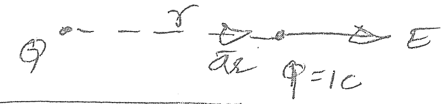
Fourth Semester BE Degree Exam. Dec 2015 / Jan. 2016

Sub: Field Theory  Sub. code: 10ES36.  M.Marks: 100.

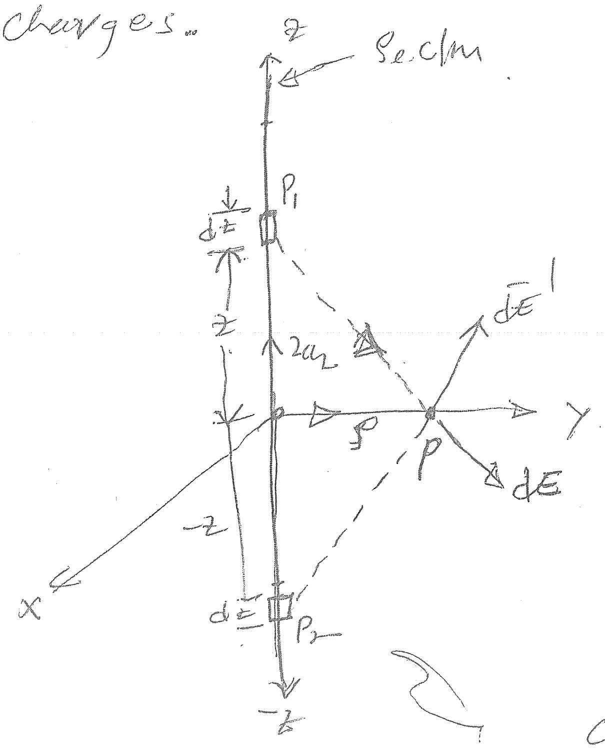
Q.1a. Explain the terms "electric field intensity" & derive expression for the field due to infinite line of charge. (08)

Soln Electric field intensity at a point is defined as the force experienced by a unit positive charge placed at that point. It is denoted by E & its unit is ~~V/m or N/C~~ Volts/metre or Newton/coulomb.




$$\boxed{\bar{E} = \frac{Q}{4\pi\epsilon r^2}\,\bar{a}_r}$$

$\bar{a}_r$ is the unit vector along the line joining the two point charges.




Consider an infinite line charge with charge density of $\rho_e$ C/m lies along z-axis. Let us find the electric field at point P at a distance of $\rho$ mt from from the ~~point~~ line conductor. Consider an elemental length of dz mt at a distance of z mt from the origin. The charge at $P_1$ is $dq = \rho_e \cdot dz \rightarrow (1)$

The ~~ele~~ electric field at P due to an elemental charge at $P_1$ is given by

$$d\bar{E} = \frac{dq}{4\pi\epsilon (P_1P)^2}\,\bar{a}_{P_1P} \rightarrow (2)$$

From the above fig,

$$z\bar{a}_z + \overline{P_1P} = \rho \bar{a}_\rho$$

$$\overline{P_1P} = \rho \bar{a}_\rho - z\bar{a}_z$$

$$|\overline{P_1P}| = \sqrt{\rho^2+z^2} \rightarrow 3$$

$$\bar{a}_{P_1P} = \frac{\rho \bar{a}_\rho - z\bar{a}_z}{\sqrt{\rho^2+z^2}} \rightarrow 4$$

putting 1, 3 & 4 in (2)

$$d\bar{E} = \frac{\rho_l \cdot dz}{4\pi\epsilon(\sqrt{\rho^2+z^2})^2} \times \frac{\rho \bar{a}_\rho - z\bar{a}_z}{\sqrt{\rho^2+z^2}}$$

$$d\bar{E} = \frac{\rho_l \cdot dz}{4\pi\epsilon(\rho^2+z^2)^{3/2}} (\rho \bar{a}_\rho - z\bar{a}_z) \rightarrow (5)$$

By symmetry, the electric field at P due to elemental charge at $P_2$ is

$$d\bar{E}' = \frac{\rho_l \cdot dz}{4\pi\epsilon(\rho^2+z^2)^{3/2}} (\rho \bar{a}_\rho + z\bar{a}_z) \rightarrow (6)$$

From eqn. 5 & 6 it is clear that z-component of electric field is equal in magnitude & opposite in direction & hence it cancels & the $\rho$-component of electric field will add up.

$$\therefore \quad d\bar{E} = \frac{\rho_l \cdot dz}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \cdot \rho \bar{a}_\rho$$

$$\therefore \quad \bar{E} = \int_{-\infty}^{+\infty} d\bar{E}$$

$$= \int_{-\infty}^{\infty} \frac{\rho_l \cdot dz}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \cdot \rho \bar{a}_\rho$$

put: $z = \rho \tan\theta$, $dz = \rho \sec^2\theta \, d\theta$, ~~$z = \infty$, $\theta = \theta_1$~~,

where $z = -\infty$, $\theta = \theta_1 = \tan^{-1}(-\infty) = -\pi/2$

$z = +\infty$, $\theta = \theta_2 = \tan^{-1}(\infty) = \pi/2$

$$\bar{E} = \rho_l$$

$$\therefore \bar{E} = \frac{\rho_L \cdot \rho\, \bar{a}_\rho}{4\pi\epsilon} \int_{-\pi/2}^{\pi/2} \frac{\rho \sec^2\theta \, d\theta}{(\rho^2 + \rho^2 \tan^2\theta)^{3/2}}$$

$$= \frac{\rho_L \cdot \rho^2\, \bar{a}_\rho}{4\pi\epsilon \cdot \rho^3} \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^{3/2}}$$

$$= \frac{\rho_L}{4\pi\epsilon\rho} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{\rho_L}{4\pi\epsilon} (-\sin\theta)_{-\pi/2}^{\pi/2}$$

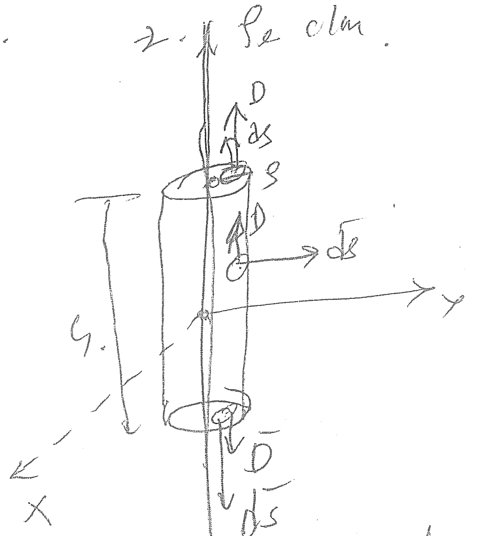
$$= \frac{\rho_L}{4\pi\epsilon} \left(\sin(\pi/2) - \sin(-\pi/2)\right)$$

$$= \frac{\rho_L}{4\pi\epsilon\rho} \bar{a}_\rho \times 2$$

$$\boxed{\bar{E} = \frac{\rho_L}{2\pi\epsilon\rho} \bar{a}_\rho} \ N/C.$$

6. Use Gauss law to determine electric field intensity due to infinite line charge (6)

Soln.




Consider an $\infty$ conductor with charge density $\rho_L$ c/m. Let us find the $E_{\parallel}$ ~~due to~~ at a $\perp$r distance from cond. Consider a closed Gaussian cylinder of radius $\rho$ mt & height 'h' mt. Apply Gauss law

$$\oint_S \bar{D} \cdot \bar{ds} = Q_{en}.$$

$$\oint_{T.S} \bar{D} \cdot \bar{ds} + \oint_{B.S} \bar{D} \cdot \bar{ds} + \oint_{C.S} \bar{D} \cdot \bar{ds} = Q_{en} \rightarrow (1)$$

from the above fig $\oint_{TS} \bar{D}\cdot\bar{ds} = \oint_{BS} \bar{D}\cdot\bar{ds} = 0.$

$\therefore \oint_{L.S} \bar{D}\cdot\bar{ds} = q_{en}.$

$\oint_s D_\rho \bar{a}_\rho \cdot \rho\, d\phi\, dz\, \bar{a}_\rho = \rho_e \cdot h.$

$\oint_s \rho D_\rho\, d\phi\, dz = \rho_e \cdot h.$

$\rho D_\rho \int_0^{2\pi} d\phi \int_0^{h} dz = \rho_e \cdot h.$

$\rho D_\rho (\phi)_0^{2\pi} (z)_0^{h} = \rho_e \cdot h.$

$\rho D_\rho (2\pi)(\cancel{h}) = \rho_e \cancel{h}$

$D_\rho = \frac{\rho_e}{2\pi\rho}$

$\bar{D} = D_\rho \bar{a}_\rho$

$\boxed{\bar{D} = \frac{\rho_e}{2\pi\rho}\bar{a}_\rho}$ C/m$^2$

$\bar{E} = \frac{\bar{D}}{\epsilon}$

$\boxed{\bar{E} = \frac{\rho_e}{2\pi\epsilon\rho}\bar{a}_\rho \text{ V/m}}$

Q.1.C The flux density $\bar{D} = \frac{r}{3}\bar{a}_r$ nC/m$^2$ in free space.

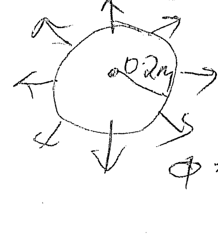
i) Find $\bar{E}$ at $r = 0.2$m.

ii) Find the total electric flux leaving the sphere at $r = 0.2$mt

iii) Find the total charge within the sphere of $r = 0.3$m

Sol Given $\bar{D} = \frac{r}{3}\bar{a}_r$ nC/m$^2$ $= \frac{r}{3}\times 10^{-9}\bar{a}_r$ C/m$^2$, $\epsilon = \epsilon_0$

i) $\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{r}{3\epsilon_0}\times 10^{-9}\bar{a}_r = \frac{0.2 \times 10^{-9}}{3\epsilon_0}\bar{a}_r = 7.529\,\bar{a}_r$ V/m.

ii) 


$\phi = \oint_s \bar{D}\cdot\bar{ds} = \oint_s D_r \bar{a}_r \cdot r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$

$\phi = q.$ $\phi = \oint_s \frac{r}{3}\times 10^{-9}\bar{a}_r \cdot r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r.$

③

$$\phi = \int_S \frac{r^3}{3}\times 10^{-9} \cdot \sin\theta\, d\theta\, d\phi$$
$r = 0.2m$

$$= \frac{(0.2)^3\times 10^{-9}}{3} \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 2.6667\times 10^{-12}\left(-(\cos\theta)_0^{\pi}\right)(\phi)_0^{2\pi}$$

$$= 2.6667\times 10^{-12}\times(2)(2\pi)$$

$$\boxed{\phi = 33.51\times 10^{-12}\ \text{Coulombs}}$$

iii) $Q = ?$ at $r = 0.3mt$.

$$Q = \phi = \oint_S \bar{D}\cdot \bar{ds}$$

$$= \oint_S \frac{r}{3}\times 10^{-9}\,\bar{a_r}\cdot r^2 \sin\theta\, d\theta\, d\phi\,\bar{a_r}$$
$r = 0.3M$

$$= \frac{r^3\times 10^{-9}}{3}\int_S \sin\theta\, d\theta\, d\phi$$
$r = 0.3$.

$$= \frac{(0.3)^3\times 10^{-9}}{3}\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi} d\phi$$

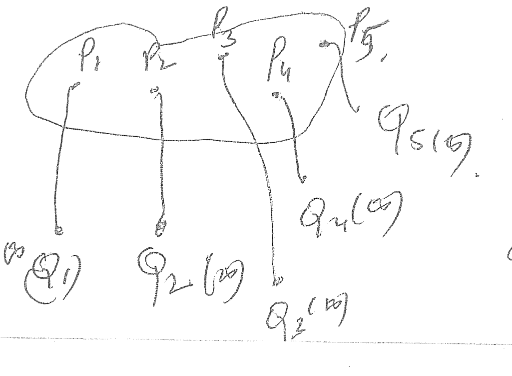
$$= 9\times 10^{-12}(-\cos\theta)_0^{\pi}(\phi)_0^{2\pi}$$

$$Q = 9\times 10^{-12}(2)(2\pi)$$

$$\boxed{Q = 113.097\ PC.}$$ //

Q.2a. Show that the energy required to assemble n number of point charges is $W_E = \frac{1}{2}\sum_{m=1}^{n} Q_m V_m$. ⑧
& hence derive expressions for energy in electric field in terms of field quantities $\bar{D}$ & $\bar{E}$.

Soln



Consider a region shown by the fig. which is free of charge. When the point charge $Q_1$ is moved from $\infty$ to $P_1$

it does not requires work to do, as there is no electric field exists. When $Q_2$ is moved ~~from~~ to ..., it has to over come the field produced by $Q_1$. So work is done to move $Q_2$ to $P_2$ to overcome the field produced by $Q_1$.

∴ Work done to move $Q_2$ to $P_2 = Q_2 V_{21}$
where $V_{21}$ = potential at $P_2$ due to $P_1$

Illy work done to move $Q_3$ to $P_3 = Q_3 V_{31} + Q_3 V_{32}$.

" " " $Q_4$ to $P_4 = Q_4 V_{41} + Q_4 V_{42} + Q_4 V_{43}$

" " $Q_n$ to $P_n = Q_n V_{n1} + Q_n V_{n2} + Q_n V_{n3} + \cdots + Q_n V_{nm}$

∴ Total work done in positioning all charges is

$$W_E = Q_2 V_{21} + Q_3 V_{31} + Q_3 V_{32} + Q_4 V_{41} + Q_4 V_{42} + Q_4 V_{43} + \cdots \to ①$$

The total work done is nothing but the energy in the system of charges. If the charges are placed in the reverse order,

$$W_E = Q_3 V_{31} + Q_2 V_{24} + Q_2 V_{23} + Q_1 V_{14} + Q_1 V_{13} + Q_1 V_{12} + \cdots \to ②$$

Adding eqn ① & ②

$$2W_E = Q_1 (V_{12} + V_{13} + V_{14} + \cdots) + Q_2 (V_{21} + V_{23} + V_{24} + \cdots) + Q_3 (V_{31} + V_{32} + V_{34} + \cdots) + Q_4 (V_{41} + V_{42} + V_{43} + \cdots)$$

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \cdots + Q_n V_n$$

Where $V_1 = V_{12} + V_{13} + V_{14} + \cdots$
$V_2 = V_{21} + V_{23} + V_{24} + \cdots$
$V_3 = V_{31} + V_{32} + V_{34} + \cdots$

$$\therefore 2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + Q_4 V_4 + \cdots$$

$$2W_E = \frac{1}{2} \sum_{m=1}^{n} Q_m V_m$$

$$\boxed{W_E = \frac{1}{2} \sum_{m=1}^{n} Q_m V_m} \to ③$$

Eqn. 3 can be written ~~as~~ in differential form as (4)

$W_E = \frac{1}{2}\int_l \rho_l dl \cdot V$  $(\because Q = \rho_l \cdot dl)$

$W_E = \frac{1}{2}\int_v \rho_v dv \cdot V$  $(Q = \rho_v \cdot dv)$ → (4)

$W_E = \frac{1}{2}\int_s \rho_s \cdot ds \cdot V$  $(Q = \rho_s \cdot ds)$

~~WKT~~ From 4

$W_E = \frac{1}{2}\int_v (\rho_v \cdot dv) V$.  $(\because \rho_v = \nabla \cdot \bar{D})$

$= \frac{1}{2}\int ((\nabla \cdot \bar{D}) V) \, dv$. → (5)

For a scalar potential V, & vector $\bar{D}$, WKT,

$\nabla \cdot V\bar{A} = \bar{A} \cdot \nabla V + V(\nabla \cdot \bar{A})$

$(\nabla \cdot \bar{A}) V = \nabla \cdot V\bar{A} - \bar{A} \cdot \nabla V$

$\therefore (\nabla \cdot \bar{D}) V = \nabla \cdot V\bar{D} - \bar{D} \cdot \nabla V$ → (6)

putting (6) in (5)

$W_E = \frac{1}{2}\int_v (\nabla \cdot V\bar{D} - \bar{D} \cdot \nabla V) \, dv$.

$= \frac{1}{2}\int_v (\nabla \cdot \bar{D} V) \, dv - \frac{1}{2}\int_v \bar{D} \cdot (-\bar{E}) \, dv$  $(\because \bar{E} = -\nabla V)$

By Divergence theorem $\int_v \nabla \cdot \bar{D} \, dv = \oint_s \bar{D} \cdot \bar{ds}$

$W_E = \frac{1}{2}\oint_s V(\bar{D} \cdot \bar{ds}) + \frac{1}{2}\int_v \bar{D} \cdot \bar{E} \, dv$.

WKT, $V \propto \frac{1}{r}$, $D \propto \frac{1}{r^2}$  $VD \propto \frac{1}{r^3}$ & $\bar{ds} \propto r^2$ $\therefore V\bar{D} \cdot \bar{ds} \propto \frac{1}{r}$.

As the surface becomes $\infty$ $V\bar{D} \cdot \bar{ds} \propto \frac{1}{r} \propto 0$ // $\therefore \oint_s V\bar{D} \cdot \bar{ds} = 0$

$\therefore W_E = \frac{1}{2}\int$

$\therefore W_E = \frac{1}{2}\int_v (\bar{D} \cdot \bar{E}) \, dv$

$= \frac{1}{2}\epsilon \cdot \int_v |E|^2 dv$  $(\because \bar{D} = \epsilon\bar{E})$

$\boxed{W_E = \frac{1}{2\epsilon}\int_v (D^2) \, dv}$ Hence the proof.

Q.26. Potential is given by $V = 2(x+1)^2(y+1)^2(z+3)^2$ volts in free space. At a point P(2, −1, 4) calculate i) potential ii) Electric field intensity iii) Flux density iv) volume charge density.

Soln. Given: $V = 2(x+1)^2(y+2)^2(z+3)^2$ volts, $\epsilon = \epsilon_0$, $P(2, -1, 4)$

i) $V = ?$

$$V = 2(1+2)^2(-1+2)^2(4+3)^2 = 882 \text{ volts}$$

ii) $\bar{E} = -\nabla V$

$$= -\left\{ \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z \right\}$$

$$= -\left\{ \frac{\partial}{\partial x}\left(2(x+1)^2(y+2)^2(z+3)^2\right)\bar{a}_x + \frac{\partial}{\partial y}\left(2(x+1)^2(y+2)^2(z+3)^2\right)\bar{a}_y + \frac{\partial}{\partial z}\left(2(x+1)^2(y+2)^2(z+3)^2\right)\bar{a}_z \right\}$$

$$\bar{E} = 2(y+1)^2(z+3)^2 \times 2(x+1)\,\bar{a}_x + (4(x+1)^2(y+2)(z+3)^2\,\bar{a}_y + 4(x+1)^2(y+2)^2(z+3)\,\bar{a}_z$$

$$\bar{E} = 2(-2+1)^2(4+3)^2 \times 2(2+1)\,\bar{a}_x + 4(3)^2(1)(7)^2\,\bar{a}_y + 4\cdot 3^2\cdot 1\cdot 7\,\bar{a}_z$$

$$\boxed{\bar{E} = 392\,\bar{a}_x + 1764\,\bar{a}_y + 252\,\bar{a}_z \quad N/C.}$$

$\bar{D} = \epsilon\bar{E}$

iii) $\bar{D} = \epsilon_0(392\,\bar{a}_x + 1764\,\bar{a}_y + 252\,\bar{a}_z)\ C/m^2$

iv) $\rho_v = \nabla\cdot\bar{D}$

$$= \nabla\cdot\epsilon\bar{E}$$

$$= \epsilon_0\,\nabla\cdot\bar{E}$$

$$= \epsilon_0\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

$$= \epsilon_0\left(\frac{\partial}{\partial x}\left(4(x+1)(y+2)^2(z+3)^2\right) + \frac{\partial}{\partial y}\left(4(x+1)^2(y+2)(z+3)^2\right) + \frac{\partial}{\partial z}\left(4\cdot(x+1)^2(y+2)^2(z+3)\right)\right)$$

$$\rho_v = \epsilon_0(4(1)^2(7)^2 + 4\cdot 3^2\cdot 1\cdot 7^2 + 4\cdot 3^2\cdot 1^2\cdot 7)$$

$$\rho_v = \epsilon_0(1092)$$

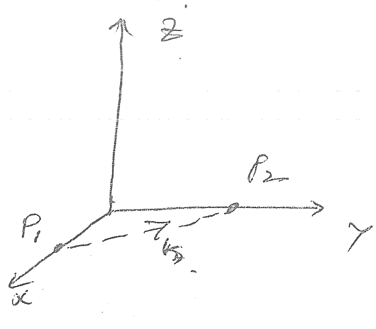
$$\boxed{\rho_v = 9.668 \times 10^{-9}\ C/m^2}$$

2.C. Find the work done in moving a charge of +2C from (2, 0, 0) m to (0, 2, 0) m along the straight line path joining two points, if the electric field is $\bar{E} = 12x\bar{a}_x - 4y\bar{a}_y$ V/m

Soln Given. $Q = 2C$, $\bar{E} = 12x\bar{a}_x - 4y\bar{a}_y$ V/m

$P_1 \equiv (2, 0, 0)$ $P_2 = (0, 2, 0)$

$W = ?$




$$W = -Q \int_l \bar{E} \cdot \bar{dl}$$

$$= -Q \int_l (12x\bar{a}_x - 4y\bar{a}_y) \cdot (dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z)$$

$$W = -2 \int_l (12x\,dx - 4y\,dy) \longrightarrow (1)$$

Equation of the Straight line joining the points $P_1$ & $P_2$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Considering the first two ratios

$$\frac{x - 2}{0 - 2} = \frac{y - 0}{2 - 0}$$

$$\frac{x - 2}{-2} = \frac{y}{2} \Rightarrow x - 2 = -y$$
$$x = 2 - y$$
$$dx = -dy$$

∴ Equation ① becomes

$$W = -Q \int_0^2 12(2 - y)(-dy) - 4y\,dy$$

$$= -2 \int_0^2 (-24 + 12y - 4y)\,dy$$

$$= -2 \int_0^2 (8y - 24)\,dy$$

$$= -2 \left(8\frac{y^2}{2} - 24y\right)_0^2$$

$$= -2(4(4 - 0) - 24(2 - 0))$$

$$= -2(16 - 48)$$

$$\boxed{W = 64 \text{ Joules}}$$

Q.3a. Arrive at the poissons equation in Cartesian coordinates. Deduce Laplace equation from poisson equation.

Soln. From Gauss law in point form

$\nabla \cdot \bar{D} = \rho_v$

$\nabla \cdot \epsilon \bar{E} = \rho_v \qquad (\bar{D} = \epsilon \bar{E})$

$\nabla \cdot \bar{E} = \frac{\rho_v}{\epsilon}$

$\nabla \cdot (-\nabla V) = \frac{\rho_v}{\epsilon} \qquad (\bar{E} = -\nabla V)$

$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}} \longrightarrow$ poisson's equation.

where $\nabla^2 V$ is Laplacian operator.

$\cancel{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} +}$

$\nabla^2 V = \nabla \cdot \nabla V$

$= \nabla \cdot \frac{\partial V}{\partial x} \bar{a}_x + \frac{\partial V}{\partial y} \bar{a}_y + \frac{\partial V}{\partial z} \bar{a}_z$

$= \left(\frac{\partial}{\partial x} \bar{a}_x + \frac{\partial}{\partial y} \bar{a}_y + \frac{\partial}{\partial z} \bar{a}_z\right) \cdot \left(\frac{\partial V}{\partial x} \bar{a}_x + \frac{\partial V}{\partial y} \bar{a}_y + \frac{\partial V}{\partial z} \bar{a}_z\right)$

$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$ //.

If the region is free of charge, then $\rho_v = 0$.

$\therefore \boxed{\nabla^2 V = 0} \longrightarrow$ Laplace equation.

3.b. Verify the potential field $V = 2x^2 - 3y^2 + z^2$ satisfies Laplace equation.

Soln. Given: $V = 2x^2 + 3y^2 + z^2$.

$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

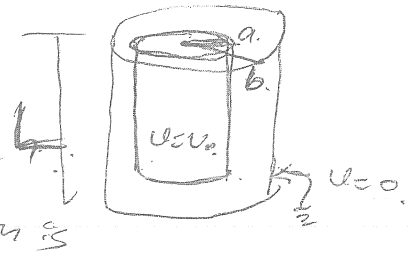
$= \frac{\partial^2}{\partial x^2}(2x^2 - 3y^2 + z^2) + \frac{\partial^2}{\partial y^2}(2x^2 - 3y^2 + z^2) + \frac{\partial^2}{\partial z^2}(2x^2 - 3y^2 + z^2)$

$\nabla^2 V = \frac{\partial}{\partial x}(4x - \cancel{0} + 0) + \frac{\partial}{\partial y}(-6y) + \frac{\partial}{\partial z}(+2z)$

$= 4 - 6 + 2 = 0$ // satisfies the Laplace equation.

Q. 3c Using Laplace equation, derive an expression for capacitance of a co-axial cable. 6.

Soln. Consider a co-axial cable of inner radius 'a' mt of cond. & outer radius 'b' mt of cond. Let the length of the cable is L mt. The Laplace equation is




$$\nabla^2 v = \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 z}{\partial z^2} \longrightarrow (1)$$

As the potential field varies only in r-direction

$\therefore \frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial z} = 0$ $\therefore$ equation (1) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial v}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = 0$$

Integrating the above expression w.r.t r.

$$\int \frac{\partial}{\partial r}\left(r\cdot\frac{\partial v}{\partial r}\right) dr = 0$$

$$r\cdot\frac{\partial v}{\partial r} = C_1$$

$$\frac{\partial v}{\partial r} = \frac{C_1}{r}.$$

Again integrating

$$\boxed{v = C_1 \cdot \ln r + C_2.} \longrightarrow 2.$$

$v = V_0$ at $r = a$. $\quad V_0 = C_1 \cdot \ln a + C_2. \rightarrow 3$

$v = 0$ at $r = b$ $\quad \underline{0 = C_1 \ln b + C_2}$

$$V_0 = C_1(\ln a - \ln b)$$

$$C_1 = V_0 / \ln(a/b)$$

putting $C_1$ in (3)

$$V_0 = \frac{V_0}{\ln(a/b)} \ln a + C_2.$$

$$C_2 = V_0 - \frac{V_0 \cdot \ln a}{\ln(a/b)}$$

$$= \frac{V_0(\ln a - \ln b - \ln a)}{\ln(a/b)}$$

$$C_2 = -V_0 \ln(b) / \ln((a/b)$$

$$C_2 = -V_0 . \ln(b) / \ln(a/b)$$

$\therefore$ equation (1) becomes

$$\boxed{V = \frac{V_0 . \ln r}{\ln(a/b)} - \frac{V_0 \ln(b)}{\ln(a/b)}} \rightarrow (3)$$

$$\bar{E} = -\nabla V$$

$$= -\frac{\partial V}{\partial r} a_r$$

$$= -\frac{\partial}{\partial r}\left(\frac{V_0 . \ln r}{\ln(a/b)} - \frac{V_0 \ln(b)}{\ln(a/b)}\right) \bar{a}_r$$

$$\bar{E} = \left(-\frac{V_0}{r \ln(a/b)} - 0\right) \bar{a}_r$$

$$\bar{E} = \frac{-V_0}{r \ln(a/b)} \bar{a}_r = \frac{V_0}{r \ln(b/a)} \bar{a}_r$$

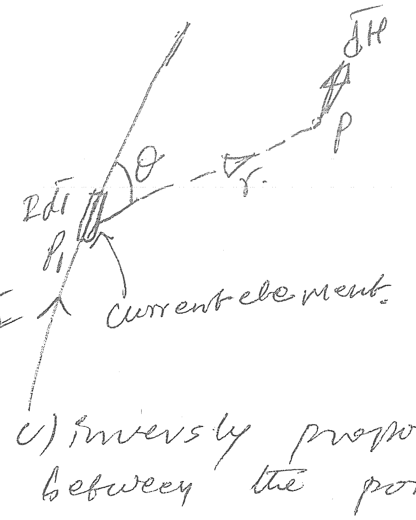
$$\rho_s = |D| = \epsilon |E| = \frac{\epsilon V_0}{r \ln(b/a)}$$

$$Q = \rho_s . A = \frac{\epsilon V_0}{\cancel{r} . \ln(b/a)} \times 2\pi \cancel{r} . L$$

$$\boxed{C = \frac{Q}{V_0} = \frac{2\pi \epsilon L}{\ln(b/a)} \text{ Farad}}$$

Q.4a. State & explain Biot-savart law. Using this, find the magnetic field intensity in the vicinity of long, straight filamentary current I Amperes along z-axis. (7)

Soln.




The magnetic field intensity at a point 'P' due to current element is

a) $d\bar{H}$ proportional to the product of current I & differential length '$dl$', $d\bar{H} \propto Id\bar{l}$

b) The sine of the angle between the element & the line joining point P to the element, $d\bar{H} \propto \sin\theta$

c) inversely proportional to the square of the distance between the point P & the current element. i.e. $dH \propto \frac{1}{r^2}$

$$\therefore \quad d\bar{H} \propto \frac{I\,d\bar{l}\,\sin\theta}{r^2}$$

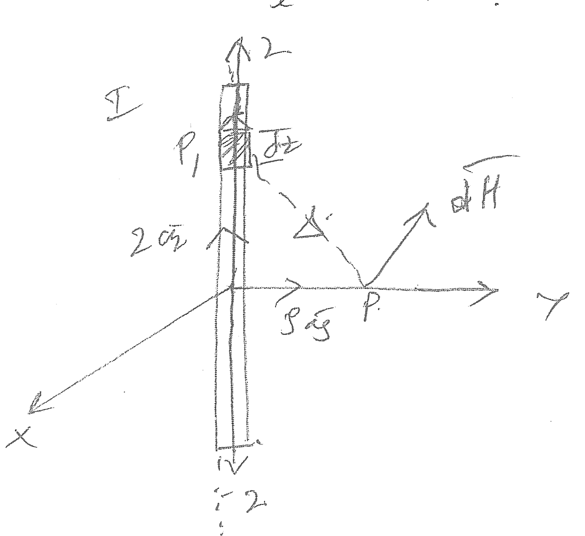
$$d\bar{H} = \frac{K\,I\,dl\,\sin\theta}{r^2}$$

$$d\bar{H} = \frac{I\,d\bar{l}\,\sin\theta}{4\pi r^2} \qquad \left(K = \frac{1}{4\pi}\right)$$

The direction of magnetic field is normal to the plane containing the differential element $Idl$ & the line joining the element & the point 'P'

$$\therefore \boxed{d\bar{H} = \frac{I\,d\bar{l} \times \bar{a}_r}{4\pi r^2}} \ A/m$$

$$\therefore \bar{H} = \int_l \frac{I\,d\bar{l} \times \bar{a}_r}{4\pi r^2}.$$




Consider an infinite conductor carrying a current of I Amps along z-axis. We wish to find the $\bar{H}$ at a $\perp$r distance from the conductor. Consider a current element $Id\bar{l}$ at $P_1$.

The magnetic field intensity at P due to current element at $P_1$ is

$$d\bar{H} = \frac{I\,d\bar{l} \times \bar{a}_r}{4\pi r^2}$$

$$d\bar{H} = \frac{I\,d\bar{l} \times \bar{a}_{P_1P}}{4\pi |\bar{P_1P}|^2}$$

$$\bar{P_1P} = r\bar{a}_r - z\bar{a}_z$$

$$|P_1P| = \sqrt{r^2+z^2}$$

$$\bar{a}_{P_1P} = \bar{P_1P}/\bar{a}_{P_1P} = \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}}, \quad I d\bar{l} = I dz \bar{a}_z.$$

$$d\bar{H} = \frac{I\,dz\bar{a}_z \times (r\bar{a}_r - z\bar{a}_z)}{4\pi(\sqrt{r^2+z^2})^2 \cdot \sqrt{r^2+z^2}}$$

$$d\bar{H} = \frac{I\,dz\,(r\bar{a}_\phi - 0)}{4\pi(r^2+z^2)^{3/2}}$$

$$d\bar{H} = \frac{I\,r\,dz}{4\pi(r^2+z^2)^{3/2}}\bar{a}_\phi$$

$$\bar{H} = \int_{-\infty}^{\infty} \frac{I r dz}{4\pi(r^2+z^2)^{3/2}}\bar{a}_\phi$$

put $z = r\tan\theta$, $dz = r\sec^2\theta\, d\theta$,
$z = -\infty$, $\theta = \theta_1 = -\pi/2$ & $z = +\infty$, $\theta = \theta_2 = \pi/2$.

$$\therefore \bar{H} = \int_{-\pi/2}^{\pi/2} \frac{I\, r^2 \sec^2\theta\, d\theta}{4\pi(r^2 + r^2\tan^2\theta)^{3/2}}\bar{a}_\phi$$

$$= \frac{I}{4\pi}\bar{a}_\phi \int_{-\pi/2}^{\pi/2} \frac{r^2 \sec^2\theta\, d\theta}{r^3\sec^3\theta}$$

$$\parallel.\ (r^2+r^2\tan^2\theta)^{3/2} = r^3(1+\tan^2\theta)^{3/2} = r^3(\sec^2\theta)^{3/2} = \underline{\underline{r^3\sec^3\theta}}$$

$$= \frac{I}{4\pi r}\bar{a}_\phi \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{I}{4\pi r}\bar{a}_\phi\,(\sin\theta)_{-\pi/2}^{\pi/2}$$

$$= \frac{I}{4\pi r}\bar{a}_\phi\,(\sin\pi/2 - \sin(-\pi/2))$$

$$= \frac{I}{2\pi r}\bar{a}_\phi\,(1+1)$$

$$\boxed{\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi}$$ Hence the proof

Q4.b. Discuss the concept of vector magnetic potential & hence show that, $\bar{A} = \frac{\mu_0}{4\pi}\int_{v} \frac{\bar{J}}{r} dv$ where $\bar{A}$ is the vector magnetic potential & $\bar{J}$ is the current density. (8)

Sol. Vector magnetic potential is a quantity, which satisfies the following equations.

$$\bar{B} = \nabla \times \bar{A} \longrightarrow (1)$$

$$\nabla \cdot \bar{A} = 0. \quad \text{where } \bar{B} = \mu \bar{H}$$

W.K.T. from Biot-Savart law,

$$\bar{dH} = \frac{I\,\bar{dl} \times \bar{a_r}}{4\pi r^2}$$

$$= \frac{\bar{J}\cdot \bar{ds}\cdot \bar{dl} \times \bar{a_r}}{4\pi r^2} \qquad (\because \ dv = \cancel{ds\,dl}.\ I = \bar{J}\cdot\bar{ds}$$

$$= \frac{\bar{J}\, dv \times \bar{a_r}}{4\pi r^2} \qquad (\because \ \bar{ds}\cdot\bar{dl} = dv)$$

$$\bar{dH} = \frac{\bar{J} \times \bar{a_r}\, dv}{4\pi r^2}$$

$$\bar{H} = \oint_v \frac{\bar{J} \times \bar{a_r}}{4\pi r^2} dv.$$

$$= \oint_v \frac{\bar{J} \times \frac{\bar{a_r}}{r^2}}{4\pi} dv.$$

$$= \oint_v \frac{\bar{J} \times (-\nabla(\frac{1}{r}))}{4\pi} dv \qquad \left(\because \ \frac{\bar{a_r}}{r^2} = -\nabla(\frac{1}{r})\right)$$

$$= -\int_v \frac{\bar{J} \times \nabla(\frac{1}{r})}{4\pi} dv. \rightarrow (2)$$

W.K.T from vector identity,

$$\bar{A} \times \nabla\phi = -\nabla \times (\phi \bar{A}) + \phi(\nabla \times \bar{A})$$

$$\bar{J} \times \nabla(\tfrac{1}{r}) = -\nabla \times (\tfrac{\bar{J}}{r}) + \tfrac{1}{r}(\nabla \times \bar{J})$$

for constant current density $\nabla \times \bar{J} = 0$

$$\therefore \bar{J} \times \nabla(\tfrac{1}{r}) = -\nabla \times \tfrac{\bar{J}}{r}. \rightarrow (3)$$

Putting 3 in 2.

$$\bar{H} = \oint_v \frac{\nabla \times (\frac{\bar{J}}{r})}{4\pi} dv.$$

$$= \nabla \times \oint_v \frac{\frac{\bar{J}}{r}\, dv}{4\pi}$$

Now, $\bar{B} = \mu \bar{H}$

$$= \nabla \times \frac{\mu}{4\pi} \oint_v \frac{\bar{J}}{r} dv \rightarrow (4)$$

Comparing (4) & (1)

$$\boxed{\bar{A} = \frac{\mu}{4\pi} \int_v \frac{\bar{J}}{r} dv.}$$

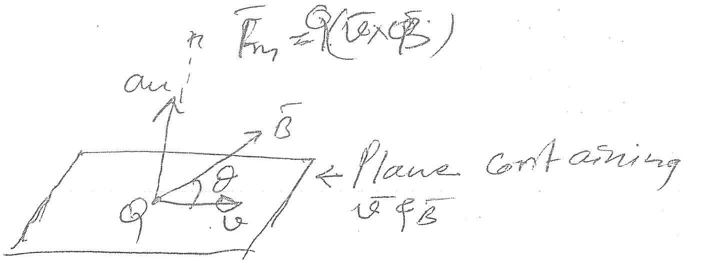
Q.4.c. At point $P(x, y, z)$, the components of vector magnetic potential are given as $A_x = 4x + 3y + 2z$, $A_y = 5x + 6y + 3z$, $A_z = 2x + 3y + 5z$. Determine B at point P & state its nature.

Soln.

PART-B

Q.5a. State & explain Lorentz force equation.

Soln




Consider a ~~for~~ a charge of Q coulombs placed in the presence of Electric field intensity. The force exerted on a charge of Q. Coulombs is given by

$\bar{F} = Q\bar{E} \longrightarrow (1).$

The direction of the force is same as the direction of electric field.

Now if a moving charge is placed in a magnetic field of flux density, $\bar{B}$, it experiences a force. This force is a function of charge Q, velocity $v$, & the angle b/w $\bar{v}$ & $\bar{B}$

~~The direction of this force is~~ ⊥ to ~~the~~ $\bar{v}$ & $\bar{B}$ & the angle b/w $\bar{v}$ & $\bar{B}$. i.e. $F = Q|v||B|\sin\theta = Q.\bar{v}\times\bar{B}\sin\theta a_n$

Now, the total force on a moving charge due to electric field 'E' & magnetic flux density 'B' is

$$\boxed{\bar{F} = Q(\bar{E} + \bar{v}\times\bar{B})}$$

The above equation is known as Lorentz force equation.

5b. A conductor of 4m long lies along the y axis with a current of 10A in the $\bar{a}_y$ direction. Find the force on the conductor if the field in the region is $\bar{B} = 0.005\ \bar{a}_x$, Tesla

Soln $l = 4\,m$, $I = 10A$, $\bar{B} = 0.005\ \bar{a}_x$ Tesla

$d\bar{F} = I d\bar{l}\times\bar{B}$

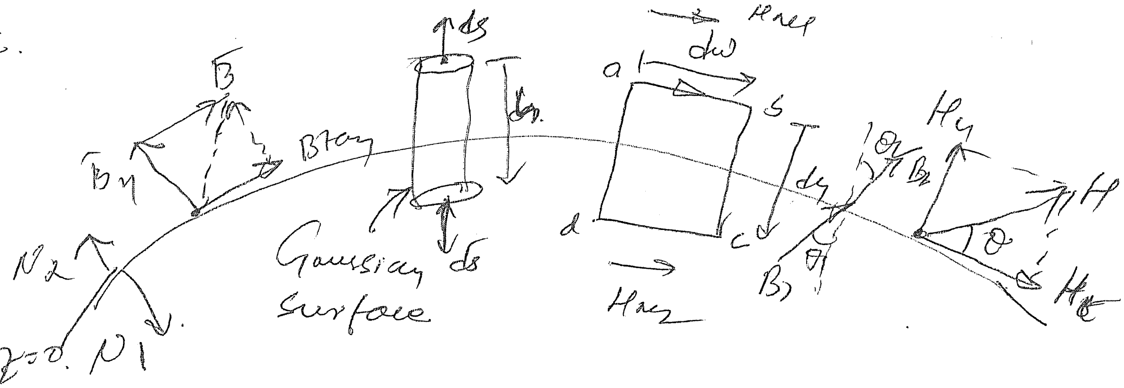
$\bar{F} = \int_l I d\bar{l}\times\bar{B}$

$= \int_l 10 \times dy\,\bar{a}_y \times 0.05\ \bar{a}_x$

$\bar{F} = -10\times 0.005 \int_0^4 dy = -0.05\bar{a}_z (y)_0^4 = \underline{-0.2\bar{a}_z \text{ Newton}}$

x
z y

Q.5c.




Consider a boundary between the two magnetic medium, at $z=0$, with di magnetic permeability $\mu_1$ & $\mu_2$ as shown above.

1. Boundary conditions for tangential components.

The magnetic field intensity at the interface b/w the two medium is resolved in to two components i Horizontal & vertical components to the interface. Considering closed path abcd as shown above, using Ampere's law

$\therefore \oint_l \bar{H}\cdot d\bar{l} = I_{enc}$

$\oint_{ab} \bar{H}\cdot d\bar{l} + \oint_{bc} \bar{H}\cdot d\bar{l} + \oint_{cd} \bar{H}\cdot d\bar{l} + \oint_{da} \bar{H}\cdot d\bar{l} = I \rightarrow (1)$

At the interface $dh \to 0$ $\therefore \oint_{bc} \bar{H}\cdot d\bar{l} = \oint_{da} \bar{H}\cdot d\bar{l} = 0$

$\oint_a^b H_{tan_1}\, dw + (-H_{tan_2})\, dw = \bar{K}\, dw$

where K is the sheet current of K A/m normal to the path.

$\therefore$ ~~$H_{t1} - H_{t2}$~~

$(\bar{H}_{tan_1} - \bar{H}_{tan_2})\, dw = K\, dw$

$H_{tan_1} - H_{tan_2} = \bar{K} \longrightarrow (2)$

If the surface is free from sheet currents, $K=0$

$\therefore \bar{H}_{tan_1} - \bar{H}_{tan_2} = 0$

$\boxed{\bar{H}_{tan_1} = \bar{H}_{tan_2}} \longrightarrow (3)$

2. Boundary conditions for normal components.
Consider a closed Gaussian cylinder, as shown above, Applying Gauss law,

$$\oint_s \bar{B} \cdot \bar{ds} = 0$$

$$\oint_{T.S} \bar{B} \cdot \bar{ds} + \oint_{B.S} \bar{B} \cdot \bar{ds} + \oint_{C.S} \bar{B} \cdot \bar{ds} = 0$$

The flux density at the interface is resolved in to horizontal & vertical components to the interface.

$$\oint_{C.S} \bar{B} \cdot \bar{ds} = 0.$$

$$\oint_{T.S} \bar{B} \cdot \bar{ds} + 0 + \oint_{C.S} \bar{B} \cdot \bar{ds} = 0$$

$$B_{n_2} \cdot ds + (-B_{n_1})\, ds = 0$$

$$(\bar{B}_{n_2} - \bar{B}_{n_1})\, ds = 0$$

$$\bar{B}_{n_2} - \bar{B}_{n_1} = 0$$

$$\boxed{\bar{B}_{n_2} = \bar{B}_{n_1}} \; //.$$

Q.6.9 Show that $\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$, where $\bar{J}$ = conduction current density, $\rho_v \rightarrow$ volume charge

Solun From Ampere's circuital law

$$\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}$$

Taking divergence on both sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

From vector identity $\nabla \cdot \nabla \times \bar{H} = 0$.

$$\therefore \nabla \cdot \bar{J} + \nabla \cdot \frac{\partial \bar{D}}{\partial t} = 0$$

$$\nabla \cdot \bar{J} = -\nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

$$\nabla \cdot \bar{J} = -\frac{\partial}{\partial t} \nabla \cdot \bar{D}$$

$$\boxed{\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}}$$ Hence the proof.

Q.6.6 $B = 0.04\,\bar{a}_y$ $\quad v = 2.5 \sin 10^3 t\, a_z$ m/s, $B = 0.$

Assuming the conductor of length $l$ placed along x-axis

$\therefore e_1 = \int_l (\bar{v} \times \bar{B}) \cdot \bar{dl}$

$= \int_l (2.5 \sin 10^3 t\, \bar{a}_z \times 0.04\,\bar{a}_y) \cdot dl\,\bar{a}_x$

$= 2.5 \sin 10^3 t \times 0.04 \int (-\bar{a}_x) \cdot dl\,\bar{a}_x$

$= -~~2.5~~\, 0.1 \sin 10^3 t\, l.$

$\boxed{e = -0.1 l \sin 10^3 t \text{ volts}}$ //

$B = 0.04\,\bar{a}_x$

$e_2 = \int_l 2.5 \sin 10^3 t\, \bar{a}_z \times 0.04\,\bar{a}_x \cdot dl\,\bar{a}_x$

$= 2.5 \times 0.04 \sin 10^3 t \int \bar{a}_y \cdot dl\,\bar{a}_x$

$\boxed{e_2 = 0}$ $\qquad \because \bar{a}_y \cdot \bar{a}_x = 0$

Q.No79 Maxwell's 2nd & 3rd equation are,

$\nabla \times \bar{E} = -\mu \frac{\partial H}{\partial t} \longrightarrow 1$

$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \longrightarrow 2.$

Taking curl to equation ① on both sides.

$\nabla \times \nabla \times \bar{E} = -\mu\, \nabla \times \frac{\partial \bar{H}}{\partial t}$

WKT, $\nabla \times \nabla \times \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E}$

$\therefore \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \bar{H})$

$\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -\mu \frac{\partial}{\partial t} (\bar{J}_c + \frac{\partial \bar{D}}{\partial t})$ $\quad$ from eqn ②

$\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -\mu\sigma \frac{\partial \bar{E}}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2}$ $\qquad \bar{J} = \sigma \bar{E}$

$\bar{D} = \epsilon \bar{E}$

$$\nabla^2 E - \mu\sigma \frac{\partial E}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2} = \nabla(\nabla\cdot E)$$

$$\nabla^2 \bar{E} - \mu\sigma \frac{\partial \bar{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = \nabla\cdot\left(\frac{\rho_v}{\epsilon}\right) \quad \left(\begin{array}{l}\nabla\cdot D = \rho_v \\ \nabla\cdot D = \frac{\rho_v}{\epsilon}\end{array}\right)$$

for free space conditions $\sigma = 0$, $\rho_v = 0$.

$$\nabla^2 E - \mu\epsilon \frac{\partial^2 E}{\partial t^2} = 0$$

$$\boxed{\nabla^2 E = \mu\epsilon \frac{\partial^2 E}{\partial t^2}}$$

Taking curl to eqn ② on both sides

$$\nabla\times\nabla\times\bar{H} = \nabla\times\bar{J}_c + \nabla\times\frac{\partial \bar{D}}{\partial t}$$

$$\nabla(\nabla\cdot\bar{H}) - \nabla^2\bar{H} = \sigma\,\nabla\times\bar{E} + \epsilon\,\nabla\times\frac{\partial \bar{E}}{\partial t}$$

$$0 - \nabla^2\bar{H} = \sigma\left(-\mu\frac{\partial \bar{H}}{\partial t}\right) + \epsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial \bar{H}}{\partial t}\right)$$

$$-\nabla^2\bar{H} = -\mu\sigma\frac{\partial \bar{H}}{\partial t} - \epsilon\mu\frac{\partial^2 \bar{H}}{\partial t^2}$$

$$\nabla^2\bar{H} - \mu\sigma\frac{\partial \bar{H}}{\partial t} - \mu\epsilon\frac{\partial^2 \bar{H}}{\partial t^2} = 0.$$

for free space, $\sigma = 0$.

$$\nabla^2\bar{H} - \mu\epsilon\frac{\partial^2 \bar{H}}{\partial t^2} = 0$$

$$\boxed{\nabla^2\bar{H} = \mu\epsilon\frac{\partial^2 \bar{H}}{\partial t^2}}$$

Q. 7b $\mu_r = 1$, $\epsilon_r = 50$, $\sigma = 60$ ℧/m, $f = 15.9$ MHz,

$\alpha = \beta = v = \eta = ?$

$$\frac{\sigma}{\omega\epsilon} = \frac{\sigma}{2\pi f\epsilon_r\epsilon_0} = \frac{60}{2\pi\times 15.9\times 10^6\times 50\times\epsilon_0}$$

So the medium is ~~perfect~~ ~~good~~ dielectric.

$$\frac{\sigma}{\omega\epsilon} = 0.0449.$$

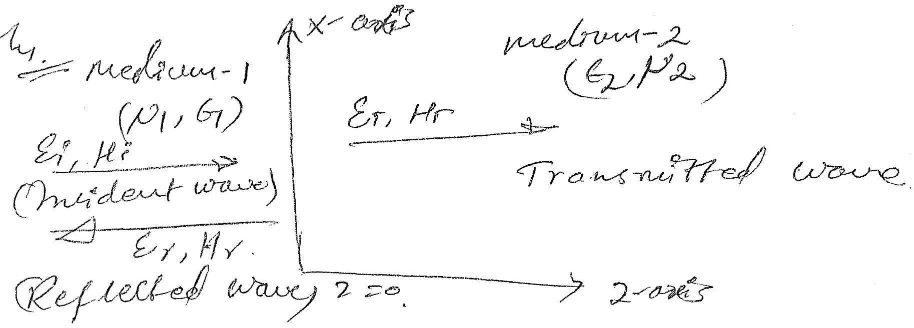
i) $\alpha = 0$ ii) $\beta = \omega\sqrt{\mu\epsilon} = 2\pi f\sqrt{\mu_0\epsilon_0\mu_r\epsilon_r}$
$= 2\pi\times 15.9\times 10^6\sqrt{\mu_0\epsilon_0\times 50\times 1} = 2.356$

iii) $v = \omega/\beta = \frac{2\pi\times 15.9\times 10^6}{2.356} = 42.40\times 10^6$ m/sec

iv) $\eta = \sqrt{\frac{\mu}{\epsilon}} = 377\sqrt{\frac{\mu_r}{\epsilon_r}} = 377\sqrt{\frac{1}{50}} = 53.31\ \Omega$

Q.8a Derive the expression for transmission Coefficient & reflection coefficient.

Soln.




Consider the interface at $z=0$, plane & the wave incident in medium-1. Part of the wave is transmitted in medium-2 & part of it is reflected back, as shown above. Now, In medium-1

$$\left.\begin{aligned} E_1 &= E_i + E_r \\ H_1 &= H_i + H_r \end{aligned}\right\} \rightarrow 1)$$

where, $E_i, H_i \rightarrow$ incident wave
$E_r, H_r \rightarrow$ Reflected wave.

In medium-2.

$$\left.\begin{aligned} E_2 &= E_t \\ H_2 &= H_t \end{aligned}\right\} \rightarrow (2)$$

$H_t, E_t \rightarrow$ Transmitted electric & magnetic fields.

~~At~~ According to the boundary conditions,

$E_{tan_1} = E_{tan_2}$ & $H_{tan_1} = H_{tan_2}$.

$\therefore \quad E_i + E_r = E_t \rightarrow 3$

$H_i + H_r = H_t \rightarrow 4$

Also,
$$\left.\begin{aligned} E_i &= \eta_1 H_i \\ E_t &= \eta_2 H_t \\ E_r &= -\eta_1 H_r \end{aligned}\right\} \rightarrow 5$$

putting (5) in (3)

$\cancel{\eta_1 H_i - \eta_1 H_r = \eta_2 H_t}$

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

$$E_i - E_r = \frac{\eta_1}{\eta_2} E_t \longrightarrow 6.$$

Adding equation ~~⑤~~ & 6

$$2\mathcal{E}_i = \mathcal{E}_t\left(1 + \frac{\eta_1}{\eta_2}\right)$$

$$= \mathcal{E}_t\frac{(\eta_2 + \eta_1)}{\eta_2}$$

$$\frac{\mathcal{E}_t}{\mathcal{E}_i} = \frac{2\eta_2}{\eta_1 + \eta_2}$$

$$\boxed{\tau = \frac{\mathcal{E}_t}{\mathcal{E}_i} = \frac{2\eta_2}{\eta_1 + \eta_2}}$$

where $\tau \rightarrow$ Transmission coefficient.

putting 3 in 6

$$\mathcal{E}_i - \mathcal{E}_r = \frac{\eta_1}{\eta_2}(\mathcal{E}_i + \mathcal{E}_r)$$

$$\mathcal{E}_i\left(1 - \frac{\eta_1}{\eta_2}\right) = \mathcal{E}_r\left(1 + \frac{\eta_1}{\eta_2}\right)$$

$$\mathcal{E}_i\frac{(\eta_2 - \eta_1)}{\cancel{\eta_2}} = \mathcal{E}_r\frac{(\eta_1 + \eta_2)}{\cancel{\eta_2}}$$

$$\frac{\mathcal{E}_r}{\mathcal{E}_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}.$$

$$\boxed{\Gamma = \frac{\mathcal{E}_r}{\mathcal{E}_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}.}$$

Where $\Gamma \rightarrow$ Reflection coefficient

Q.6. Write necessary expression, explain standing wave ratio.

Soln. When the reflecting medium is perfect conductor then the entire wave is reflected back. When the reflecting medium is not a perfect conductor, then small part of the incident wave is subjected to transmission & the remaining part gets reflected back in medium & the standing wave is setup & is given by

$$\mathcal{E}_1 = 2\mathcal{E}_i \sin\omega t \cdot \sin\beta z.$$

Now there will be maximum & minimum amplitude locations. However the degree to which the field is shared between the travelling & the standing wave is identified by considering the ratio of maximum & minimum amplitudes of the wave after reflection & this ratio is known as standing wave ratio.

Consider, $\bar{E_i} = E_i\, e^{j(\omega t - \beta z)}$

Let, $\delta \rightarrow$ phase difference introduced due to reflection

then, $\bar{E_r} = E_r\, e^{j(\omega t + \beta z + \delta)}$

$\bar{E_r} = \Gamma E_i\, e^{j(\omega t + \beta z + \delta)}$ $\quad \left( \because \Gamma = \frac{E_r}{E_i} \right)$

The resultant wave is,

$$\bar{E_1} = \bar{E_i} + \bar{E_r}$$
$$= E_i\, e^{j(\omega t - \beta z)} + \Gamma E_i\, e^{j(\omega t + \beta z + \delta)}$$
$$= E_i \left( e^{j(\omega t - \beta z)} + \Gamma e^{j(\omega t + \beta z + \delta)} \right)$$

Add subtract $\Gamma e^{j(\omega t - \beta z)}$

$$\bar{E_1} = E_i \left[ e^{j(\omega t - \beta z)} + \Gamma e^{j(\omega t + \beta z + \delta)} + \Gamma e^{j(\omega t - \beta z)} - \Gamma e^{j(\omega t - \beta z)} \right]$$
$$= E_i \left[ (1+\Gamma)\, e^{j(\omega t - \beta z)} + \Gamma\left(- e^{j(\omega t - \beta z)} + \Gamma e^{j(\omega t + \beta z + \delta)}\right) \right]$$

$\delta = \pi$ for reflection, & considering the real part of above eqn

$$\bar{E_1} = E_i \left[ (1+\Gamma)\cos(\omega t - \beta z) + 2\Gamma \sin\omega t \cdot \sin\beta z \right]$$

$$\bar{E_1} = E_i\, \tau \underbrace{\cos(\omega t - \beta z)}_{I} + \underbrace{2\Gamma E_i \sin\omega t \cdot \sin\beta z}_{II}.$$

The second term is called standing wave.

Now $E_{max} = E_i + E_r$, & $E_{min} = E_i - E_r$.

$$\therefore SWR = \frac{E_{max}}{E_{min}} = \frac{E_i + E_r}{E_i - E_r} = \frac{E_i + \Gamma E_r}{E_i - \Gamma E_r} = \frac{1+\Gamma}{1-\Gamma}$$

$$\boxed{SWR = \frac{1+\Gamma}{1-\Gamma}}$$