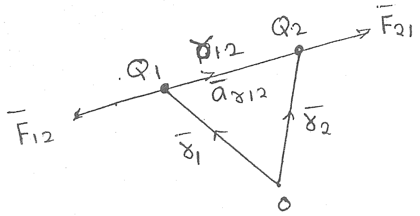
10EE34

Fourth Semester B.E Degree Examination, Dec 2014/Jan. 2015

Field Theory.

1] a) State and explain the coulomb's law of electrostatic force between two point charges.

⇒ Statement :- " The Forces between two very small object separated in vaccum or free space by a distance which is large compared to their size is proportional to the charge on each & inversely proportional to the square of the distance between them."



consider the vector r_1 Located at Q_1 & r_2 located at Q_2 The vector r_{12} represents the directed line segment from Q_1 to Q_2 and is given by,

$$\bar{r}_{12} = \bar{r}_2 - \bar{r}_1$$

According to the statement,

$$F \propto Q_1 Q_2$$

$$F \propto \frac{1}{r_{12}^2}$$

$$F = K \frac{Q_1 Q_2}{r_{12}^2}$$

$$F = \frac{Q_1 Q_2}{4\pi\epsilon (r_{12})^2} \quad \text{Newtons}$$

In vector form

$$\bar{F} = \frac{Q_1 Q_2}{4\pi\epsilon(\bar{r}_{12})^2} \cdot \bar{a}_{r_{12}}$$

From above figure

$$|\bar{r}_{12}| = \sqrt{r_2^2 + r_1^2}$$

$$\bar{a}_{r_{12}} = \frac{\bar{r}_{12}}{|\bar{r}_{12}|}$$

$$\bar{a}_{r_{12}} = \frac{\bar{r}_2 - \bar{r}_1}{\sqrt{\bar{r}_2^2 + r_1^2}}$$

$$\therefore \bar{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon\left(\sqrt{r_2^2+r_1^2}\right)^2} \cdot \frac{(\bar{r}_2-\bar{r}_1)}{\sqrt{r_2^2+r_1^2}}$$

$$\bar{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon(r_2^2+r_1^2)^{3/2}} \cdot (\bar{r}_2-\bar{r}_1) \quad — ①$$

III'ly force on Q_1 due to a charge of Q_2 is given by

$$\bar{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon(r_2^2+r_1^2)^{3/2}} \cdot (\bar{r}_1-\bar{r}_2) \quad — ②$$

From equation ① and ②

$$\vec{F}_{21} = -\vec{F}_{12}$$

where,

Q, Q_2 = +ve or -ve quantities of charge (coulombs)

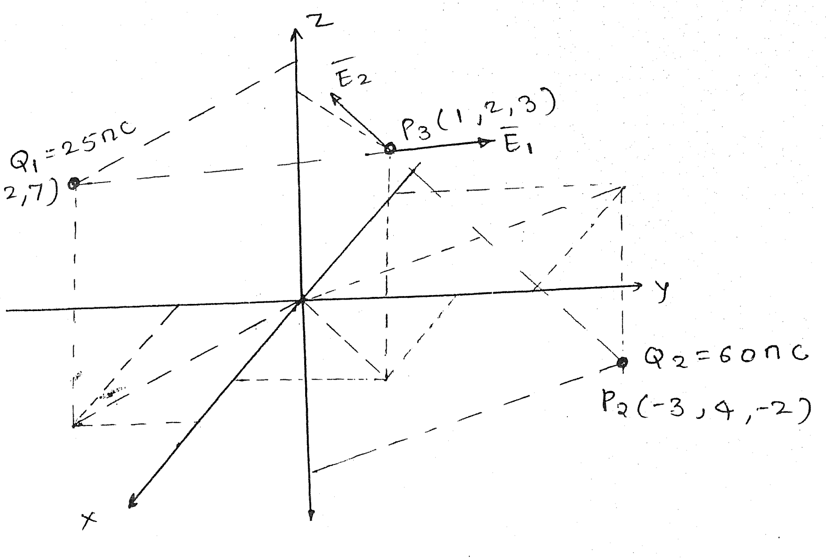
R_{12} = Distance between the charges (meter)

ϵ_0 = Absolute permitivity of the medium (F.m)

$\hat{a}_{12}$ = Unit vector in the direction of R_{12}.

b] A point charge $Q_1 = 25$ nC is located at $P_1(4,-2,7)$ and a charge $Q_2 = 60$ nC is at $P_2(-3,4,-2)$ in free space. Find electric field $\bar{E}$ at $P_3(1,2,3)$.

⇒



$$\bar{E}_1 = \frac{Q_1}{4\pi\epsilon|\overline{P_1P_3}|^2} \cdot \bar{a}_{P_1P_3}$$

$$\overline{P_1P_3} = -3\bar{a}_x + 4\bar{a}_y - 4\bar{a}_z$$

$$|\overline{P_1P_3}| = \sqrt{9+16+16} = 6.4$$

$$\bar{a}_{P_1P_3} = \frac{\overline{P_1P_3}}{|\overline{P_1P_3}|} = \frac{-3\bar{a}_x + 4\bar{a}_y - 4\bar{a}_z}{6.4}$$

$$\bar{E}_1 = \frac{25n}{4\pi\epsilon_0 (6.4)^2} \cdot \frac{-3\bar{a}_x + 4\bar{a}_y - 4\bar{a}_z}{6.4}$$

$$\bar{E}_1 = 0.857(-3\bar{a}_x + 4\bar{a}_y - 4\bar{a}_z)$$

$$\bar{E}_1 = -2.571\,\bar{a}_x + 3.428\,\bar{a}_y - 3.428\,\bar{a}_z \quad N/C$$

$$\bar{E}_2 = \frac{Q_2}{4\pi\epsilon|\overline{P_2P_3}|^2}\,\bar{a}_{P_2P_3}$$

$$\overline{P_2P_3} = 4\bar{a}_x - 2\bar{a}_y + 5\bar{a}_z$$

$$|\overline{P_2P_3}| = \sqrt{16+4+25} = 6.71$$

$$\bar{a}_{P_2P_3} = \frac{\overline{P_2P_3}}{|\overline{P_2P_3}|} = \frac{4\bar{a}_x - 2\bar{a}_y + 5\bar{a}_z}{6.71}$$

$$\bar{E}_2 = \frac{60n}{4\pi\epsilon_0 (6.71)^2} \cdot \frac{(4\bar{a}_x - 2\bar{a}_y + 5\bar{a}_z)}{6.71}$$

$$\bar{E}_2 = 1.785(4\bar{a}_x - 2\bar{a}_y + 5\bar{a}_z)$$

$$\bar{E}_2 = 7.14\,\bar{a}_x - 3.57\,\bar{a}_y + 8.925\,\bar{a}_z \quad N/C$$

Total Field

$$\bar{E} = \bar{E}_1 + \bar{E}_2$$

$$\bar{E} = (-2.571 + 7.14)\bar{a}_x + (4 - 3.57)\bar{a}_y + (-4 + 8.925)\bar{a}_z$$

$$\bar{E} = 4.569\,\bar{a}_x + 0.43\,\bar{a}_y + 4.925\,\bar{a}_z$$

$$|\bar{E}| = \sqrt{(4.569)^2 + (0.43)^2 + (4.925)^2}$$

$$|\bar{E}| = 6.73 \text{ V/m}$$

c] Evaluate both sides of the divergence theorem for the field $\bar{D} = 2xy\,\bar{a}_x + x^2\,\bar{a}_y$ c/m^2, the surface is a rectangular parallelepiped formed by planes $x=0$ and $x=1$, $y=0$ and $y=2$ and $z=0$ and $z=3$

=> The divergence theorem states that

$$\oint_S \bar{D}\cdot d\bar{s} = \int_V (\nabla\cdot\bar{D})\,dv$$

$$R.H.S = \int_V (\nabla\cdot\bar{D})\,dv$$

In rectangular co-ordinate system

$$\nabla \cdot \bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\nabla \cdot \bar{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) + 0$$

$$\nabla \cdot \bar{D} = 2y + 0$$

$$\nabla \cdot \bar{D} = 2y$$

$$\therefore \int_V (\nabla \cdot \bar{D})\, dv$$

$$= \int_V 2y \; dx\,dy\,dz$$

$$= \int_0^1 dx \int_0^2 2y\,dy \int_0^3 dz$$

$$= [x]_0^1 \; [2y]_0^2 \; [z]_0^3$$

$$= [1]\,[4]\,[3]$$

$$= 12$$

$\therefore$ L.H.S = R.H.S,

$\therefore$ Divergence theorem is prooved.

b] Given the potential $V = 2x^2y - 5z$. Determine the expression for electric field intensity $\bar{E}$, the flux density $\bar{D}$ and Volume charge density ρ_v. Find the numerical values of V, E, D, ρ_v at a given point $P(-4,3,6)$. Given

$\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$\Rightarrow$ $V = 2x^2y - 5z$

V at $P(-4,3,6)$

$V = 2(-4)^2(3) - 5(6)$

$V = 66$ V

$\bar{E} = -\nabla V$

$\bar{E} = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$

$\bar{E} = -\left[\frac{\partial}{\partial x}(2x^2y-5z)\bar{a}_x + \frac{\partial}{\partial y}(2x^2y-5z)\bar{a}_y + \frac{\partial}{\partial z}(2x^2y-5z)\bar{a}_z\right]$

$\bar{E} = -\left[4xy\,\bar{a}_x + 2x^2\bar{a}_y - 5\bar{a}_z\right]$

$\bar{E} = -4xy\,\bar{a}_x - 2x^2\bar{a}_y + 5\bar{a}_z$

At $P(-4,3,6)$.

$\bar{E} = -4(-4)(3)\bar{a}_x - 2(-4)^2\bar{a}_y + 5\bar{a}_z$

$\bar{E} = 48\bar{a}_x - 32\bar{a}_y + 5\bar{a}_z$ V/m

$|\bar{E}| = \sqrt{48^2 + 32^2 + 5^2} = 57.91$ V/m

$\bar{D} = \epsilon_0\bar{E}$

$\bar{D} = 8.854 \times 10^{-12}(48\bar{a}_x - 32\bar{a}_y + 5\bar{a}_z)$

$\bar{D} = 4.25 \times 10^{-10}\bar{a}_x - 2.83 \times 10^{-10}\bar{a}_y + 4.43 \times 10^{-11}\bar{a}_z$ c/m²

$|\bar{D}| = \sqrt{(4.25^2 + 2.83^2 + 0.443^2)(10^{-10})^2}$

$|\bar{D}| = 5.125 \times 10^{-10}$ c/m²

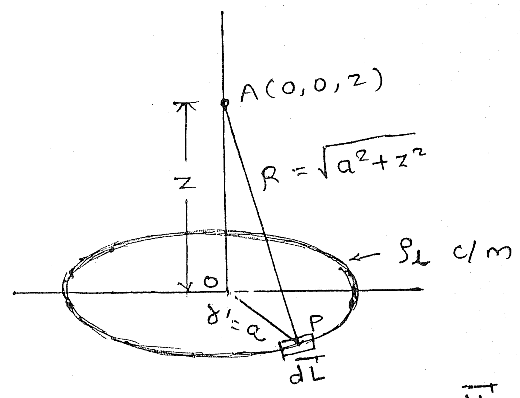
$\rho_v = \nabla \cdot \bar{D}$

$\bar{D} = \epsilon_0\bar{E} = \epsilon_0[48\bar{a}_x - 32\bar{a}_y + 5\bar{a}_z]$

$\therefore \rho_v = 0$ c/m²

2] a] Find the potential V due to a Line charge density ρ_L c/m, bent in the form of a circular ring of radius 'a'.

⇒ Consider a point A (0,0,z) is on z-axis, at a distance z from the origin while radius of the ring is 'a'.



Consider differential length $\bar{dL}$ at point P on the ring. The ring is in z=0 plane hence $\bar{dL}$ in cylindrical system is,

$$\bar{dL} = \bar{r}\, d\phi = a\, d\phi$$

The distance of point A from the differential charge is $R = l(PA)$.

$$R = \sqrt{a^2+z^2}$$

The charge $dQ = \rho_L a\, d\phi$

$$dV_A = \frac{dQ}{4\pi\epsilon_0 R}$$

$$dV_A = \frac{\rho_L a\, d\phi}{4\pi\epsilon_0\sqrt{a^2+z^2}}$$

Hence the potential of A is to be obtained by integrating dV_A ove the circular ring i.e. path with radius $\bar{r} = a$ and ϕ varies from 0 to 2π

$$V_A = \int_{\phi=0}^{2\pi} \frac{\rho_L a\, d\phi}{4\pi\epsilon_0\sqrt{a^2+z^2}}$$

$$V_A = \frac{\rho_L a}{4\pi\epsilon_0\sqrt{a^2+z^2}} [\phi]_0^{2\pi}$$

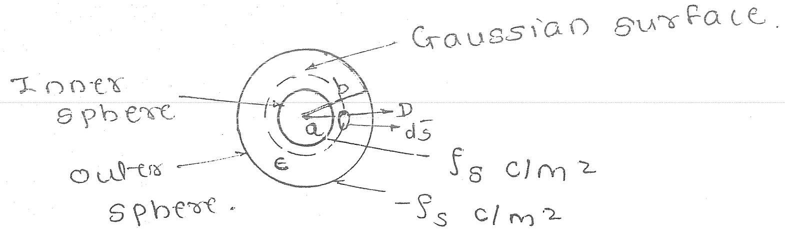
$$V_A = \frac{\rho_L a}{2\epsilon_0\sqrt{a^2+z^2}} \quad \text{Volts}$$

Potential at origin where z = 0

$V_A = \frac{\rho_L}{2\epsilon_0}$ volts.

c] Define capacitance and evaluate capacitance of two concentric spherical conducting shells of radius 'a' and 'b' with $b > a$

⇒ "Capacitance is defined as, the ratio of an impressed charge on a conductor to the corresponding change in potential."



$C = \frac{Q}{V}$ —① by the definition of capacitance.

$Q = \rho_s A$

$Q = \rho_s 4\pi a^2$ —②

$V = -\int_l \bar{E} \cdot \bar{dl}$

$\oint_s \bar{D} \cdot \bar{ds} = Q_{enc}$ —③

$\oint_{s_{r=K}} D_r \bar{a}_r \cdot r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r = Q_{enc}$

$D_r r^2 \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi = Q_{enc}$

$D_r r^2 [-\cos\theta]_0^{\pi} \; [\phi]_0^{2\pi} = Q_{enc}$

$D_r r^2 [1+1][2\pi] = Q_{enc}$

$D_r 4\pi r^2 = \rho_s 4\pi a_2^2$

$D_r = \frac{\rho_s a^2}{r^2}$

$\bar{D} = D_r \cdot \bar{a}_r$

$\bar{D} = \frac{\rho_s a^2}{r^2} \cdot \bar{a}_r$

$\bar{E} = \frac{\bar{D}}{\epsilon}$

$\bar{E} = \frac{\rho_s a^2}{\epsilon r^2} \bar{a}_r$ —④

Putting (4). in (3)

$$V = -\int_l \left(\frac{\rho_s a^2}{\epsilon r^2} \cdot \bar{a}_r \right) \cdot \left(dr\, \bar{a}_r + r\, d\theta\, \bar{a}_\theta + r \sin\theta\, d\phi\, \bar{a}_\phi \right)$$

$$V = -\int_l \frac{\rho_s a^2}{\epsilon r^2}\, dr$$

$$V = -\frac{\rho_s a^2}{\epsilon} \int_b^a \frac{dr}{r^2}$$

$$V = \frac{-\rho_s a^2}{\epsilon} \left[-\frac{1}{r} \right]_b^a$$

$$V = \frac{-\rho_s a^2}{\epsilon} \left[-\frac{1}{a} + \frac{1}{b} \right]$$

$$V = \frac{\rho_s a^2}{\epsilon} \left[\frac{1}{a} - \frac{1}{b} \right] \quad -(5)$$

Putting (5) & (2) in (1) we get

$$C = \frac{\rho_s 4\pi a^2}{\frac{\rho_s a^2}{\epsilon} \left[\frac{1}{a} - \frac{1}{b} \right]}$$

$$C = \frac{4\pi \epsilon ab}{b-a}$$

3] a] Derive Poisson's and Laplace's equation.

⇒ From the Gauss law in point form, Poissons equation can be derived.

Consider the Gauss law in point form as;

$$\nabla \cdot \bar{D} = \rho_v$$

$$\bar{D} = \epsilon \bar{E}$$

where, $\bar{D}$ = Flux density, ρ_v = Volume charge density

$$\nabla \cdot \epsilon \bar{E} = \rho_v$$

$$\epsilon\, \nabla \cdot \bar{E} = \rho_v$$

$$\nabla \cdot \bar{E} = \frac{\rho_v}{\epsilon}$$

From the gradient relationship, we have

$$\bar{E} = -\nabla V$$

$$\nabla \cdot (-\nabla V) = \frac{\rho_v}{\epsilon}$$

$$\nabla^2 V = \frac{-\rho_v}{\epsilon}$$ is the required poissons equation.

For charge free region ; $\rho_v = 0$

$\therefore \nabla^2 V = 0$ Laplace equation.

Rectangular form:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylendrical form:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical form:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2}$$

b] State and prove uniqueness theorem.

=> Uniqueness theorem can be stated as :

If the solution of Laplace's equation satisfies the boundary condition then that solution is unique, by whatever method it is obtained.

The solution of Laplace's equation gives the field which is unique, satisfying the same boundary conditions in a given region.

Assume that the Laplace's equation has two solutions say V_1 and V_2, both are function of the co-ordinates of the system used. These solutions must satisfy Laplace's equation.

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0 \qquad \text{— (1)}$$

At the boundary, the potentials at different points are same due to equipotential surface then,

$$V_1 = V_2 \qquad \text{— (2)}$$

Let the difference between the two solutions is V_d.

$$V_d = V_2 - V_1 \qquad \text{— (3)}$$

Using Laplace's equation for the difference V_d,

$$\nabla^2 V_d = \nabla^2 (V_2 - V_1) = 0 \qquad \text{— (4)}$$

$$\nabla^2 V_2 - \nabla^2 V_1 = 0 \qquad \text{— (5)}$$

On the boundary $V_d = 0$ from the equation (2) & (3)

Now, the divergence theorem states that,

$$\int_{vol} \nabla \cdot \bar{A} \, dv = \oint_S \bar{A} \cdot \bar{ds} \qquad \text{— (6)}$$

Let $\bar{A} = V_d \nabla . V_d$ and from vector algebra

$\nabla \cdot (\alpha \bar{B}) = \alpha (\nabla \cdot \bar{B}) + \bar{B} \cdot (\nabla \alpha)$

$\therefore \nabla \cdot (V_d \nabla V_d)$ with $\alpha = V_d$ and $\nabla V_d = \bar{B}$

$\nabla \cdot (V_d \nabla V_d) = V_d (\nabla \cdot \nabla V_d) + \nabla V_d \cdot (\nabla V_d)$

But $\nabla . \nabla = \nabla^2$

$\therefore \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d \cdot \nabla V_d \quad — (7)$

using (4)

$\nabla \cdot (V_d \nabla V_d) = \nabla V_d \cdot \nabla V_d \quad — (8)$

$\therefore$ Equation (6)

Let $\bar{A} = V_d \nabla V_d$

$\nabla \cdot V_d \nabla V_d = \nabla \cdot \bar{A} = \nabla V_d \cdot \nabla V_d$

$\int_{vol} \nabla V_d \cdot \nabla V_d \, dv = \oint_{s} V_d \nabla V_d \cdot \bar{ds} \quad — (9)$

But $V_d = 0$ on boundary

$\int_{vol} \nabla V_d \cdot \nabla V_d \, dv = 0 \quad — (10)$

$\therefore \int_{vol} |\nabla V_d|^2 dv = 0$ as ∇V_d is vector

Now integration can be zero under two conditions,

i) The quantity under integral sign is zero.

ii) The quantity is positive in some regions and negative in other regions by equal amount and hence zero.

But square term can not be negative in any region

$\therefore |\nabla V_d|^2 = 0$

$\nabla V_d = 0 \quad — (12)$

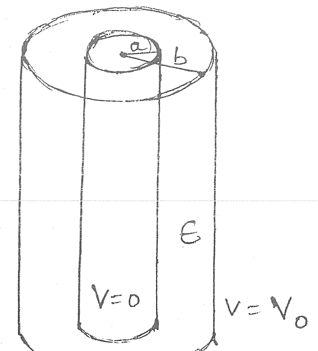
As the gradient of $V_d = V_2 - V_1$ is zero means $V_2 - V_1$ is constant $=$ zero.

$\therefore \quad V_2 = V_1 \quad — (13)$

This proves that both the solutions are equal and Can not be different.

c] Find the capacitance of a co-axial cable with inner radius 'a' and outer radius 'b' where b>a, using Laplace equation.

⇒



From Laplace's equation

$\nabla^2 V = 0$

From above figure the potential varies along only ρ-axis.

$$\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial z} = 0$$

$$\therefore \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

Integrate w.r.t. ρ

$$\rho \frac{\partial V}{\partial \rho} = C_1$$

$$\frac{\partial V}{\partial \rho} = \frac{C_1}{\rho}$$

Again integrate w.r.t. ρ

$$V = \int \frac{C_1}{\rho}\, d\rho$$

$$V = C_1 \ln(\rho) + C_2$$

At $V = V_0$, $\rho = b$

$V = 0$, $\rho = a$

$$0 = C_1 \ln(a) + C_2$$

$$V_0 = C_1 \ln(b) + C_2$$

$$-V_0 = C_1 [\ln(a) - \ln(b)]$$

$$V_0 = -C_1 \ln(a/b)$$

$$C_1 = \frac{V_0}{\ln(b/a)}$$

$$0 = \frac{V_0}{\ln(\frac{b}{a})} \ln(a) + C_2$$

$$C_2 = \frac{-V_0 \ln(a)}{\ln(b/a)}$$

$$V = \frac{V_0}{\ln(\frac{b}{a})} \ln(\rho) - \frac{V_0 \ln(a)}{\ln(\frac{b}{a})}$$

$$\bar{E} = -\nabla V$$

$$\bar{E} = -\left[\frac{\partial V}{\partial \rho} \bar{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \bar{a}_\phi + \frac{\partial V}{\partial z} \bar{a}_z \right]$$

$$\bar{E} = -\left[\frac{\partial}{\partial \rho} \left[\frac{V_0}{\ln(b/a)} \ln(\rho) - \frac{V_0 \ln(a)}{\ln(b/a)} \right] \right] \bar{a}_\rho$$

$$\bar{E} = -\frac{V_0}{\ln(b/a)} \times \frac{1}{\rho} \bar{a}_\rho$$

$$|\bar{E}| = \frac{V_0}{\rho \ln(\frac{b}{a})}$$

$$|\bar{D}| = \epsilon |\bar{E}|$$

$$|\bar{D}| = \epsilon \frac{V_0}{\rho \ln(\frac{b}{a})}$$

$$Q = \rho_S A$$

$$Q = |\bar{D}| A$$

$$Q = \frac{\epsilon V_0}{\rho \ln(\frac{b}{a})} \times 2\pi \rho h$$

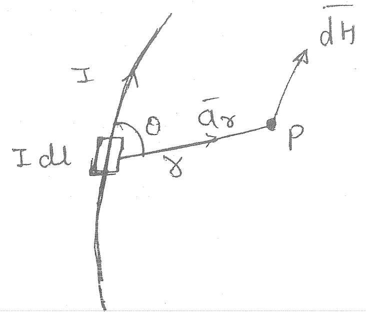
$$\frac{Q}{V_0} = \frac{2\pi \epsilon h}{\ln(\frac{b}{a})}$$

$$\because C = \frac{Q}{V}$$

$$\therefore C = \frac{2\pi \epsilon h}{\ln(\frac{b}{a})}$$

4]
a) State and explain Biot - Savart law.

$\Rightarrow$



Biot - Savart law state that, the magnetic field intensity $\overline{dH}$ produced at point P due to a different current element Idl is,

i) Directly proportional to the product of current I and differential length dl.

ii) Directly proportional to the sine of the angle between the element and the line joining to the point P.

iii) Inversly proportional to the square of distance between point P and element.

$\therefore dH \propto I|\bar{dl}|$

$dH \propto \sin\theta$

$dH \propto \frac{1}{r^2}$

$\therefore dH \propto \frac{I\,dl\,\sin\theta}{r^2}$

$\overline{dH} = \frac{K\,I\,dl\,\sin\theta}{r^2}$

$K = \frac{1}{4\pi}$

$\overline{dH} = \frac{I|\bar{dl}|\sin\theta}{4\pi r^2}$

$d\bar{H} = \frac{I\,\bar{dl} \times \bar{a}_r}{4\pi r^2}$ A/m

$\bar{H} = \oint \frac{I\,\bar{dl} \times \bar{a}_r}{4\pi r^2}$ A/M b] Calculate vector current density at a given point P(2,3,4) if $\bar{H} = x^2 z \, \bar{a}_y - y^2 x \, \bar{a}_z$.

⇒ $\bar{J} = \nabla \times \bar{H}$

$$\bar{J} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\bar{J} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2 z & -y^2 x \end{vmatrix}$$

$$\bar{J} = \bar{a}_x \left(-2xy - x^2\right) - \bar{a}_y \left(-y^2 - 0\right) + \bar{a}_z \left(2xz - 0\right)$$

$$\bar{J} = (-2xy - x^2)\,\bar{a}_x + y^2 \bar{a}_y + 2xz\,\bar{a}_z$$

At $P = (2, 3, 4)$

$$\bar{J} = (-2 \times 2 \times 3 - 2^2)\,\bar{a}_x + 3^2 \bar{a}_y + 2(2)(4)\,\bar{a}_z$$

$$\bar{J} = -16\,\bar{a}_x + 9\,\bar{a}_y + 16\,\bar{a}_z \quad A/m^2$$

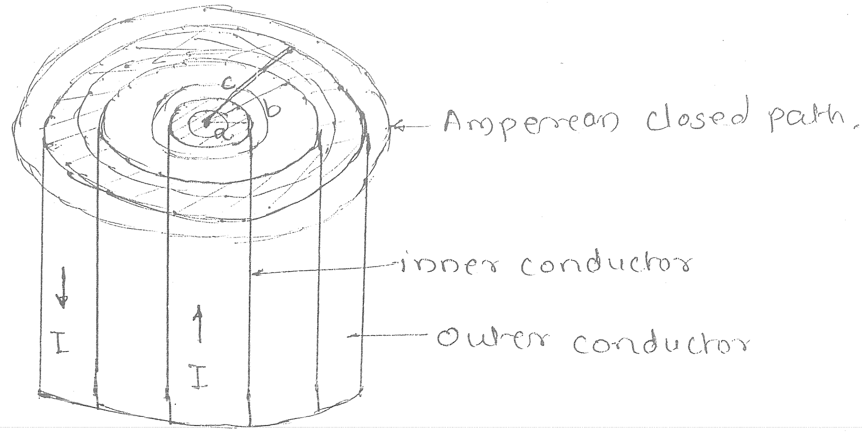
$$|\bar{J}| = \sqrt{16^2 + 9^2 + 16^2}$$

$$|\bar{J}| = 24.35 \ A/m^2$$

c] State Ampere's circuit law. Apply it to a co-axial cable with inner conductor of radius 'a' carrying current I. The outer conductor carries return current -I. The inner radius of outer conductor is 'b' and its outer radius is 'c'. Evaluate magnetic field intensity.

⇒ Ampere's circuit law state that, line integral of the magnetic field intensity over closed path around a conductor is equal to the current enclosed.

$$\oint \bar{H} \cdot d\bar{l} = I_{enclosed}.$$



i] magnetic Field intensity inside the inner conductor

$(r < a)$

$$\oint_l \bar{H} \cdot \bar{dl} = I_{enc}$$

$$\oint_l H_\phi \, \bar{a}_\phi \cdot r \, d\phi \, \bar{a}_\phi = I_{enc}$$

$$\oint_l H_\phi \, r \, d\phi = I_{enc}$$

$$H_\phi \, r \, [\phi]_0^{2\pi} = J \pi r^2$$

$$H_\phi \, 2\pi r = J \pi r^2$$

$$H_\phi = \frac{J r}{2}$$

$$\bar{H} = H_\phi \, \bar{a}_\phi$$

$$\bar{H} = \frac{J r}{2} \, \bar{a}_\phi$$

$$\bar{H} = \frac{I}{2\pi a^2} \cdot r \, \bar{a}_\phi$$

ii] magnetic Field intensity at $a < r < b$

$$\bar{H} = \frac{I}{2\pi r} \, \bar{a}_\phi$$

iii] magnetic field intensity at $b < r < c$

$$\oint_l \bar{H} \cdot \bar{dl} = I_{enc}$$

$$\oint_l H_\phi \, \bar{a}_\phi \cdot r \, d\phi \, \bar{a}_\phi = I_{enc}$$

$$\oint_l H_\phi \, r \, d\phi = I_{enc}$$

$$H_\phi \, r \, 2\pi = I + J \pi (r^2 - b^2)$$

$\therefore r H_\phi 2\pi = I + J\pi (r^2 - b^2)$

$$= I + \frac{(-I)}{\pi(c^2-b^2)} \cdot \pi (r^2 - b^2)$$

$$r H_\phi 2\pi = I \left[\frac{c^2 - b^2 - r^2 + b^2}{c^2 - b^2} \right]$$

$$r H_\phi 2\pi = I \frac{(c^2 - r^2)}{(c^2 - b^2)}$$

$$H_\phi = \frac{I}{2\pi r} \frac{(c^2 - r^2)}{(c^2 - b^2)}$$

$$\bar{H} = H_\phi \bar{a}_\phi$$

$$\bar{H} = \frac{I}{2\pi r} \cdot \frac{(c^2 - r^2)}{(c^2 - b^2)} \bar{a}_\phi$$

iv] magnetic field intensity at $r > c$

$$\oint_l \bar{H} \cdot \bar{dl} = I_{enc}$$

$$\oint_l H_\phi \bar{a}_\phi \cdot r d\phi \, \bar{a}_\phi = I_{enc}$$

$$H_\phi r \oint_l d\phi = I - I$$

$$2\pi r H_\phi = 0$$

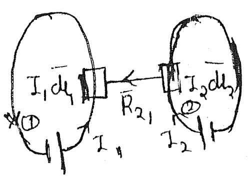
$$H_\phi = 0$$

$$\therefore \bar{H} = 0$$

5]

a] Derive the equation for force between two differential current carrying elements.

$\Rightarrow$ Let us now consider two current elements $I_1 \bar{dl}_1$ and $I_2 \bar{dl}_2$



Both the current elements produces their own magnetic fields. The force exerted on a differential current element is given by,

$\bar{dF} = I\,\bar{dl} \times \bar{B}$

$d(\bar{dF_1}) = I_1 \bar{dL_1} \times dB_2$ — ①

According to Biot - savart law the magnetic field produced by current element is given by,

$$\bar{dB_2} = \mu d\bar{H_2} = \mu \frac{(I_2 \bar{dL_2} \times \bar{a}_{R21})}{4\pi \bar{R}_{21}^2} \quad — ②$$

Substituting dB_2 in equation ①

$$d(d\bar{F_1}) = I_1 d\bar{L_1} \times \mu \frac{(I_2 d\bar{L_2} \times \bar{a}_{R21})}{4\pi \bar{R}_{21}^2} \quad -③$$

$$d(d\bar{F_1}) = \frac{\mu I_1 I_2}{4\pi} \frac{\bar{dl_1} \times d\bar{l_2} \times \bar{a}_{R21}}{|\bar{R}_{21}|^2}$$

Integrating the above equation twice, the total force $\bar{F_1}$ on current element 1 due to current element 2 is given by,

$$\bar{F_1} = \frac{\mu I_1 I_2}{4\pi} \oint_{l_1} \oint_{l_2} \frac{d\bar{l_1} \times d\bar{l_2} \times \bar{a}_{R21}}{|\bar{R}_{21}|^2}$$

$\bar{F_2}$ Force on loop 2 due to flux density produced by $I_1 dL_1$

$$\bar{F_2} = \frac{\mu I_1 I_2}{4\pi} \oint_{l_2} \oint_{l_1} \frac{d\bar{l_2} \times d\bar{L_1} \times \bar{a}_{R12}}{|\bar{R}_{12}|^2}$$

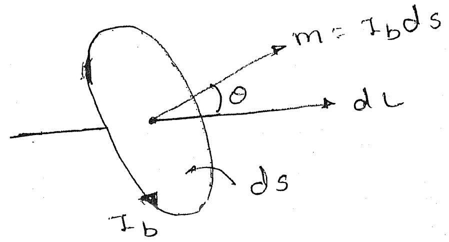
b] Explain the terms magnetization and permeability.

⇒ The movement of the orbital electrons, electron spin and nuclear spin produce internal magnetic field similar to that produced by current loop. The current produced by the bound charges is called bound current represented by I_b. The bound charges are charges which are bound to nucleus. The field produced due to movement of bound charges is called magnetization represented by $\bar{M}$.

The magnetization is defined as the magnetic dipole moment per unit volume.

$$\bar{M} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{a=1}^{n\Delta V} \bar{m}_a$$

Let us consider alignment of a magnetic dipole along a closed path as shown in the fig. below



∴ Bound current,

$$dI_b = nI_b d\bar{s} \cdot d\bar{L} = \bar{M} \cdot d\bar{L}$$

$$I_b = \oint \bar{M} \cdot d\bar{L}$$

Ampere circuit law for the total current I_T

$$I_T = I_b + I = \oint \bar{H} \cdot d\bar{L}$$

$$I_T = \oint \frac{\bar{B}}{\mu_0} \cdot d\bar{L}$$

$$I = I_T - I_b$$

$$I = \oint \frac{\bar{B}}{\mu_0} \cdot d\bar{L} - \oint \bar{M} \cdot d\bar{L}$$

$$I = \oint \left(\frac{\bar{B}}{\mu_0} - \bar{M} \right) \cdot d\bar{L}$$

Compare this equation with ampere's circuit law

$$I = \oint \bar{H} \cdot d\bar{L}$$

$$\bar{H} = \frac{\bar{B}}{\mu_0} - \bar{M}$$

$$\bar{B} = \mu_0 (\bar{H} + \bar{M})$$

For linear, isotropic magnetic materials,

$$\bar{M} = \chi_m \bar{H}$$

where, χ = magnetic susceptibility.

$$\therefore \bar{B} = \mu_0 (\bar{H} + \chi_m \bar{H}) = \mu_0 (1 + \chi_m) \bar{H}$$

$$\bar{B} = \mu \bar{H} = \mu_0 \mu_r \bar{H}$$

$\therefore \mu_r = (1 + \chi_m) = \frac{\mu}{\mu_0}$

In general $\mu = \mu_0 \mu_r$ is called permeability of a material.

Consider again the expressions for the current,

$$I_b = \oint_s \bar{J}_b \cdot d\bar{s}$$

$$I_T = \oint_s \bar{J}_T \cdot d\bar{s}$$

$$I = \oint_s \bar{J} \cdot d\bar{s}$$

From the curl definition

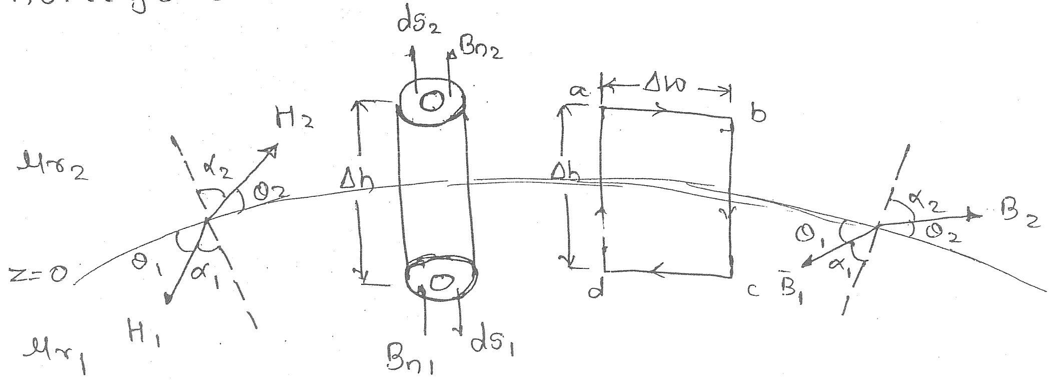
$$\nabla \times \bar{M} = \bar{J}_b$$

$$\nabla \times \frac{\bar{B}}{\mu_0} = \bar{J}_T$$

$$\nabla \times \bar{H} = \bar{J}$$

c] Derive the boundary condition between two isotropic homogeneous materials with permeability μ_1 and μ_2.

⇒



Expression for B_{n1} and B_{n2}

$$\oint_s \bar{B} \cdot d\bar{s} = 0$$

$$\oint_{B.S} \bar{B} \cdot d\bar{s} + \int_{T.S} \bar{B} \cdot d\bar{s} + \oint_{CS} \bar{B} \cdot d\bar{s} = 0$$

$$B_{n1}(-\pi a^2) + B_{N2}(\pi a^2) + 0 = 0$$

$$(-B_{n1} + B_{n2})\pi a^2 = 0$$

$$B_{n2} - B_{n1} = 0$$

$$\boxed{B_{n2} = B_{n1}}$$

$B_{n1} = \mu_1 H_{n1}$ From $B = \mu H$ $\therefore H = \frac{B}{\mu}$.

$\mu_2 H_{n2} = \mu_1 H_{n1}$

$$\boxed{\frac{H_{n2}}{H_{n1}} = \frac{\mu_{r1}}{\mu_{r2}}}$$

Tangential component.

Applying ampere circuit law to the closed loop.

$\oint_l \bar{H} \cdot d\bar{l} = I_{enc}$.

$$\boxed{H_{tan2} - H_{tan1} = K}$$

If the interface is free of sheet current. i.e $K=0$

$$\boxed{\bar{H}_{tan2} = \bar{H}_{tan1}}$$

$$\boxed{\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\mu_{r1}}{\mu_{r2}}}$$

$$\boxed{\therefore \frac{B_{tan1}}{B_{tan2}} = \frac{\mu_{r1}}{\mu_{r2}}}$$

6]

a] State and explain Faraday's law.

Faraday's laws:-

Whenever a flux linking with the coil changes an emf is induced in that coil due to change in flux linkages.

Magnitude of emf induced is equal to the rate of change of flux linkages.

$$e \propto \frac{d\phi}{dt}$$

$$e = -N\frac{d\phi}{dt}$$

where, N = Number of conductors

$$\oint_l \bar{E} \cdot d\bar{l} = -\frac{d}{dt}\int_s \bar{B} \cdot d\bar{s}$$

$$\oint_l \bar{E} \cdot d\bar{l} = -\frac{\partial}{\partial t}\int_s \bar{B} \cdot d\bar{s}$$

Differential form of Faraday's law

From stokes theorem

$$\oint_l \bar{H} \cdot d\bar{l} = \oint_s (\nabla \times \bar{H})\, d\bar{s}$$

$$\oint_l \bar{E}\cdot d\bar{l} = \oint_s (\nabla\times\bar{E})\cdot d\bar{s}$$

$$\oint_s (\nabla\times\bar{E})\, ds = -\frac{\partial}{\partial t}\int_s \bar{B}\cdot d\bar{s}$$

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

Faraday's Law in point form.

b) Write Maxwell's equation in integral and point form for time varying fields.

=> Maxwells equations for time varying fields.

i) Gauss's law (electrical field):

$$\oint_s \bar{D}\cdot d\bar{s} = \int_{vol} \rho_v \, dv$$

$$\nabla\cdot\bar{D} = \rho_v$$

ii) Amperes circuit law:

$$\oint \bar{H}\cdot d\bar{l} = \oint_s \frac{\partial\bar{D}}{\partial t}\cdot d\bar{s} + I$$

where $I = \oint_s \bar{J}_c \, d\bar{s}$

$$\nabla\times\bar{H} = \bar{J}_c + \frac{\partial\bar{D}}{\partial t}$$

iii) Gauss's law (magnetic field):

$$\oint_s \bar{B}\cdot d\bar{s} = 0$$

$$\nabla\cdot\bar{B} = 0$$

iv) Faraday's law:

$$\oint_l \bar{E}\cdot d\bar{l} = -\oint_s \frac{\partial\bar{B}}{\partial t}\cdot d\bar{s}$$

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

These are the required maxwell's equation in integral and point form.

c] Derive the concept of displacement current density.

$\Rightarrow$ For static electromagnetic fields, according to Ampere's circuit law

$$\nabla \times \bar{H} = \bar{J}$$

Taking divergence on both the sides,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J}$$

W.K.T. divergence of curl of any vector field is zero

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} = 0$$

Continuity equation is given by

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

Let $\nabla \times \bar{H} = \bar{J} + \bar{N}$

Again taking divergence on both sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \bar{N} = 0$$

As $\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$

$$\therefore \nabla \cdot \bar{N} = \frac{\partial \rho_v}{\partial t}$$

From Gauss law,

$$\rho_v = \nabla \cdot \bar{D}$$

$$\therefore \nabla \cdot \bar{N} = \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$\nabla \cdot \bar{N} = \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

$$\bar{N} = \frac{\partial \bar{D}}{\partial t}$$

$\therefore$ Ampere's circuit law in point form

$$\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t} = \bar{J}_c + \bar{J}_D$$

where, $\bar{J}_c = \sigma \bar{E}$ = conduction current density

$\bar{J}_D$ = Displacement current density.

7]
9] Derive the wave equation for uniform plane wave propogation in perfect dielectric and explain the concept of loss tangent.

⇒ If a medium, through which the uniform plane wave is propagating, is perfect dielectric, then the conductivity is zero. i.e. $\sigma = 0$. Let the permittivity permeability of the medium be $\epsilon = \epsilon_0 \epsilon_r$ and $\mu = \mu_0 \mu_r$ respectively.

The propagation constant γ is given by,

$$\gamma = \sqrt{j\omega\mu(0 + j\omega\epsilon)} = \pm j\omega\sqrt{\mu\epsilon}$$

$$\gamma = \alpha + j\beta = \pm j\omega\sqrt{\mu\epsilon} \quad \text{—①}$$

From equation ① it is clear that, propagation constant is purely imaginary. It indicates in a perfect dielectric medium, attenuation constant α is zero. Let us select value of β which gives propogation of wave in positive z-direction.

$$\therefore \alpha = 0 \quad , \beta = \omega\sqrt{\mu\epsilon} \quad \text{—②}$$

Similarly an intrinsic impedance for a perfect dielectric medium is given by,

$$\eta = \sqrt{\frac{j\omega\mu}{0 + j\omega\epsilon}}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \ \Omega \quad \text{—③}$$

Thus intrinsic impedance η is real resistive. That means phase angle of intrinsic impedance is zero. But the phase angle of intrinsic impedance is zero means phase difference between $\bar{E}$ and $\bar{H}$ is zero. In other words, for a perfect dielectric, both the field $\bar{E}$ and $\bar{H}$ are in phase.

As in perfect dielectric, $\sigma = 0$, attenuation constant (α) is also zero. As wave propagates only the phase (β) changes. Thus no attenuation i.e. $\alpha = 0$ means no loss.

b] Derive the wave equation for uniform plane wave propagation in perfect conductor and explain the concept of skin effect.

⇒ A practical or good conductor is the material which has very high conductivity. In general, the conductivity is of the order of 10^7 ℧/m in the good conductors like copper, aluminium etc.

For good conductors,

$$\frac{\sigma}{\omega\epsilon} >> 1$$

The propagation constant γ is given by,

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} \quad \text{—(1)}$$

As $\sigma >> \omega\epsilon$, we can neglect imaginary part $(j\omega\epsilon)$

$$\gamma = \sqrt{j\omega\mu\sigma}$$

$$\gamma = \sqrt{\omega\mu\sigma}\,\sqrt{j}$$

but $j = 1\angle 90^\circ$

$$\gamma = \sqrt{\omega\mu\sigma}\,\sqrt{1\angle 90^\circ}$$

$$\gamma = \sqrt{\omega\mu\sigma}\,\angle 45^\circ$$

$$\gamma = \sqrt{\omega\mu\sigma}\,[\cos 45 + j\sin 45^\circ]$$

$$\gamma = \sqrt{\omega\mu\sigma}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$\gamma = \sqrt{2\pi f\mu\sigma}\left[\frac{1}{\sqrt{2}}(1+j1)\right]$$

$$\gamma = \alpha + j\beta = \sqrt{\pi f\mu\sigma} + j\sqrt{\pi f\mu\sigma} \quad \text{—(2)}$$

Thus for good conductor,

$$\alpha = \sqrt{\pi f\mu\sigma} \quad \text{Np/m}$$

$$\beta = \sqrt{\pi f\mu\sigma} \quad \text{rad/m}$$

For good conductor, α and β are equal and both are directly proportional to the square root of frequency and conductivity.

The intrinsic impedance of a good conductor is given by

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \quad -③$$

But for good conductor, $\sigma >> j\omega\epsilon$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\sqrt{j}$$

But $\sqrt{j} = \sqrt{1\angle 90^\circ} = 1\angle 45^\circ = \cos 45 + j\sin 45$

$$\sqrt{j} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$\therefore \eta = \sqrt{\frac{\omega\mu}{\sigma}}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

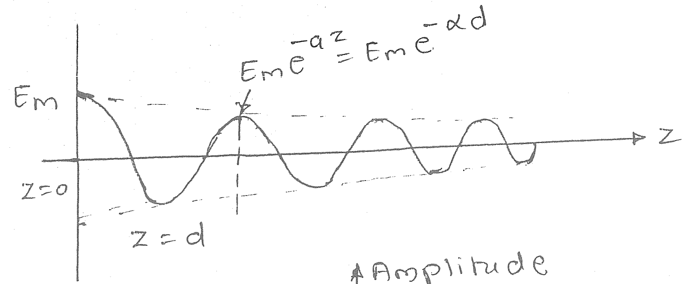
$$\eta = \sqrt{\frac{\omega\mu}{2\sigma}}(1+j1)$$

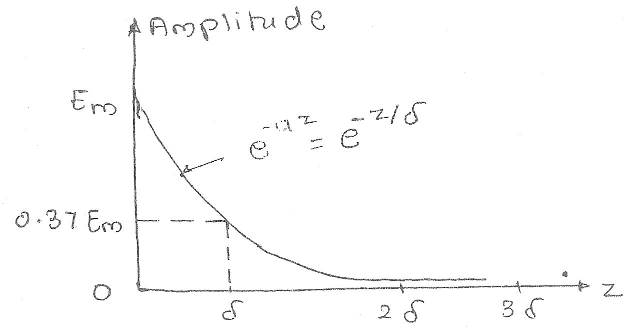
$$\eta = \sqrt{\frac{2\pi f\mu}{2\sigma}}(1+j1)$$

$$\eta = \sqrt{\frac{\pi f\mu}{\sigma}}(1+j1). \quad -④$$

Consider only the component of the electric field E_x travelling in positive z-direction. When it travels in good conductor, the conductivity is very high and attenuation constant α is also high. Thus we can write such a component in phasor form as

$$E_x = E_m^+ e^{-\alpha z} e^{-j\beta z} \quad -⑤$$





At $z=0$, $E_x = E_m$ At $z=d$, amplitude is $E_m e^{-\alpha d}$

If we select $d = \frac{1}{\alpha}$, then the factor becomes $e^{-1} = 0.368$

So over a distance $d = \frac{1}{\alpha}$ the amplitude of the wave decreases to approximately 37% of its original value.

The distance through which the amplitude of the travelling wave decreases to 37% of the original amplitude is called skin depth or depth of penetration. It is denoted by δ.

$$\therefore \text{Skin depth} = \delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f \mu \sigma}} \; m \qquad \text{— (6)}$$

From the expression of the skin depth, it is clear that δ is inversely proportional to the square root of frequency. So for the frequencies in the microwave range, the skin depth or depth of penetration is very small for good conductors. And all the fields and currents may be considered as confined to a very thin layer near the surface of the conductor. This thin layer is nothing but the skin of the conductor, hence this effect is called skin effect.