Module 2
MATHEMATICAL INDUCTION

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Mathematical induction, is a technique for proving results or establishing statements for natural numbers. This part illustrates the method through a variety of examples.

Definition

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.

The technique involves two steps to prove a statement, as stated below:

Step 1(Base step): It proves that a statement is true for the initial value.

Step 2(Inductive step): It proves that if the statement is true for the n^{th} iteration (or number *n*), then it is also true for $(n+1)^{th}$ iteration (or number *n+1*).

How to Do It

Step 1: Consider an initial value for which the statement is true. It is to be shown that the statement is true for *n*=initial value.

Step 2: Assume the statement is true for any value of *n=k*. Then prove the statement is true for *n=k+1*. We actually break *n=k+1* into two parts, one part is n=k (which is already proved) and try to prove the other part.

Problem 1

3^n-1 is a multiple of 2 for n=1, 2, ...

Solution

Step 1: For n=1, 3^1-1 = 3-1 = 2 which is a multiple of 2

Step 2: Let us assume 3^n-1 is true for n=k, Hence, 3^k -1 is true (It is an assumption) We have to prove that 3^{k+1}-1 is also a multiple of

2 $3^{k+1} - 1 = 3 \times 3^k - 1 = (2 \times 3^k) + (3^k - 1)$

The first part (2×3^k) is certain to be a multiple of 2 and the second part $(3^k$ -1) is also true as our previous assumption.

Hence, $3^{k+1} - 1$ is a multiple of 2.

So, it is proved that $3^n - 1$ is a multiple of 2.

Problem 2

$1 + 3 + 5 + ... + (2n-1) = n^2$ for n=1, 2, ...

Solution

Step 1: For n=1, $1 = 1^2$, Hence, step 1 is satisfied.
Step 2: Let us assume the statement is true for n=k.

Hence, $1 + 3 + 5 + ... + (2k-1) = k^2$ is true (It is an assumption)

We have to prove that $1 + 3 + 5 + ... + (2(k+1)-1) = (k+1)^2$ also holds $1 + 3 + 5 + ... + (2(k+1) - 1)$

$= 1 + 3 + 5 + ... + (2k+2 - 1)$

$= 1 + 3 + 5 + ... + (2k + 1)$

$= 1 + 3 + 5 + ... + (2k - 1) + (2k + 1)$

$= k^2 + (2k + 1)$

$= (k + 1)^2$

So, $1 + 3 + 5 + ... + (2(k+1) - 1) = (k+1)^2$ hold which satisfies the step 2. Hence, $1 + 3 + 5 + ... + (2n - 1) = n^2$ is proved.

Problem 3

Prove that $(ab)^n = a^n b^n$ is true for every natural number *n*

Solution

Step 1: For n=1, $(ab)^1 = a^1 b^1 = ab$, Hence, step 1 is satisfied.
Step 2: Let us assume the statement is true for n=k, Hence, $(ab)^k = a^k b^k$ is true (It is an assumption).

We have to prove that $(ab)^{k+1} = a^{k+1} b^{k+1}$ also hold

Given, $(ab)^k = a^k b^k$
Or, $(ab)^k (ab) = (a^k b^k) (ab)$ [Multiplying both side by 'ab']
Or, $(ab)^{k+1} = (aa^k) (bb^k)$
Or, $(ab)^{k+1} = (a_{k+1} b_{k+1})$
Hence, step 2 is proved.
So, $(ab)^n = a^n b^n$ is true for every natural number n.

Strong Induction

Strong Induction is another form of mathematical induction. Through this induction technique, we can prove that a propositional function, *P(n)* is true for all positive integers, *n*, using the following steps:

- **Step 1(Base step):** It proves that the initial proposition *P(1)* true.
- **Step 2(Inductive step):** It proves that the conditional statement

13. RECURRENCE RELATION

In this chapter, we will discuss how recursive techniques can derive sequences and be used for solving counting problems. The procedure for finding the terms of a sequence in a recursive manner is called **recurrence relation**. We study the theory of linear recurrence relations and their solutions. Finally, we introduce generating functions for solving recurrence relations.

Definition

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i<n$).

Example: Fibonacci series: $F_n = F_{n-1} + F_{n-2}$, Tower of Hanoi: $F_n = 2F_{n-1} + 1$

Linear Recurrence Relations

A linear recurrence equation of degree k is a recurrence equation which is in the format $x_n= A_1\ x_{n-1}+\ A_2\ x_{n-1}+\ A_3\ x_{n-1}+\ldots\ A_k\ x_{n-k}$ (A_n is a constant and $A_k \neq 0$) on a sequence of numbers as a first-degree polynomial.

These are some examples of linear recurrence equations:

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1=a_2=1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1=1$, $a_2=3$	Lucas number
$F_n = F_{n-2} + F_{n-3}$	$a_1=a_2=a_3=1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1=0$, $a_2=1$	Pell number

How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is: $F_n = AF_{n-1} + BF_{n-2}$ where A and B are real numbers.

The characteristic equation for the above recurrence relation is:

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots:

Case 1: If this equation factors as $(x- x_1)(x- x_1) = 0$ and it produces two distinct real roots x_1 and x_2, then $F_n = ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]

Case 2: If this equation factors as $(x- x_1)^2 = 0$ and it produces single real root x_1, then $F_n = a\ x_1^n + bn\ x_1^n$ is the solution.

Case 3: If the equation produces two distinct real roots x_1 and x_2 in polar form $x_1 = r \angle \theta$ and $x_2 = r \angle(- \theta)$, then $F_n = r^n (a \cos(n\theta)+ b \sin(n\theta))$ is the solution.

Problem 1

Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

Solution

The characteristic equation of the recurrence relation is:

$$x^2 - 5x + 6=0,$$

So, $(x-3)(x-2) = 0$

Hence, the roots are:

$$x_1 = 3 \text{ and } x_2= 2$$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is:

$$F_n = ax_1^n + bx_2^n$$

Here, $F_n = a3^n + b2^n$ (As $x_1 = 3$ and $x_2=$
2) Therefore,

$1=F_0 = a3^0 + b2^0 = a+b$

1 1

Solving these two equations, we get a = 2 and b = -1

Hence, the final solution is:

$$F_n = 2.3^n + (-1) . 2^n = 2.3^n - 2^n$$

Problem 2

Solve the recurrence relation $F_n = 10F_{n-1} - 25F_{n-2}$ where $F_0 = 3$ and $F_1 = 17$

Solution

The characteristic equation of the recurrence relation is:

$x^2 -10x -25 =0,$

So, $(x - 5)^2 = 0$

Hence, there is single real root $x_1 = 5$

As there is single real valued root, this is in the form of case
2 Hence, the solution is:

$F_n = ax_1^n + bnx_1^n$

$3 = F_0= a.5^0 + b.0.5^0 = a$

$17 = F_1= a.5^1 + b.1.5^1 = 5a+5b$

Solving these two equations, we get a = 3 and b = 2/5

Hence, the final solution is:

$$F_n = 3.5^n + (2/5) .n.2^n$$

Problem 3

Solve the recurrence relation $F_n = 2F_{n-1} - 2F_{n-2}$ where $F_0 = 1$ and $F_1 = 3$

Solution

The characteristic equation of the recurrence relation is:

$$x^2 - 2x - 2 = 0$$

Hence, the roots are:

$$x_1 = 1 + i \quad \text{and} \quad x_2 = 1 - i$$

In polar form,

$x_1 = r \angle \theta$ and $x_2 = r \angle(-\theta)$, where $r = \sqrt{2}$ and $\theta = \pi / 4$

The roots are imaginary. So, this is in the form of case 3.

Hence, the solution is:

$$F_n = (\sqrt{2})^n (a \cos(n. \pi / 4) + b \sin(n. \pi / 4))$$

$$1 = F_0 = (\sqrt{2})^0 (a \cos(0. \pi / 4) + b \sin(0. \pi / 4)) = a$$

$$3 = F_1 = (\sqrt{2})^1 (a \cos(1. \pi / 4) + b \sin(1. \pi / 4)) = \sqrt{2} (a/\sqrt{2} + b/\sqrt{2})$$

Solving these two equations we get a = 1 and b = 2

Hence, the final solution is:

$$F_n = (\sqrt{2})^n (\cos(n. \pi / 4) + 2 \sin(n. \pi / 4))$$

Particular Solutions

A recurrence relation is called non-homogeneous if it is in the form

$$F_n = AF_{n-1} + BF_{n-2} + F(n) \quad \text{where } F(n) \neq 0$$

The solution (a_n) of a non-homogeneous recurrence relation has two parts. First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the particular solution (a_t). So, $a_n = a_h + a_t$

Let $F(n) = cx^n$ and x_1 and x_2 are the roots of the characteristic equation: $x^2 = Ax + B$ which is the characteristic equation of the associated homogeneous recurrence relation:

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $a_t = An^2x^n$

Problem

Solve the recurrence relation $F_n = 3F_{n-1} + 10F_{n-2} + 7.5^n$ where $F_0 = 4$ and $F_1 = 3$

Solution

The characteristic equation is:

$$x^2 - 3x - 10 = 0$$

Or, $(x - 5)(x + 2) = 0$

Or, $x_1 = 5$ and $x_2 = -2$

Since, $x = x_1$ and $x \neq x_2$, the solution is:

$$a_t = Anx^n = An5^n$$

After putting the solution into the non-homogeneous relation, we get:

$$An5^n = 3A(n-1)5^{n-1} + 10A(n-2)5^{n-2} + 7.5^n$$

Dividing both sides by 5^{n-2}, we get:

$$An5^2 = 3A(n-1)5 + 10A(n-2)5^0 + 7.5^2$$

Or, $25An = 15An - 15A + 10An - 20A + 175$

Or, $35A = 175$

Or, $A = 5$

So, $F_n = n5^{n+1}$

Hence, the solution is:

$$F_n = n5^{n+1} + 6.(-2)^n - 2.5^n$$

Generating Functions

Generating Functions represents sequences where each term of a sequence is expressed as a coefficient of a variable *x* in a formal power series.

Mathematically, for an infinite sequence, say 0, 1, 2, … … … …, , … … … , the generating function will be:

Some Areas of Application:

Generating functions can be used for the following purposes:

- For solving a variety of counting problems. For example, the number of ways to make change for a Rs. 100 note with the notes of denominations Rs.1, Rs.2, Rs.5, Rs.10, Rs.20 and Rs.50
- For solving recurrence relations
- For proving some of the combinatorial identities
- For finding asymptotic formulae for terms of sequences

Problem 1

What are the generating functions for the sequences { } with

Solution

When , generating function, G(x) = ∑∞ =0 2 = 2 + 2 + 2 2 + 2 3 + … … … = 3 , G( ) = ∑∞ =0 3 = 0 + 3 + 6 2 + 9 3 + … … …

When

=

2

Problem 2

What is the generating function of the infinite series; 1, 1, 1, 1, ……….?

Solution

Some Useful Generating Functions

5. PROPOSITIONAL LOGIC

The rules of mathematical logic specify methods of reasoning mathematical statements. Greek philosopher, Aristotle, was the pioneer of logical reasoning. Logical reasoning provides the theoretical base for many areas of mathematics and consequently computer science. It has many practical applications in computer science like design of computing machines, artificial intelligence, definition of data structures for programming languages etc.

Propositional Logic is concerned with statements to which the truth values, “true” and “false”, can be assigned. The purpose is to analyze these statements either individually or in a composite manner.

Prepositional Logic – Definition

A proposition is a collection of declarative statements that has either a truth value "true” or a truth value "false". A propositional consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B, etc). The connectives connect the propositional variables.

Some examples of Propositions are given below:

- "Man is Mortal", it returns truth value “TRUE”
- "12 + 9 = 3 – 2", it returns truth value “FALSE”

The following is not a Proposition:

- "A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true or false.

Connectives

In propositional logic generally we use five connectives which are: OR (V), AND (Λ), Negation/ NOT (¬), Implication / if-then (→), If and only if (⇔).

OR (V): The OR operation of two propositions A and B (written as A V B) is true if at least any of the propositional variable A or B is true.

The truth table is as follows:

A	**B**	**A V B**
True	True	True
True	False	True
False	True	True
False	False	False

AND (Λ): The AND operation of two propositions A and B (written as A Λ B) is true if both the propositional variable A and B is true.

The truth table is as follows:

A	B	A ∧ B
True	True	True
True	False	False
False	True	False
False	False	False

Negation (¬): The negation of a proposition A (written as ¬A) is false when A is true and is true when A is false.

The truth table is as follows:

A	¬A
True	False
False	True

Implication / if-then (→): An implication A →B is False if A is true and B is false. The rest cases are true.

The truth table is as follows:

A	B	A → B
True	True	True
True	False	False
False	True	True
False	False	True

If and only if (⇔): A ⇔B is bi-conditional logical connective which is true when p and q are both false or both are true.

The truth table is as follows:

A	B	A⇔B
True	True	True
True	False	False
False	True	False
False	False	True

Tautologies

A Tautology is a formula which is always true for every value of its propositional variables.

Example: Prove [(A → B) ∧ A] →B is a tautology

The truth table is as follows:

A	**B**	**A → B**	**(A → B) ∧ A**	**[(A → B) ∧ A] →B**
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

As we can see every value of [(A → B) ∧ A] →B is "True", it is a tautology.

Contradictions

A Contradiction is a formula which is always false for every value of its propositional variables.

Example: Prove (A V B) ∧ [(¬A) ∧ (¬B)] is a contradiction

The truth table is as follows:

A	**B**	**A V B**	**¬A**	**¬B**	**(¬A) ∧ (¬B)**	**(A V B) ∧ [(¬A) ∧ (¬B)]**
True	True	True	False	False	False	False
True	False	True	False	True	False	False
False	True	True	True	False	False	False
False	False	False	True	True	True	False

As we can see every value of (A V B) ∧ [(¬A) ∧ (¬B)] is "False", it is a contradiction.

Contingency

A Contingency is a formula which has both some true and some false values for every value of its propositional variables.

Example: Prove (A V B) ∧ (¬A) a contingency

The truth table is as follows:

A	**B**	**A V B**	**¬A**	**(A V B) ∧ (¬A)**
True	True	True	False	False
True	False	True	False	False
False	True	True	True	True
False	False	False	True	False

As we can see every value of (A V B) Λ (¬A) has both "True" and "False", it is a contingency.

Propositional Equivalences

Two statements X and Y are logically equivalent if any of the following two conditions hold:

- The truth tables of each statement have the same truth values.
- The bi-conditional statement X ⇔ Y is a tautology.

Example: Prove ¬ (A V B) and [(¬A) Λ (¬B)] are equivalent

Testing by 1st method (Matching truth table):

A	**B**	**A V B**	**¬ (A V B)**	**¬A**	**¬B**	**[(¬A) Λ (¬B)]**
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

Here, we can see the truth values of ¬ (A V B) and [(¬A) Λ (¬B)] are same, hence the statements are equivalent.

Testing by 2nd method (Bi-conditionality):

A	**B**	**¬ (A V B)**	**[(¬A) Λ (¬B)]**	**[¬ (A V B)] ⇔[(¬A) Λ (¬B)]**
True	True	False	False	True
True	False	False	False	True
False	True	False	False	True
False	False	True	True	True

As [¬ (A V B)] ⇔ [(¬A) Λ (¬B)] is a tautology, the statements are equivalent.

Inverse, Converse, and Contra-positive

A conditional statement has two parts: **Hypothesis** and **Conclusion**.

Example of Conditional Statement: "If you do your homework, you will not be punished." Here, "you do your homework" is the hypothesis and "you will not be punished" is the conclusion.

Inverse: An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If not *p*, then not *q*". The inverse of "If you do your homework, you will not be punished" is "If you do not do your homework, you will be punished."

Converse: The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If *q*, then *p*". The converse of "If you do your homework, you will not be punished" is "If you will not be punished, you do not do your homework".

Contra-positive: The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is "If p, then q", the inverse will be "If not *q*, then not *p*". The Contra-positive of " If you do your homework, you will not be punished" is" If you will be punished, you do your homework".

Duality Principle

Duality principle set states that for any true statement, the dual statement obtained by interchanging unions into intersections (and vice versa) and interchanging Universal set into Null set (and vice versa) is also true. If dual of any statement is the statement itself, it is said **self-dual** statement.

Example: The dual of (A ∩ B) ∪ C is (A ∪ B) ∩ C

Normal Forms

We can convert any proposition in two normal forms:

- Conjunctive normal form
- Disjunctive normal form

Conjunctive Normal Form

A compound statement is in conjunctive normal form if it is obtained by operating AND among variables (negation of variables included) connected with ORs.

Examples

- (P ∪Q) ∩ (Q ∪ R)
- (¬P ∪Q ∪S ∪¬T)

Disjunctive Normal Form

A compound statement is in conjunctive normal form if it is obtained by operating OR among variables (negation of variables included) connected with ANDs.

Examples

- (P ∩ Q) ∪ (Q ∩ R)
- (¬P ∩Q ∩S ∩¬T)

6. PREDICATE LOGIC

Discrete Mathematics

Predicate Logic deals with predicates, which are propositions containing variables.

Predicate Logic – Definition

A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.

The following are some examples of predicates:

- Let E(x, y) denote "x = y"
- Let X(a , b, c) denote "a + b + c = 0"
-

Well Formed Formula

Well Formed Formula (wff) is a predicate holding any of the following -

- All propositional constants and propositional variables are wffs
- If x is a variable and Y is a wff, $\forall x$ Y and $\exists x$ Y are also wff
- Truth value and false values are wffs
- Each atomic formula is a wff
- All connectives connecting wffs are wffs

Quantifiers

The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic: Universal Quantifier and Existential Quantifier.

Universal Quantifier

Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol $\forall$.

$\forall x$ P(x) is read as for every value of x, P(x) is true.

Example: "Man is mortal" can be transformed into the propositional form $\forall x$ P(x) where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

Existential Quantifier

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol $\exists$.

$\exists x$ P(x) is read as for some values of x, P(x) is true.

Example: "Some people are dishonest" can be transformed into the propositional form $\exists x\ P(x)$ where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

Nested Quantifiers

If we use a quantifier that appears within the scope of another quantifier, it is called nested quantifier.

Examples

- $\forall a\ \exists b\ P(x, y)$ where P (a, b) denotes $a + b = 0$
- $\forall a\ \forall b\ \forall c\ P(a, b, c)$ where P (a, b) denotes $a + (b+c) = (a+b) + c$

Note: $\forall a\ \exists b\ P(x, y) \neq \exists a\ \forall b\ P(x, y)$

7. RULES OF INFERENCE

Discrete Mathematics

To deduce new statements from the statements whose truth that we already know, **Rules of Inference** are used.

What are Rules of Inference for?

Mathematical logic is often used for logical proofs. Proofs are valid arguments that determine the truth values of mathematical statements.

An argument is a sequence of statements. The last statement is the conclusion and all its preceding statements are called premises (or hypothesis). The symbol "∴ ", (read therefore) is placed before the conclusion. A valid argument is one where the conclusion follows from the truth values of the premises.

Rules of Inference provide the templates or guidelines for constructing valid arguments from the statements that we already have.

Addition

If P is a premise, we can use Addiction rule to derive P V Q.

P

∴ P V Q

Example

Let P be the proposition, "He studies very hard" is true

Therefore: "Either he studies very hard Or he is a very bad student." Here Q is the proposition "he is a very bad student".

Conjunction

If P and Q are two premises, we can use Conjunction rule to derive P Λ Q.

P

Q

Example

Let P: "He studies very hard"

Let Q: "He is the best boy in the class"

Therefore: "He studies very hard and he is the best boy in the class"

Simplification

If P ∧ Q is a premise, we can use Simplification rule to derive P.

$$\frac{P \land Q}{\therefore P}$$

Example

"He studies very hard and he is the best boy in the

class" Therefore: "He studies very hard"

Modus Ponens

If P and P→Q are two premises, we can use Modus Ponens to derive Q.

$$\frac{\begin{array}{c}P \rightarrow Q \\ P\end{array}}{\therefore Q}$$

Example

"If you have a password, then you can log on to facebook"

"You have a password"

Therefore: "You can log on to facebook"

Modus Tollens

If P→Q and ¬Q are two premises, we can use Modus Tollens to derive ¬P.

$$\frac{\begin{array}{c}P \rightarrow Q \\ \neg Q\end{array}}{\therefore \neg P}$$

Example

"If you have a password, then you can log on to facebook"

"You cannot log on to facebook"

Therefore: "You do not have a password "

Disjunctive Syllogism

If ¬P and P V Q are two premises, we can use Disjunctive Syllogism to derive

Q. ¬P

P V Q

Example

"The ice cream is not vanilla flavored"

"The ice cream is either vanilla flavored or chocolate flavored"

Therefore: "The ice cream is chocolate flavored"

Hypothetical Syllogism

If P → Q and Q → R are two premises, we can use Hypothetical Syllogism to derive P →

R P → Q

Q → R

Example

"If it rains, I shall not go to school"

"If I don't go to school, I won't need to do homework"

Therefore: "If it rains, I won't need to do homework"

Constructive Dilemma

If (P → Q) Λ (R → S) and P V R are two premises, we can use constructive dilemma to derive Q V S.

(P → Q) Λ (R → S)

P V R

Example

"If it rains, I will take a leave"

"If it is hot outside, I will go for a shower"

"Either it will rain or it is hot outside"

Therefore: "I will take a leave or I will go for a shower"

Destructive Dilemma

If $(P \rightarrow Q) \land (R \rightarrow S)$ and $\neg Q \lor \neg S$ are two premises, we can use destructive dilemma to derive $P \lor R$.

$$(P \rightarrow Q) \land (R \rightarrow S)$$
$$\neg Q \lor \neg S$$

Example

"If it rains, I will take a leave"

"If it is hot outside, I will go for a shower"

"Either I will not take a leave or I will not go for a shower"

Therefore: "It rains or it is hot outside"