1 Operations on Languages

Operations on Languages

- Recall: A language is a set of strings
- We can consider new languages derived from operations on given languages
  - e.g., $L_1 \cup L_2$, $L_1 \cap L_2$, ...
- A simple but powerful collection of operations:
  - Union, Concatenation and Kleene Closure

Union is a familiar operation on sets. We define and explain the other two operations below.

Concatenation of Languages

Definition 1. Given languages L_1 and L_2, we define their *concatenation* to be the language $L_1 \circ L_2 = \{xy \mid x \in L_1,\ y \in L_2\}$

Example 2.
- $L_1 = \{\text{hello}\}$ and $L_2 = \{\text{world}\}$ then $L_1 \circ L_2 = \{\text{helloworld}\}$
- $L_1 = \{00, 10\}$; $L_2 = \{0, 1\}$. $L_1 \circ L_2 = \{000, 001, 100, 101\}$
- $L_1 =$ set of strings ending in 0; $L_2 =$ set of strings beginning with 01. $L_1 \circ L_2 =$ set of strings containing 001 as a substring
- $L \circ \{\epsilon\} = L$. $L \circ \emptyset = \emptyset$.

Kleene Closure

Definition 3.

$$L^n = \begin{cases} \{\epsilon\} & \text{if } n = 0 \\ L^{n-1} \circ L & \text{otherwise} \end{cases} \qquad L^* = \bigcup_{i \geq 0} L^i$$

i.e., L^i is $L \circ L \circ \cdots \circ L$ (concatenation of i copies of L), for $i > 0$.

L^*, *the Kleene Closure of* L: set of strings formed by taking any number of strings (possibly none) from L, possibly with repetitions and concatenating all of them.

- If $L = \{0, 1\}$, then $L^0 = \{\epsilon\}$, $L^2 = \{00, 01, 10, 11\}$. $L^* =$ set of *all* binary strings (including ϵ).
- $\emptyset^0 = \{\epsilon\}$. For $i > 0$, $\emptyset^i = \emptyset$. $\emptyset^* = \{\epsilon\}$
- $\emptyset$ is one of only two languages whose Kleene closure is finite. Which is the other? $\{\epsilon\}^* = \{\epsilon\}$.

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2 Regular Expressions

2.1 Definition and Identities

Regular Expressions
A Simple Programming Language

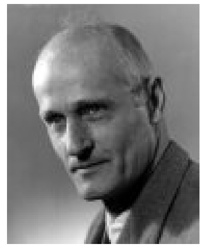

Figure 1: Stephen Cole Kleene

A *regular expression* is a formula for representing a (complex) language in terms of "elementary" languages combined using the three operations union, concatenation and Kleene closure.

Regular Expressions
Formal Inductive Definition

Syntax and Semantics
A regular expression over an alphabet Σ is of one of the following forms:

	Syntax	Semantics
	$\emptyset$	$\mathbf{L}(\emptyset) = \{\}$
Basis	ϵ	$\mathbf{L}(\epsilon) = \{\epsilon\}$
	a	$\mathbf{L}(a) = \{a\}$
	$(R_1 \cup R_2)$	$\mathbf{L}((R_1 \cup R_2)) = \mathbf{L}(R_1) \cup \mathbf{L}(R_2)$
Induction	$(R_1 \circ R_2)$	$\mathbf{L}((R_1 \circ R_2)) = \mathbf{L}(R_1) \circ \mathbf{L}(R_2)$
	(R_1^*)	$\mathbf{L}((R_1^*)) = \mathbf{L}(R_1)^*$

Notational Conventions
Removing the brackets To avoid cluttering of parenthesis, we adopt the following conventions.

- Precedence: $*, \circ, \cup$. For example, $R \cup S^* \circ T$ means $(R \cup ((S^*) \circ T))$
- Associativity: $(R \cup (S \cup T)) = ((R \cup S) \cup T) = R \cup S \cup T$ and $(R \circ (S \circ T)) = ((R \circ S) \circ T) = R \circ S \circ T$.

Also will sometimes omit $\circ$: e.g. will write RS instead of $R \circ S$

Regular Expression Examples

R	$\mathbf{L}(R)$
$(0 \cup 1)^*$	$= (\{0\} \cup \{1\})^* = \{0,1\}^*$
$0\emptyset$	$\emptyset$
$0^* \cup (0^*10^*10^*10^*)^*$	Strings where the number of 1s is divisible by 3
$(0 \cup 1)^*001(0 \cup 1)^*$	Strings that have 001 as a substring
$(10)^* \cup (01)^* \cup 0(10)^* \cup 1(01)^*$	Strings that consist of alternating 0s and 1s
$(\epsilon \cup 1)(01)^*(\epsilon \cup 0)$	Strings that consist of alternating 0s and 1s
$(0 \cup \epsilon)(1 \cup 10)^*$	Strings that do not have two consecutive 0s

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Regular Languages

Definition 4. A language $L \subseteq \Sigma^*$ is a *regular language* iff there is a regular expression R such that $\mathbf{L}(R) = L$.

Some Regular Expression Identities

We say $R_1 = R_2$ if $\mathbf{L}(R_1) = \mathbf{L}(R_2)$.

- *Commutativity:* $R_1 \cup R_2 = R_2 \cup R_1$ (but $R_1 \circ R_2 \neq R_2 \circ R_1$ typically)
- *Associativity:* $(R_1 \cup R_2) \cup R_3 = R_1 \cup (R_2 \cup R_3)$ and $(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$
- *Distributivity:* $R \circ (R_1 \cup R_2) = R \circ R_1 \cup R \circ R_2$ and $(R_1 \cup R_2) \circ R = R_1 \circ R \cup R_2 \circ R$
- *Concatenating with ϵ:* $R \circ \epsilon = \epsilon \circ R = R$
- *Concatenating with $\emptyset$:* $R \circ \emptyset = \emptyset \circ R = \emptyset$
- $R \cup \emptyset = R$. $R \cup \epsilon = R$ iff $\epsilon \in L(R)$
- $(R^*)^* = R^*$
- $\emptyset^* = \epsilon$

Useful Notation

Definition 5. Define $R^+ = RR^*$. Thus, $R^* = R^+ \cup \epsilon$. In addition, $R^+ = R^*$ iff $\epsilon \in L(R)$.
