pi + \frac{\sin(4\pi - 2\beta z)}{2\omega} \right) - 0 - \sin\left(0 - \frac{2\beta z}{2\omega}\right) \right]$$

$$= \frac{E_m^2}{4\pi\eta} \left(2\pi - \frac{\sin 2\beta z}{2\omega} + \frac{\sin 2\beta z}{2\omega} \right)$$

$$\boxed{P_{avg} = \frac{E_m^2}{2\eta}} \quad w/m^2.$$

Integral & point form of poynting theorem:

Consider maxwells equations, $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t} \rightarrow (1)$

& $\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} = \sigma \bar{E} + \epsilon \frac{\partial \bar{E}}{\partial t} \rightarrow (2)$

Taking dot product on both sides of eqn (2) with $\bar{E}$

$$\bar{E} \cdot (\nabla \times \bar{H}) = \bar{E} \cdot \sigma \bar{E} + \bar{E} \cdot \epsilon \frac{\partial \bar{E}}{\partial t} \rightarrow (3)$$

From the vector identity, $\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$

Let $\bar{A} = \bar{E}$ & $\bar{B} = \bar{H}$

$$\nabla \cdot (\bar{E} \times \bar{H}) = \bar{H} \cdot (\nabla \times \bar{E}) - \bar{E} \cdot (\nabla \times \bar{H})$$

$$\therefore \bar{E} \cdot (\nabla \times \bar{H}) = -\nabla \cdot (\bar{E} \times \bar{H}) + \bar{H} \cdot (\nabla \times \bar{E}) \rightarrow (4)$$

putting (4) in (3)

$$-\nabla \cdot (\bar{E} \times \bar{H}) + \bar{H} \cdot (\nabla \times \bar{E}) = \bar{E} \cdot \sigma \bar{E} + \bar{E} \cdot \epsilon \frac{\partial \bar{E}}{\partial t} \rightarrow (5)$$

Now, $\bar{H} \cdot (\nabla \times \bar{E}) = \bar{H} \cdot \left(-\mu \frac{\partial \bar{H}}{\partial t} \right) = -\mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t}$

$$= -\mu \cdot \frac{1}{2} \frac{\partial H^2}{\partial t}$$

$$\frac{\partial}{\partial t}(\bar{H} \cdot \bar{H}) = \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} + \bar{H} \cdot \frac{\partial \bar{H}}{\partial t}$$

$$\frac{\partial}{\partial t}(H^2) = 2\bar{H} \cdot \frac{\partial \bar{H}}{\partial t}$$

$$\therefore \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} = \frac{1}{2} \frac{\partial H^2}{\partial t} \;//$$

Illy $\bar{E} \cdot \frac{\partial \bar{E}}{\partial t} = \frac{1}{2} \frac{\partial E^2}{\partial t}$

so eqn. 5 becomes,

$$-\nabla\cdot \bar{E}\times\bar{H} + \left(-\mu\frac{1}{2}\frac{\partial H^2}{\partial t}\right) = \sigma E^2 + \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t}$$

$$-\nabla\cdot \bar{E}\times\bar{H} = \sigma E^2 + \frac{1}{2}\frac{\partial}{\partial t}[\mu H^2 + \epsilon E^2]$$

$$\boxed{-\nabla\cdot P = \sigma E^2 + \frac{1}{2}\frac{\partial}{\partial t}(\mu H^2 + \epsilon E^2]} \rightarrow 6 \quad (\bar{P} = \bar{E}\times\bar{H})$$

Eqn 6 is known as poynting theorem in point form

Taking volume integration on both sides

$$-\int_v \nabla\cdot\bar{P}\,dv = \int_v \sigma E^2\,dv + \int_v \frac{1}{2}\frac{\partial}{\partial t}(\mu H^2 + \epsilon E^2)\,dv.$$

By divergence theorem, $\int_v (\nabla\cdot\bar{P})dv = \oint_s \bar{P}\cdot\bar{ds}$.

$$\therefore -\oint_s \bar{P}\cdot\bar{ds} = \int_v \sigma E^2\,dv + \int_v \frac{1}{2}\frac{\partial}{\partial t}(\mu H^2 + \epsilon E^2)\,dv. \rightarrow 7$$

Eqn 7 is known as poynting theorem in integral form.

The -ve sign on LHS indicates the power is flowing in to the surface.

||ly The power flowing out of the surface can be written as

$$\oint_s \bar{P}\cdot\bar{ds} = -\int_v \sigma E^2\,dv - \int_v \frac{1}{2}\frac{\partial}{\partial t}(\mu H^2 + \epsilon E^2)\,dv.$$

-ve sign indicates that the power is flowing out of volume or rate of decrease of energy stored with in volume.

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Fourth Semester B.E. Degree Examination, June/July 2016
Field Theory

Time: 3 hrs. Max. Marks: 100

Note: *Answer any FIVE full questions, selecting atleast TWO questions from each part.*

PART – A

1 a. State Gauss theorem of electrostatics. List characteristics of Gaussian surface. **(05 Marks)**

b. Determine electric flux density 'D' in Cartesian coordinates caused at p(6, 8, –10) by i) a point charge of 30 mc at origin ii) infinite line charge with $\rho_r = 40\ \mu c/m$ ii) A surface charge with $\rho_s = 57.2\ \mu c/m^2$ on a plane z = – 9m. **(08 Marks)**

c. Evaluate both side of divergence theorem for the region $r \le a$ (spherical coordinates) having flux density $D = \frac{5r}{3} a_r\ c/m^2$ **(07 Marks)**

2 a. Prove that : $E = -\nabla V$ **(05 Marks)**

b. Determine work done in carrying a charge of –2C from (2, 1, –1) to (8, 2, –1) in an electric field $E = y a_x + x a_y$ v/m along the path $x = 2y^2$. **(07 Marks)**

c. Three point charges 3 coul, 4 coul and 5 coul are to be situated at corner of an equilateral triangle of side 5 m. Find energy density at the centre of triangle. **(08 Marks)**

3 a. Derive Poisson's and Laplace equation. **(06 Marks)**

b. A potential field is given by $v = x^2yz + Ay^3z$ volts determine of 'A' such that v satisfies Laplace equation and hence find electric field E at p(2, 1, –1). **(06 Marks)**

c. A spherical capacitor has a capacitance of 54 pF. It consists of two concentric spheres with inner and outer radii differing by 4 cm. Dielectric in between is air. Determine inner and outer radii. **(08 Marks)**

4 a. State and explain Ampere's circuital law. **(05 Marks)**

b. Determine magnetic flux density 'B' at 'P' for a current loop shown in Fig.Q4(b). **(09 Marks)**

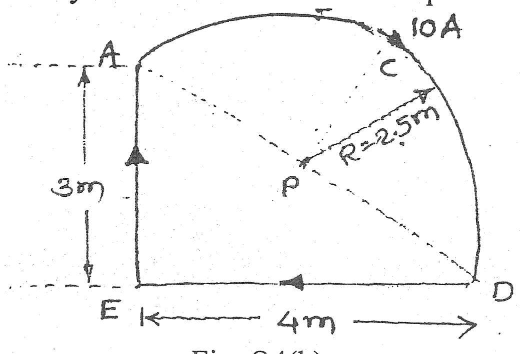

Fig. Q4(b)

c. Clearly distinguish between scalar magnetic potential and vector magnetic potential. **(06 Marks)**

Important Note : 1. On completing your answers, comporily draw diagonal cross lines on the remaining blank ps. 2. Any revealing of identification, app to evaluator and /or equations written eg, 42+8 = 50, wi treated as malpractice.

PART – B

5 a. Derive Lorentz force equation for a moving change placed in a combined electric and magnetic field. **(06 Marks)**

b. A point charge Q = 18 nc moves with a velocity of 5×10^6 m/sec in the direction of $0.06a_x + 0.75a_y + 0.3a_z$. Determine magnitude of force experienced by the charge when placed in i) electric field $E = -3a_x + 4a_y + 6a_z$ kv/m ii) magnetic field $E = -3a_x + 4a_y + 6a_z$ mT iii) combined E and B. **(08 Marks)**

c. An air cored toroid has a cross sectional area of 6 cm^2, a mean radius of 15 cm and is wound with 500 turns and carries a current of 4A. Find the magnetic field intensity at the mean radius. **(06 Marks)**

6 a. Explain Faraday's laws applied to : i) stationary path, changing field and ii) steady field, moving circuit. **(06 Marks)**

b. List Maxwell's equations for both : i) steady and ii) Time varying fields in differential and integral form, also mention the relevant laws they demonstrate. **(08 Marks)**

c. A straight conductor of length 0.2m, lies on x-axis with one end at origin. The conductor is subjected to a magnetic flux density $B = 0.04a_y$ Tesla and the velocity $v = 2.5 \sin 10^3 t\, a_z$ m/sec. Determine motional emf induced in the conductor. **(06 Marks)**

7 a. Derive wave equation for E in a general medium. **(06 Marks)**

b. State and explain Poynting theorem. **(06 Marks)**

c. A lossless dielectric medium has $\sigma = 0$, $\mu_r = 1$ $\varepsilon_r = 1$. A electromagnetic wave has field as $H = -0.1 \cos(\omega t - z)a_x + 0.5 \sin(\omega t - z)a_y$ A/m. Find : i) phase constant, ii) angular velocity iii) the wave impedance iv) components of electric field intensity of the wave. **(08 Marks)**

8 a. Derive an expression for transmission coefficient and reflection coefficient and relate them. **(08 Marks)**

b. Define standing wave ratio. Write an expression for it. **(04 Marks)**

c. Determine the amplitude of reflected and transmitted 'E' and 'H' at the interface between two regions. Characteristics of region 1 are $\varepsilon_{r_1} = 8$, $\mu_{r1} = 0$; $\sigma_1 = 0$ and region 2 is free space. The incident E_0^i in region 1 is of 1.5 V/m. Assume normal incidence. Also find average power in two regions. **(08 Marks)**

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10EE44

Fourth Semester B.E. Degree Examination, Dec.2016/Jan.2017
Field Theory

Time: 3 hrs. Max. Marks:100

Note: ***Answer any FIVE full questions, selecting atleast TWO questions from each part.***

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

PART – A

1 a. State and explain Coulomb's law in vector form. **(05 Marks)**

b. Two point charges $Q_1 = -0.3nC$ at [25, −30, −15], and $Q_2 = 0.5nC$ at [−10, 8, 12] present in free space determine $\vec{E}$ at P(15, 20, 50). **(05 Marks)**

c. Given $D = 4y^2\hat{a}_x + 3x^2y\hat{a}_y + 15\hat{a}_z\ C/m^2$ verify both sides of Divergence theorem and evaluate charge enclosed within region $0 < x, y, z < 2$. **(10 Marks)**

2 a. Find out the work done in moving a charge $\rho = a$ to $\rho = b$ along with radial direction due to infinite line charge. **(06 Marks)**

b. Given a potential $V = 3x^2 + 4y^2(V)$, find the energy stored in volume described by $0 \le x \le 1m$, $0 \le y \le 1m$ and $0 \le z \le 1m$. **(06 Marks)**

c. Obtain the boundary condition between conductor and free space. **(08 Marks)**

3 a. State and prove uniqueness theorem. **(08 Marks)**

b. In spherical co-ordinates V = 0 at r = 0.1 m and V = 100 V at r = 2m. Assuming free space between the concentric spherical shell find $\vec{E}$ and $\vec{D}$. **(06 Marks)**

c. Use Laplace equation to find the capacitance between two plate of a parallel plate capacitor, separated by distance 'd" and maintained at potential "o" and "V_0" respectively. **(06 Marks)**

4 a. Find the magnetic field intensity and flux density at the centre, of a circular wire carrying a current 'I' and of radius 'a' by using Biot – Savart's law. **(06 Marks)**

b. In cylindrical co-ordinates a magnetic field is given as $\vec{H} = [4\rho - 2\rho^2]\hat{a}_\phi\ A/m\ 0 \le \rho \le 1$

i) Find the current density as a function of ρ within the cylinder

ii) Find the total current that passes through the surface z = 0 and $0 \le \rho \le 1m$ in $\hat{a}_z$ direction. **(06 Marks)**

c. Define vector magnetic potential and prove that $A = \frac{\mu_0}{4\pi}\int_v \frac{J}{r}\cdot dv$. **(08 Marks)**

6 - 6
6
8
7 - 10
10
8 - 10
10

PART – B

5 a. Derive an expression for the force between two differential current elements. **(06 Marks)**

b. The z = 0 marks the boundary between two magnetic materials. For region 1, (z > 0), $\mu_1 = 4\ \mu H$ and region 2, (z < 0), $\mu_2 = 6\ \mu H$. The surface current density at the boundary is given as $\vec{K} = 12\hat{a}_y\ A/m$, find $\vec{H}_2$ if $\vec{H}_1 = 40\hat{a}_x + 50\hat{a}_y + 12\hat{a}_z\ kA/m$. **(06 Marks)**

c. Calculate the inductance of a solenoid of 200 turns wound tightly on a cylindrical type of length 60 cm and of diameter 6 cm. Given that the medium is air. Derive the expression used. **(08 Marks)**

6 a. List Maxwell's equations for time varying field in point and integral form. **(06 Marks)**
b. Starting from Ampere's circuital law derive an expression for displacement current density for time varying fields. **(06 Marks)**
c. What is retarded potential? Obtain an expression for retarded potential V and A. **(08 Marks)**

7 a. State and prove Poynting's theorem. **(10 Marks)**
b. With respect to wave propagation in good conductors, describe what is skin effect, derive an expression for the depth of penetration. If $\sigma = 58 \times 10^6$ ℧/m at frequency 10 MHz determine depth of penetration. **(10 Marks)**

8 a. The plane x = 0 is the boundary between two perfect dielectric. For x < 0, $\mu_1 = \mu_0$, $\varepsilon_1 = 3.6\pi$ pf/m and $\sigma_1 = 0$; for x > 0, $\mu_2 = \mu_0$, $\varepsilon_2 = 14.4\pi$ pf/m and $\sigma_2 = 0$.
If $E_i^+ = 60\cos(10^9 t - \beta_1 x) V/m$ find :
i) Incident magnetic field H_i
ii) Reflected electric and magnetic field E_r and H_r
iii) Transmitted electric and magnetic field E_t and H_t **(10 Marks)**
b. What is a standing wave? Derive an expression for standing wave ratio. **(10 Marks)**

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USN | | | | | | | | | | **10EE44**

Fourth Semester B.E. Degree Examination, Dec.2014/Jan.2015
Field Theory

Time: 3 hrs.   Max. Marks: 100

Note: *Answer any FIVE full questions, selecting atleast TWO questions from each part.*

PART – A

1 a. State and explain the Coulomb's law of electrostatic force between two point charges. (05 Marks)

b. A point charge $Q_1 = 25nC$ is located at $P_1(4, -27)$ and a charge $Q_2 = 60nC$ is at $P_2(-3, 4, -2)$ in free space. Find electric field $\bar{E}$ at $P_3(1, 2, 3)$. (05 Marks)

c. Evaluate both sides of the divergence theorem for the field $\bar{D} = 2xy\bar{a}_x + x^2\bar{a}_y\ c/m^2$, the surface is a rectangular parallelepiped formed by planes x = 0 and x = 1, y = 0 and y = 2 and z = 0 and z = 3. (10 Marks)

2 a. Find the potential V due to a line charge density ρ_l c/m, bent in the form of a circular ring of radius 'a'. (05 Marks)

b. Given the potential $V = 2x^2y - 5z$. Determine the expression for electric field intensity $\bar{E}$, the flux density $\bar{D}$ and volume charge density ρ_v. Find the numerical values of V, E, D, ρ_v at a given point P(-4, 3, 6). Given $\epsilon_0 = 8.854 \times 10^{-12}$ F/m. (10 Marks)

c. Define capacitance and evaluate capacitance of two concentric spherical conducting shells of radius a and b with b > a. (05 Marks)

3 a. Derive Poisson's and Laplace's equation. (06 Marks)

b. State and prove uniqueness theorem. (08 Marks)

c. Find the capacitance of a co-axial cable with inner radius a and outer radius b where b > a, using Laplace equation. (06 Marks)

4 a. State and explain Biot-Savart law. (05 Marks)

b. Calculate vector current density at a given point P(2, 3, 4) if $\bar{H} = x^2z\bar{a}_y - y^2x\bar{a}_z$. (05 Marks)

c. State Ampere's circuital law. Apply it to a co-axial cable with inner conductor of radius 'a' carrying current I. The outer conductor carries return current –I. The inner radius of outer conductor is 'b' and its outer radius is 'c'. Evaluate magnetic field intensity. (10 Marks)

PART – B

5 a. Derive the equation for force between two differential current carrying elements. (06 Marks)

b. Explain the terms magnetization and permeability. (06 Marks)

c. Derive the boundary condition between two isotropic homogeneous materials with permeability μ_1 and μ_2. (08 Marks)

6 a. State and explain Faraday's law. (06 Marks)

b. Write Maxwell's equation in integral and point form for time varying fields. (08 Marks)

c. Derive the concept of displacement current density. (06 Marks)

7 a. Derive the wave equation for uniform plane wave propagation in perfect dielectric and explain the concept of loss tangent. (10 Marks)

b. Derive the wave equation for uniform plane wave propagation in perfect conductor and explain the concept of skin effect. (10 Marks)

8 a. Derive reflection coefficient and transmission coefficient equations for a uniform plane wave incident normally at the boundary. (10 Marks)

b. Two media are characterized by intrinsic impedances $\eta_1 = 100\Omega$ and $\eta_2 = 300\Omega$ respectively. For a incident electric field of magnitude 100 v/m calculate reflected and transmitted wave magnitude. Calculate the value of standing wave ratio. (10 Marks)

E_t, E_r $S = \frac{1+\Gamma}{1-\Gamma}$

H_t, H_r

$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

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Vanu Basavaraj

USN 2AN11EE080

10EE44

Fourth Semester B.E. Degree Examination, June/July 2013
Field Theory

Time: 3 hrs. Max. Marks: 100

Note: ***Answer FIVE full questions, selecting at least TWO questions from each part.***

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

PART – A

1 a. State and explain Coulomb's law of force between the two point charges. Also indicate the units of quantities in the force equation. (05 Marks)

b. State and apply Gauss law to obtain an expression for the electric field intensity due to an infinite sheet of charge with a surface charge density ρ_s C/m^2 and area A m^2. (10 Marks)

c. Find : i) Electric field intensity and ii) Electric flux density at the origin due to $Q_1 = 0.35\,\mu C$ at (0, 4, 0) m and $Q_2 = -0.55\,\mu C$ at (3, 0, 0) m. (05 Marks)

2 a. Explain with mathematical expressions: i) Potential difference ii) Absolute potential iii) Potential gradient. (06 Marks)

b. Derive an expression for the equation of continuity of current. (06 Marks)

c. At the boundary between glass ($\epsilon_r = 4$) and air, the lines of electric field make an angle of 40° with normal to the boundary. If electric flux density in air is 0.25 $\mu C/m^2$, determine the orientation and magnitude of, i) Electric flux density and ii) Electric field intensity, in glass. (08 Marks)

3 a. Derive Poisson's and Laplace equations starting from point form of Gauss law. (06 Marks)

b. Using Laplace equation derive an expression for the capacitance of a concentric spherical capacitor. The inner spherical conductor is of radius 'a' and potential V, while outer conductor is of radius 'b' and potential zero. (08 Marks)

c. Determine whether or not the following potential fields satisfy Laplace's equation :
i) $V = 2x^2 - 3y^2 + z^2$ ii) $V = r^2 + z^2$ (06 Marks)

4 a. Write an explanatory note on Biot Savarts law. (04 Marks)

b. Discuss the concept of scalar and vector magnetic potential and arrive at the expressions for Poissons equation in magnetostatics. (08 Marks)

c. State and prove ampere's circuital law and apply it to a straight solid conductor to calculate the magnetic field intensity. (08 Marks)

PART – B

5 a. Find the expression for the force on differential current carrying elements. (06 Marks)

b. Define Lorentz force equation and mention the application of its solution. (06 Marks)

c. Calculate the inductance of a Solenoid of 200 turns wound tightly on a cylindrical tube of length 60 cm and of diameter 6 cm, with air as media. Derive the expression used. (08 Marks)

6 a. With necessary relationships, explain Faradays law of electromagnetic induction for both static and time varying conditions. (10 Marks)

b. Starting from Faradays law of electromagnetic induction derive $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. (06 Marks)

c. Find the frequency at which conduction current density and displacement current density are equal in a medium with $\sigma = 2\times10^{-4}$ s/m and $\epsilon_r = 81$. (04 Marks)

7 a. What is uniform plane wave? Explain its propagation in free space with necessary equation. (08 Marks)

b. Define skin depth and depth of penetration. (08 Marks)

c. For copper the conductivity is 58 mega-s/m. Find the skin depth at a frequency of 10 MHz. (04 Marks)

8 a. With necessary equations, explain standing wave ratio. (10 Marks)

b. Find weather the wet, marshy soil characterized by $\sigma = 10^{-2}$ s/m, $\epsilon_r = 15$ and $\mu_r = 1$ may be considered as a conductor, a dielectric or neither for the frequencies: i) 60 Hz ii) 1 MHz iii) 100 MHz iv) 10 GHz. (10 Marks)

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USN | | | | | | | | | |

10EE44

Fourth Semester B.E. Degree Examination, June 2012

Field Theory

Time: 3 hrs. Max. Marks:100

Note: *Answer FIVE full questions, selecting at least TWO questions from each part.*

PART – A

1 a. State and explain Coulomb's law in vector form. Derive an expression for the electric field intensity at a point due to 'x' number of charges. **(06 Marks)**

b. A charge of 1 C is at (2, 0, 0). What charge must be placed at (–2, 0, 0) which will make Y-component of total electric field intensity zero at the point (1, 2, 2). **(05 Marks)**

c. Calculate the divergence of $\overline{D}$ at the point specified if

i) $\overline{D} = \frac{1}{z^2}\left[10xyz\overline{a}_x + 5x^2z\overline{a}_y + (2z^3 + 5x^2y)\overline{a}_x\right]$ at P[–2, 3, 5]

ii) $\overline{D} = 5z^2\overline{a}_\rho + 10\rho z\overline{a}_z$ at P(3, –45°, 45)

iii) $\overline{D} = 2r\sin\theta\sin\phi + r\cos\theta\sin\overline{a}_\theta + r\cos\phi\overline{a}_\phi$ at $P(3, 45°, -45°)$ **(09 Marks)**

2 a. Derive an expression for energy expended in moving a point charge in an electric field. **(06 Marks)**

b. Calculate the potential difference between A and B for a line charge density $\rho_L = 0.25$ nc on the z-axis when point A(12m, $\frac{\pi}{2}$, 0) and point B(4m, $\frac{\pi}{2}$, 3m). **(04 Marks)**

c. Potential is given by $V = 2(x+1)^2(y+2)^2(z+3)^2 V$ in free space. Calculate:

i) Electric potential

ii) Flux density at a point A(1, 2, 3) **(04 Marks)**

d. Obtain the boundary condition between conductor and free space. **(06 Marks)**

3 a. State and prove uniqueness theorem. **(06 Marks)**

b. Use Laplace's equation to find the capacitor per unit length of a co-axial cable of inner radius 'a' m and outer radius 'b' m. Assume $V = V_o$ at r = a and V = 0 at r = b. **(08 Marks)**

c. Determine whether or not potential equations: i) $V = 2x^2 - 4y^2 + z^2$ and ii) $V = r^2\cos\phi + \theta$ Satisfy the Laplace's equations. **(06 Marks)**

4 a. Using Biot-Savart's law, obtain magnetic field intensity expression due to an infinite length conductor carrying current I. **(06 Marks)**

b. State and prove Ampere's circuital law. **(06 Marks)**

c. Given the magnetic field $\overline{H} = 2r^2(z+1)\sin\phi\,\overline{a}_\phi$. Verify Stoke's theorem for the portion of a cylindrical surface defined by r = 2, $\frac{\pi}{4} < \phi < \frac{\pi}{2}$, 1 < z < 1.5 and for its perimeter. **(08 Marks)**

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PART – B

5 a. Obtain expression for the force between differential current elements. **(06 Marks)**

b. A rectangular loop in z = 0 plane has corners at (0, 0, 0), (1, 0, 0), (1, 2, 0) and (0, 2, 0). The loop carries a current of 5A in $\bar{a}_x$ direction. Find total force and torque on the loop produced by the magnetic field $\bar{B} = 2\bar{a}_x + 2\bar{a}_y - 4\bar{a}_z wb/m^2$. **(08 Marks)**

c. Calculate the inductance of a solenoid of 200 turns wound tightly on a cylindrical tube of length 60 cm and of diameter 6 cm. Given that medium is air. Derive the expression used. **(06 Marks)**

6 a. State and explain Faraday's law of electromagnetic induction. **(06 Marks)**

b. List Maxwell's equations in differential form for both steady fields and time varying field. **(06 Marks)**

c. What is displacement current? Find the displacement current density within a parallel-plate capacitor having a dielectric with $\epsilon_r = 10$, area of plates = 0.01 m^2, distance of separation = 0.05 mm and the capacitor voltage is 200 sin 200t. **(08 Marks)**

7 a. Obtain the solution of wave equations for uniform plane wave propagating in free space. **(10 Marks)**

b. State and prove Poynting's theorem. **(06 Marks)**

c. The magnetic field intensity of uniform plane wave in air is 20 A/m in $\bar{a}_y$ direction. The wave is propagating in the $\bar{a}_z$ direction at an angular frequency of 2×10^9 r/s. Find:

i) Phase shift constant
ii) Wavelength
iii) Frequency
iv) Amplitude of field intensity **(04 Marks)**

8 a. Derive the expression for transmission coefficient and reflection coefficient. **(10 Marks)**

b. With necessary expression, explain standing wave ratio. **(10 Marks)**

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06EE44

Fourth Semester B.E. Degree Examination, June/July 2011
Field Theory

Time: 3 hrs. Max. Marks:100

Note: *1. Answer FIVE full questions selecting at least TWO questions from each part.*
2. Assume missing data, if any.

PART – A

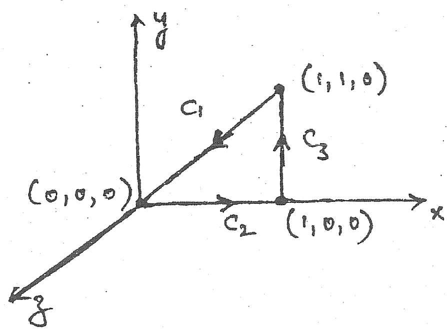
1 a. State and explain Coulomb's law for electrostatic force between two point charges. Workout the vectorial expression for electric field intensity due to a point charge. **(05 Marks)**

b. Define electronic flux density and explain the mathematical interpretation of vectorial evaluation of electric flux density on the basis of Gauss's law. **(05 Marks)**

c. Two very small conducting spheres, each of mass 1×10^{-4} kg are suspended at a common point by very thin filaments of length 0.2m. A charge Q coulomb is placed on each sphere. The electric force of repulsion separates the spheres and an equilibrium is reached when the suspending filaments make an angle of 10^0. Assuming $\epsilon_r = 1$, $g = 9.8$ Nw/kg and negligible mass for the filaments, find Q. **(10 Marks)**

2 a. Show that vector electric field, E is the negative gradient of scalar electric potential, V. **(05 Marks)**

b. The electric potential at an arbitrary point in free space is given by the expression $V = (-2xy + 3)$ volts. Show that $\oint_C \vec{E}.\vec{dl} = 0$ for the closed contour shown in Fig. Q2 (b).



(05 Marks)

c. Derive the expression for energy stored and energy density in an electrostatic field. **(10 Marks)**

3 a. Mention the properties of 'Conductor'. Distinguish conduction current density and displacement current density through the expressions derived for each. **(05 Marks)**

b. Derive the equation of continuity and show that for steady current case, divergence of $\vec{J}$ is zero. **(05 Marks)**

c. Derive the boundary conditions at the interface between the conductor and the free space. **(10 Marks)**

4 a. State and prove 'Uniqueness theorem'. **(05 Marks)**

b. Mention the statements of 'Biot-Savart's law', 'Ampere's circuit law' and 'stoke's theorm'. **(05 Marks)**

c. Two perfectly conducting planes of infinite extent in z-direction are arranged at an angle of 30^0 and are bounded by cylindrical surfaces at $\rho = 1.1$m and $\rho = 0.2$m. One plate is held at a potential of 1kV and the other is grounded. Find the potential distribution, vector E and the capacitance per unit length of the systems, with $\epsilon_r = 1$. **(10 Marks)**

PART – B

5 a. Derive the expression for the magnetic torque due to a rectangular current loop. **(05 Marks)**

b. Work out the expression for the self-inductance per unit length of an infinitely long solenoid. **(05 Marks)**

c. The $z = 0$ plane marks the boundary between two magnetic media. Medium 1 is the region $z > 0$ and medium 2 is the region $z < 0$. The magnetic flux density in the medium 1is, $B_1 = 1.5\,\hat{a}x + 0.8\,\hat{a}y + 0.6\,\hat{a}z\ \text{mT}$. Find the magnetic flux density of medium 2. Assume medium 1 as free space and μ_r of medium 2 as 100. **(10 Marks)**

6 a. Explain the interpretation of Faraday's law applicable to time-varying magnetic field and derive the expressions for 'Transformer e.m.f' and 'Motional e.m.f'. **(05 Marks)**

b. State the integral forms of Maxwell's equations applicapable to time-varying magnetic fields. Give the wordly description of each statement also. **(05 Marks)**

c. A certain material has conductivity $\sigma = 0$ and relative permeability $\mu_r = 1$. Make use of maxwell's equations to find i) $\vec{H}(z, t)$ and ii) ϵ_r. **(10 Ma)**

7 a. Obtain the relation between $\vec{E}$ and $\vec{H}$ in a perfect dielectric medium and comment on the values on intrinsic impedance in a medium other than free space. **(10 Marks)**

b. If the electronic field vector in free space is $\vec{E} = 800\text{CoS}\,(10^8 t - \beta y)\hat{a}z$ V/m, find i) β ii) λ And iii) $\vec{H}$ at the point $P\,(1, 1.5, 0.4)$ at $t = 8\text{ns}$. **(05 Marks)**

c. Discuss the wave propagation in good conductions. In this context, define "skin depth, δ". **(05 Marks)**

8 a. Discuss the reflection phenomenon of uniform plane wave at normal incidence at the boundary between two lossless media. **(10 Marks)**

b. Explain the nature of standing waves in a good conductor and obtain the expression for SWR. **(10 Marks)**

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USN | | | | | | | | | | 06EE44

Fourth Semester B.E. Degree Examination, May/June 2010
Field Theory

Time: 3 hrs. Max. Marks:100

Note: *Answer any FIVE full questions, selecting at least TWO questions from each part.*

PART – A

1 a. Define divergence of a vector. What do positive and negative divergences represent? **(04 Marks)**

$\nabla \cdot \bar{D} = \lim_{\Delta v \to 0} \frac{\oint \bar{D} \cdot d\bar{s}}{\Delta V}$

b. Two uniform line charges of density 4nc/m and 6 nc/m lie in x = 0 plane at y = +5 m and – 6m respectively. Find $\bar{E}$ at (4, 0, 5)m. $= 15.32\ a_x + 3.66\ a_y$ **(06 Marks)**

c. State and explain Gauss's law. Given that $\bar{D} = \frac{\rho^2 z^2}{3}\cos\phi\,\bar{a}_\phi$, determine the flux crossing $\phi = \pi/4$ half plane defined by $0 \le \rho \le 3$ and $2 \le z \le 4$. **(10 Marks)**

$\bar{D} \cdot d\bar{s} = \frac{\rho^2 z^2}{3}\cos\phi\ d\rho\, dz$

$= 39.6\ c$

2 a. Show that the energy required to assemble 'n' number of point charges is $W_E = \frac{1}{2}\sum_{m=1}^{n} Q_m V_m$ and hence derive expression for energy in electric field in terms of field quantities $\bar{D}$ and $\bar{E}$. **(08 Marks)**

b. For a line charge $\rho_l = \frac{10^{-9}}{2}$ c/m on the z-axis, find V_{AB} where A is (2m, $\pi/2$, 0) and B is (4m, π, 5m). **(06 Marks)**

c. Find the workdone in assembling four qual point charges of 1 μc each on X and Y axis at $\pm$3m and $\pm$4m respectively. **(06 Marks)**

3 a. Starting from Gauss's law in integral form, derive Laplace's and Poisson's equations in Cartesian coordinates. **(07 Marks)**

b. Determine the expression for $\bar{E}$, in cylindrical coordinates, between two planes insulated along Z-axis, assuming a potential of 100 V for $\phi = \alpha$ and zero reference at $\phi = 0°$. **(06 Marks)**

c. Calculate the capacitance/unit length of two co-axial cylindrical conductors in free space. If the space between the cylinders were filled with dielectric, how would the dielectric constant of the dielectric have to depend on the distance 'r' from the axis, in order that the electric field intensity be independent of 'r'. **(07 Marks)**

4 a. State and explain Ampere's circuital law. **(04 Marks)**

b. Derive the Gauss's law for the magnetic field in point form. Hence show that scalar magnetic potential follows Laplace's equation. **(04 Marks)**

c. Given the field $\bar{H} = 6r\sin\phi\,\bar{a}_r + 18r\sin\theta\cos\phi\,\bar{a}_\phi$. Evaluate each side of Stoke's theorem for portion of a spherical surface specified by r = 4, $0 \le \theta \le 0.1\pi$, $0 \le \phi \le 0.3\pi$ and a closed path forming its perimeter. **(12 Marks)**

Important Note : 1. On completing your answer ..., compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

PART – B

5 a. Derive the boundary conditions to apply to B and H at the interface between two different magnetic materials. **(08 Marks)**

b. The point charge Q = 18 nc has a velocity of 5×10^{6} m/s in the direction $\bar{a}_v = 0.60\bar{a}_x + 0.75\bar{a}_y + 0.3\bar{a}_z$. Calculate the magnitude of the force exerted on the charge by the field

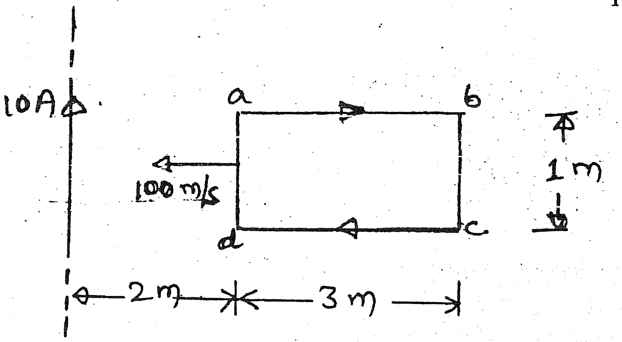
i) $\bar{E} = -3\bar{a}_x + 4\bar{a}_y + 6\bar{a}_z$ kV/m ii) $\bar{B} = -3\bar{a}_x + 4\bar{a}_y + 6\bar{a}_z$ mT iii) $\bar{B}$ and $\bar{E}$ acting together. **(08 Marks)**

c. Find the force/mtr length between two long parallel wires separated by 10 cm in air and carrying a current of 100 A in opposite directions. State the nature of force between the wires. **(04 Marks)**

6 a. Derive the continuity equation from Maxwell's equation. **(04 Marks)**

b. For time varying field, show that $\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t}$, where $\bar{A}$ is vector magnetic potential. **(08 Marks)**

c. A rectangular loop is approaching a long straight current carrying conductor as shown in Fig.Q6(c). For the position shown, find the total induced emf in the loop. **(08 Marks)**



0·4 μ now

Fig.Q6(c)

7 a. Determine the relation between E and H of an EM wave traveling in free space along x direction. **(10 Marks)**

b. A 160 MHz plane wave penetrates through aluminium of conductivity 10^{5} mhos/mtr, $\epsilon_r = \mu_r = 1$. Calculate the skin depth and also depth at which the wave amplitude decreases to 13.5% of its initial value. **(06 Marks)**

c. 8 Watts/m^2 is the Poynting vector of a plane wave traveling in free space. What is the average energy density? **(04 Marks)**

8 a. Discuss the reflection of uniform plane waves at normal incidence. Hence derive expressions for transmission and reflection coefficient. **(10 Marks)**

b. A conductor of circular cross-sectional area of radius 'a' is carrying a current of I amps. Show that the surface integral of the Poynting vector over the surface of the conductor gives the total power dissipated in the conductor. Given the conductivity of the material $\sigma = 1/\rho$. **(10 Marks)**

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Important Note: 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages. 2. Any revealing of identification, appeal to evaluator and/or equations written eg, 42+8=50, will be treated as malpractice.

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06EE44

Fourth Semester B.E. Degree Examination, Dec.09-Jan.10

Field Theory

Time: 3 hrs. Max. Marks:100

Note: *Answer any FIVE full questions, selecting at least TWO questions from each part.*

PART – A

1 a. Given $D = [10r^2 + 5e^{-r}]\ \hat{a}_r\ {}^{c}/_{m^2}$. Find the following :

i) ρ_v as a function of r

ii) The total charge enclosed by a sphere of radius a,centered at the origin. **(08 Marks)**

b. Derive an expression for electric field intensity due to circular disc of charge density $\rho_s\ {}^{c}/_{m^2}$. **(05 Marks)**

c. Derive an expression for electric field intensity due to an infinite line charge of linear charge density ρ_L, using Gauss law. **(07 Marks)**

2 a. Prove that $E = -\nabla V$. **(04 Marks)**

b. Determine the work done in carrying a -2μc charge from $P_1(2,1,-1)$ to $P_2(8,2,-1)$ in the field $\vec{E} = Y\hat{a}_x + x\hat{a}_y\ {}^{V}/_{m}$, along the parabola $x = 2y^2$. **(08 Marks)**

c. With usual notations, derive boundary conditions at the boundary between a dielectric and a conductor in an electric field. **(08 Marks)**

3 a. Using Laplace equation, derive an expression for the capacitance of a concentric spherical capacitor. **(08 Marks)**

b. State and prove uniqueness theorem. **(07 Marks)**

c. If the field of a region of space is given by $\vec{E} = \hat{a}_z(5\cos z)$, is the region free of charge? **(05 Marks)**

4 a. Given the field $\vec{H} = 20\,r^2\,\hat{a}_\phi\ {}^{A}/_{m}$;

i) Determine the current density $\vec{J}$.

ii) Integrate $\vec{J}$ over the circular surface r = 1, $0 < \phi < 2\pi$ z= 0, to determine the total current passing through that surface in the $\hat{a}_z$ direction. **(08 Marks)**

b. Derive the expressions for scalar and vector magnetic potential. **(08 Marks)**

c. Prove that vector magnetic potential satisfies Poisson's equation. **(04 Marks)**

PART – B

5 a. Define self inductance and mutual inductance with suitable formulae. **(04 Marks)**

b. A solenoid with air core has 2000 turns and a length of 500mm. Core radius is 40mm. Find its inductance. Derive the formula used. **(08 Marks)**

c. For the square loop of wire in the z = 0 plane carrying 2mA in the field of an infinite filament on the Y-axis, as shown in Fig.Q5(c), calculate the total force on the loop.

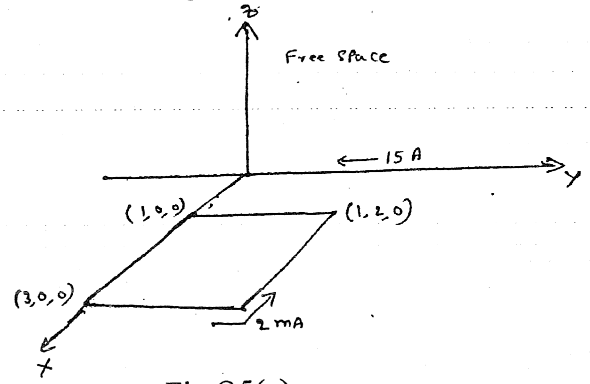

Fig.Q5(c) (08 Marks)

6 a. Starting from the concept of Faraday's law of electromagnetic induction, derive the Maxwell's equation, $\nabla \times E = -\frac{\partial B}{\partial t}$. (06 Marks)

b. Find the frequency at which conduction current density and displacement current density are equal in a medium with $\sigma = 2\times10^{-4}$ S/m and $\varepsilon_r = 81$. (04 Marks)

c. Explain the concept of retarded potentials. Derive the expressions for the same. (10 Marks)

7 a. A radio station transmits power radially around the spherical region. The desired electric field intensity at a distance of 10km from the station is 1 mv/m. Calculate the corresponding H, P and station power. (06 Marks)

b. State and prove Poynting theorem. (06 Marks)

c. For an electromagnetic wave, prove that $\alpha = \omega\sqrt{\frac{\mu \in}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega \in}\right)^2} - 1\right]^{1/2}$ and

$$\beta = \omega\sqrt{\frac{\mu \in}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega \in}\right)^2} + 1\right]^{1/2}$$

where α = attenuation constant and β = phase constant. (08 Marks)

8 a. Define the terms :

i) Reflection coefficient and

ii) Transmission coefficient.

Also bring out the relation between them. (08 Marks)

b. Write short note on SWR. (05 Marks)

c. A uniform plane wave in air partially reflects from the surface of a material whose properties are unknown. Measurements of the electric field in the region in front of the interface yield 1.5m spacing between the maxima with the first maximum occurring 0.75m from the interface. A standing wave ratio of 5 is measured. Determine the intrinsic impedance of the unknown material. (07 Marks)

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06EE44

Fourth Semester B.E. Degree Examination, June-July 2009
Field Theory

Time: 3 hrs. Max. Marks:100

Note: *Answer any FIVE full questions, selecting at least TWO questions from each part.*

PART – A

1 a. State and explain experimental law of Coulomb. **(05 Marks)**

b. Identical point charges of $3\mu C$ are located at the four corners of the square of 5cm side, find the magnitude of force on any one charge. **(08 Marks)**

c. Using Gauss law, determine electric field intensity every where due to a hollow sphere of charge. **(07 Marks)**

2 a. Obtain an expression for the energy expanded in moving a point charge in an electric field. **(05 Marks)**

b. Potential is given by $V = 2(x+1)^2(y+2)^2(z+3)^2$ volts in free space. At a point P(2, −1, 4) calculate i) Potential ii) Electric field intensity iii) Flux density and iv) Volume charge density. **(08 Marks)**

c. Obtain boundary conditions for dielectric-dielectric boundary. **(07 Marks)**

3 a. Explain Poisson's equation & Laplace equation. **(05 Marks)**

b. Given the potential field $V = [Ar^4 + Br^{-4}]\sin 4\phi$ volts. Show that $\nabla^2 v = 0$, select A & B so that $v = 100$ volts and $|\vec{E}| = 500$ v/m at $P(r = 1, \phi = 22.5°, z = 2)$. **(08 Marks)**

c. State and prove Uniqueness theorem. **(07 Marks)**

4 a. Using Biot Savart's law, obtain magnetic field intensity expression due to an infinite length conductor carrying current I. **(05 Marks)**

b. Derive the general expression for the field $\vec{B}$ at any point along the axis of a solenoid. **(08 Marks)**

c. Define vector magnetic potential. Prove that $A = \int_v \frac{\mu_0 J dv}{4\pi R}$. **(07 Marks)**

PART – B

5 a. Derive Lorentz force equation and mention the application of the solution. **(05 Marks)**

b. Derive an expression for the force on a differential current element placed in a magnetic field. Find the force per meter length between two long parallel wires separated by 10cm in air and carrying a current of 10A in the same direction. **(08 Marks)**

c. Derive differential form of continuity equation. **(07 Marks)**

6 a. What is the inconsistency of Ampere's law with the equation of continuity? Derive the modified form of Ampere's law of Maxwell. **(05 Marks)**

b. Given $\vec{E} = E_0 \sin(wt - \beta z)\vec{a}y$ v/m in free space. Find i) $\vec{D}$ ii) $\vec{B}$ iii) $\vec{H}$. Sketch $\vec{E}$ & $\vec{H}$ at $t = 0$. **(08 Marks)**

c. Write Maxwell's equation in point form and in integral form for time varying fields. **(07 Marks)**

7 a. Define wave equation. Derive the wave equation for $\vec{E}$ in a general medium. **(05 Marks)**

b. For an electromagnetic wave propagating in free space, prove that

i) $\frac{|\vec{E}|}{|\vec{H}|} = \eta$ ii) $\vec{E}$ & $\vec{H}$ are mutually perpendicular **(08 Marks)**

c. State and prove Poynting theorem. **(07 Marks)**

8 a. Define 'depth' of penetration'. Show that depth of penetration of a wave in a conductor decreases with an increase in frequency. **(05 Marks)**

b. Show that at any instant the magnetic and electric field in a reflected wave are out-of phase by 90°. **(08 Marks)**

c. Define Brewster's angle. Derive the necessary expression in terms of permittivity. **(07 Marks)**

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06EE44

Fourth Semester B.E. Degree Examination, June / July 08

Field Theory

Time: 3 hrs. Max. Marks:100

Note : ***Answer any FIVE full questions, selecting atleast two questions from each part.***

PART - A

1 a. Derive the relation between vector E and scalar V. State Maxwell's equations applicable to electrostatic fields. **(05 Marks)**

b. Consider a uniform ring charge of radius a. Derive the general expression for electric field vector E at a height h (h < a), along the axis of the ring charge and normal to its plane. **(05 Marks)**

c. Point charges of 120 nano coulomb each are located at A(0, 0, 1) and B(0, 0, -1) in free space. Find vector E at P(0, 0.5, 0). What single charge at the origin would provide the identical field strength as calculated at P? **(10 Marks)**

2 a. State Gauss's Law. Derive an expression for the electric field vector due to an infinite line charge using Gauss's law. Assume the linear charge density of the charge distribution to be ρ_l C /m. **(05 Marks)**

b. Two 6 nano coulomb point charges are located at (1, 0, 0) and (-1, 0, 0) in free space. Find electric potential V at P(0, 0, z) and what is its maximum value? **(05 Marks)**

c. If a potential of $V = x^2yz + Ay^3z$ Volts, i) Find A so that the Laplace's equation is satisfied ii) With that value A, determine the electric field at a point P whose coordinates are (2, 1, -1). **(10 Marks)**

3 a. Let $\vec{G} = 4x\hat{a}_x + 2z\hat{a}_y + 2y\hat{a}_z$, given an initial point P(2, 1, 1) and a final point M(4, 3, 1), find $\int \vec{G}.\vec{d\ell}$ using the path along a straight line : y = x – 1, z =1. **(05 Marks)**

b. State and prove Uniqueness theorem. **(05 Marks)**

c. Perform the analysis for the divergence of electric flux density vector D, with respect to differential cubical volume having a charge at its symmetric centre. Extend the analysis to work out the Maxwell's equation in the form $\nabla \cdot \vec{D} = \rho_v$. **(10 Marks)**

4 a. State Biot –Savart's law. Derive the expression for magnetic flux density at a given point due to a current carrying element of finite length. **(05 Marks)**

b. Explain the concept of vector magnetic potential. **(05 Marks)**

c. Conducting spherical shells with radius a = 10 cm and b = 30 cm are maintained at a potential difference of 100 V such that $V_{(r = a)} = 0$ and $V_{(r = b)} = 100$ V. Determine V and vector E in the region between the shells. If $\varepsilon_r = 2.5$ in the region, determine the total charge induced on the shells and the capacitance thereon. **(10 Marks)**

PART - B

5 a. Derive the expression for the force between two current loops. **(05 Marks)**

b. Given the vector $E = 10 \sin(\omega t - \beta z)\ \hat{a}_y$ V /m, in free space, determine the vectors D, B and H. **(05 Marks)**

c. Workout the Lorentz force equation for the case of moving charge in the presence of electric and magnetic fields. **(10 Marks)**

6 a. State Maxwell's equations in point and integral forms for time varying fields. **(05 Marks)**

b. Two homogeneous, linear and isotropic media have an interface at x = 0. The region x < 0 describes medium 1 and x > 0 describes medium 2. $\mu_{r1} = 2$ and $\mu_{r2} = 5$. The magnetic field in medium 1 is $150\hat{a}_x - 400\hat{a}_y + 250\hat{a}_z$ A /m. Determine i) Magnetic field in medium 2 ii) Magnetic flux density in medium 1. **(05 Marks)**

c. Derive the Maxwell's equation (based on Ampere's circuit law) $\nabla \times \hat{H} = J + J_d$ for time – varying field. The term $J_d = \partial\hat{D}/\partial t$ is known as displacement current density and J is the conduction current density. **(10 Marks)**

7 a. State and prove Poynting's theorem. **(10 Marks)**

b. If the electric field strength of a radio broadcast signal at a TV receiver is given by vector $E = 5.0 \cos(\omega t - \beta y)\ \hat{a}_z$, V /m, determine the displacement current density. If the same field exists in a medium whose conductivity is given by 2.0×10^3 (mho) /cm, find the conduction current density. **(10 Marks)**

8 a. Explain he phenomenon of Skin effect and its significance. **(05 Marks)**

b. Discuss the phenomenon of wave propagation in lossy dielectrics. **(05 Marks)**

c. Derive the wave equation for vector E and H fields in a conducting medium. **(10 Marks)**

Transverse nature of E.M waves.

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EE403

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Fourth Semester B.E. Degree Examination, July/August 2003

Electrical and Electronics Engineering

Field Theory

Time: 3 hrs.] [Max.Marks : 100

Note: *1. Answer any FIVE full questions only.*
2. Assume the missing data, if any, suitably.

1. (a) State and explain coulombs law in complete vector form and thereby obtain an expression for the electric field intensity in Cartesian coordinate system. **(10 Marks)**

(b) Find the total charge inside the volume indicate by $\rho_v = 10Z^2e^{-0.1x}\ sin\pi y$ where $-1 \leq x \leq 2,\ 0 \leq y \leq 1,\ 3 \leq z \leq 3.6$ **(5 Marks)**

(c) Three charges $Q_1 = -1\mu e,\ Q_2 = -2\mu e\ and\ Q_3 = -3\mu c$ are placed at the corners of an equilateral triangle. If the length of each side is 1m, find the magnitude and direction of electric field at the point bisecting the line between the charges Q_2 and Q_3 **(5 Marks)**

2. (a) Define and explain the following terms :

i) Electric field intensity

ii) Electric flux density

iii) Gaussian surface. **(6 Marks)**

(b) Three point charges of capacity Q coulombs each are located at the three corners of an equilatera' triangle of side 2m each. Find the location and magnitude of the charge to be placed for the charges to be in equilibrium. **(7 Marks)**

(c) A circular ring of inner radius 1m and outer radius 2m has a surface charge density $\rho_s = 100/r\ \mu c/m^2$. Determine the resulting $\vec{E}$ field on the axis of the ring 10 m away from the center of the ring. **(7 Marks)**

3. (a) State and explain Divergence theorem. Derive an expression for the same. **(6 Marks)**

(b) The plane Z=0 marks the boundary between free space and a dielectric medium with a dielectric constant of 40. the $\vec{E}$ field next to the interface in free space is $\vec{E} = 13\ \vec{a}_x + 40a\vec{y} + 50\vec{a}_z\ V/mt$. Determine the $\vec{E}$ field on the otherside of the interface. **(6 Marks)**

(c) The radii of two spheres differ by 2 cms and the capacitance of the spherical capacitor is 53.33 pF. If the outer sphere is earthed calculate the radii assuming air as the dielectric medium. **(8 Marks)**

4. (a) State and explain Biot-Savart Law. **(4 Marks)**

(b) State and prove Stokes theorem. **(6 Marks)**

(c) Using Biot-Savart law find magnetic field intensity at any point on the axis of a circular loop of radius 'a' carrying a current I. The point is at a distance h on the Z axis from the center of the loop. Also determine $\frac{\partial B_y}{\partial y}$ at a point P(0,0,Z). **(10 Marks)**

Contd.... 2

5. (a) Derive an expression for $\vec{H}$ field due to an infinitely long co-axial transmission line and draw the variation of $\vec{H}$ field as a function of radius of the line. (10 Marks

(b) The magnetic field $\vec{H} = 2\rho^2(Z+1)\ sin^2\phi\ \vec{a}\phi$ AT/m. Verify Stokes theorem for the portion of a cylindrical surface defined by $\rho = 2$, $\frac{\pi}{4} < \phi < \frac{\pi}{2}$, $1 < Z < 1.5$ and for its perimeter. (10 Marks

6. (a) Derive a general expression for the field intensity $\vec{H}$ at a point along the axis of a solenoid. (10 Marks

(b) Two homogenous, linear, isotopic materials have interface at x=0 in which there is a surface current. We have $K = 200\ a_z\ A/m$.

For $x < 0$, $\mu_n = 2$ *and* $H_1 = 150\vec{a}_x - 400\vec{a_y} + 250\vec{a}_z\ AT/m$ in region z, wher $x > 0$. Find

i) H_2 ii) Magnitudes of B_1 *and* B_2 iii) Values of α_1 *and* α_2 (0 Marks

(c) The magnetic flux density in a finitely conducting cylinder of radius 10 cms and with a relative permeability of 5 is found to vary as $\frac{0.2}{\rho}\ \vec{a}_\phi$ Tesla. If the region surrounding the cylinder is characterized by free space, determine the magnetic flux density just outside the cylinder. (4 Marks

7. (a) Derive an expression for the relation between $\vec{E}$ *and* $\vec{H}$ in a nonconducting medium of lossy dielectrics. (10 Marks

(b) Find the group velocity for a 100 MHz wave for a normally dispersive loss less medium for which the phase velocity $V = 2 \times 10^7\ \lambda^{2/3}$ m/sec. (5 Marks

(c) A plane wave is travelling in the +ve x direction in a loss less unbounded medium having permeability which is same as free space, and permitivity nine times that of free space.

i) Find the phase voltage of the wave.

ii) If the electric field intensity has only y component with an amplitude of 10V/m, find the magnitude and direction of $\vec{H}_z$ field. (5 Marks

8. (a) Enumerate the Maxwell's equations for time varying electric and magnetic fields. (10 Marks

(b) State and prove poynting theorem starting from fundamentals. (10 Marks

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EE403

Reg. No. [ ][ ][ ][ ][ ][ ][ ]

Fourth Semester B.E. Degree Examination, August 2001

Electrical And Electronics Engineering

Field Theory

Time: 3 hrs.] [Max.Marks : 100

Note: *Answer any FIVE full questions.*

1. (a) Derive an expression for the electric field intensity due to a uniformly charged circular conductor with a linear charge density of $\lambda C/M$ at any point on the axis. (8 Marks)

   (b) Three equal charges of μC are placed at the corners of a square of length 10cmS. Find the direction and magnitude of electric field intensity of the vacant corner. (8 Marks)

   (c) Derive a relation between electric field intensity and eletric potential. (4 Marks)

2. (a) State and prove Gauss theorem. (6 Marks)

   (b) Using Gauss theorem obtain an expression for electric field intensity due to infinite sheet of charge with a surface charge density of $\sigma C/M^2$. (8 Marks)

   (c) Obtain expression for volume energy density in terms of electric fielbreak intensity and flux density in a capacitor. (6 Marks)

3. (a) Obtain the conditions on the tangential and normal components of field intensity and electric flux density at the boundary between two dielectric media. (8 Marks)

   (b). Derive an expression for a capacitance of a co-axial cable using Laplace's equation. (6 Marks)

   (c) A parallel plate capacetor consists of 3 dielectric layers.

   If $\in_1 = 1, \quad d_1 = 0.4mm$ and area of cross sect...

   $\in_2 = 2, \quad d_2 = 0.6mm$

   $\in_3 = 3, \quad d_3 = 0.8mm$

   find capacitance C. (6 Marks)

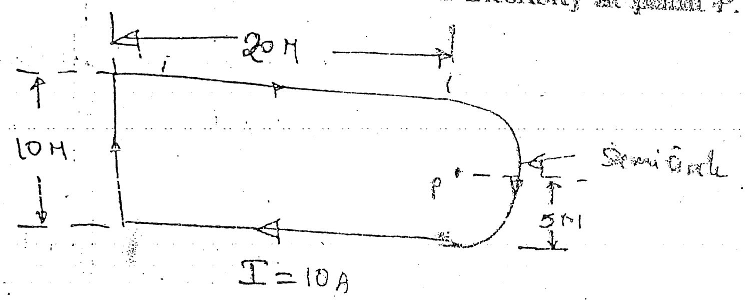
4. (a) State & explain Biot-Savart's Law. (5 Marks)

   (b) Derive an expression for magnetic flux density at any point on the axis of a solenoid. (8 Marks)

Contd.... 2

(c) For the circuit shown find the field intensity at point P.

(7 Marks)



5. (a) Explain the concept of vector magnetic potential. (5 Marks)

(b) A long conductor carries a current of I amp. Calculate the magnetic field intensity at any point using the concept of vector magnetic potential. (10 Marks)

(c) Two long parallel wires separated by 2cm in air carry steady currents of 100 amps. flowing in opposite directions. Fidn the force per mt length of the conductor. (5 Marks)

6. (a) State and explain General Ampere's law and derive the Maxwell's first equation, starting from Ampere's, circuital law. (10 Marks)

(b) Explain the following i) Motional emf. ii) transformer emf (10 Marks)

7. (a) Derive wave equations for E and H for any conducting medium. (7 Marks)

(b) Obtain a relation between the magnetic field intensity and electric field intensity for a wave travelling in free space. (7 Marks)

(c) Define Skin depth and derive an expression for it. (6 Marks)

8. (a) State and prove Poynting's theory from fundamentals. (10 Marks)

(b) The Electric held of a uniform plase wave is given by

$\vec{E} = 40sin(30\pi \times 10^6 t - 2\pi z)\vec{a_x} + 40cos(30\pi \times 10^6 t - 2\pi z)\vec{a_y}$ V/M

Find

i) Fre ... on

ii) Th ... r

iii) Di ... propogation of the wave

iv) As ... ated magnetic field $\vec{H}$

(10 Marks)

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